On the Invariance of Maximal Monotone Operators on Convex Sets and some Applications to the Porous Medium Equation and the p-Laplacian

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1 Introduction

We investigate the relation between monotone operators on Hilbert spaces, the generated semi-group and their resolvent.

In the first part of our work we give conditions on the operator under which the associated resolvent remains invariant on convex sets. Especially, we consider perturbations of a monotone operator by an unbounded linear operator. To this end, we use the invariance results on the sum of monotone operators and a lower-semicontinous function proven by Barthélemy [Ba], apply them to the perturbation results achieved by Stannat [St] and obtain several new results.

More precisely, we show invariance results on closed convex sets in three different settings. In the first basic setting we consider an operator A on a Hilbert space $\mathcal{H}$ without any perturbation. We generalize a well-known invariance result stated, for example, in [Br] and show that for an operator A on $\mathcal{H}$ not necessarily monotone $(Au, u - P_C u) \geq 0$ implies invariance of the resolvent of A on C. We show that the results of the basic setting can be transferred to the setting of [Ba] and adopt the notion of a semilinear monotone form $a(\cdot\,,\cdot)$.

Finally, we present our new results in the setting of [St]. We generalize the setting $\mathcal{V} \hookrightarrow \mathcal{H} \hookrightarrow \mathcal{V}'$ to the case where $\mathcal{V}$ is a reflexive Banach space. We consider a maximal monotone operator M perturbed by an unbounded linear operator Λ generalizing the notion of a bilinear form $\mathcal{E}(\cdot\,,\cdot)$ to the form $e(\cdot\,,\cdot)$, monotone in the first variable and linear in the second one. We show the existence of a resolvent of contractions G_α of the operator $A = M - \Lambda$ under some common assumptions and construct a semigroup of contractions on $\overline{D(A)}$. In our main theorem we show that $G_\alpha(C) \subset C$ for all $\alpha > 0$ if we assume that

$$u \in \mathcal{F} \Rightarrow P_C u \in \mathcal{V}, \|P_C u\|_{\mathcal{V}} \leq \|u\|_{\mathcal{V}} \text{ and } a(P_C u, u - P_C u) \geq 0 \text{ and}$$

$$u \in D(\Lambda, \mathcal{H}) \cap \mathcal{V} \Rightarrow \langle \Lambda u, u - P_C u \rangle \geq 0.$$

In the second part of our work we consider the porous medium equation as posed in [Sh] and the p-Laplacian.

In the porous medium equation example we show some useful invariance results while we shift the problem from the usual space $H^{-1}(G)$ to $L^2(0, T; H^{-1}(G))$. This seems to be a promising way to solve the problem, however, we are confronted with an unsolved problem in the end. We describe the crucial issue so that future research might reveal new insights.

In the second example, we show that the p-Laplacian is a maximal monotone operator on $L^p(0, T; L^p(G))$ fulfilling the conditions of our main theorem. So we can apply our invariance results and obtain that for an initial condition $u_0 \in L^p(0,T;L^p(G))$ which is contained in a closed, convex set C, the unique solution to the Cauchy problem will be contained in C, too.

We relate mostly to the theory as summarized by Showalter in [Sh] and Brézis in [Br]. Van Beusekom [Be] already showed that the p-Laplacian is a maximal monotone operator even for $p > 1$, but only in the spaces $L^p(G)$ and $L^{p'}(G)$. The set of pure potentials investigated there may be an example for more complex convex sets than the ones we consider here.

Cipriani and Grillo [CG] did a lot of work on the p-Laplacian and discussed nonlinear Dirichlet forms in a more general frame than we do. However, they consider only symmetric forms while we consider forms which are not even sectorial. Future results might lead to combine these works and point out deeper connections.

2 Preliminaries

Let $\mathcal{H}$ always be a real Hilbert space and let $(\cdot\,,\cdot\,)$ denote its inner product. Set $\|\cdot\| := (\cdot\,,\cdot\,)^{\frac{1}{2}}$. Often we will consider a real reflexive Banach space $\mathcal{V}$, densely and continuously embedded in $\mathcal{H}$. If not stated otherwise, let $\langle\cdot\,,\cdot\,\rangle$ always denote the dualization between $\mathcal{V}$ and $\mathcal{V}'$.

Identifying $\mathcal{H}$ with its dual $\mathcal{H}'$ via the Riesz isometry we have

$$\mathcal{V} \hookrightarrow \mathcal{H} \cong \mathcal{H}' \hookrightarrow \mathcal{V}'$$

continuously and densely and $\langle\cdot\,,\cdot\,\rangle_{\mathcal{V}\times\mathcal{H}} = (\cdot\,,\cdot\,)_{\mathcal{H}}$.

We will present firstly some different types of operators on a Hilbert space and some well-known properties. We leave out the proofs of the propositions and refer to [Br] and [Sh] for the details.

The concept of monotone operators is fundamental in nonlinear operator theory.

Definition 2.1: An operator $A : \mathcal{H} \to \mathcal{H}$ is called *monotone* if for $u, v \in D(A)$

$$(Au - Av, u - v) \geq 0.$$

Analogously, an operator $A : \mathcal{V} \to \mathcal{V}'$ is called monotone if

$$_{\mathcal{V}'}\langle Au - Av, u - v\rangle_{\mathcal{V}} \geq 0.$$

It is called *strictly monotone* if the inequality is strict for all $u \neq v$ and *strongly monotone* if for all $u, v \in \mathcal{H}$ there is some $c > 0$ such that

$$(Au - Av, u - v) \geq c\,\|u - v\|^2$$

Definition 2.2: The function $A : \mathcal{V} \to \mathcal{V}'$ is *coercive* if

$$\frac{A(u)(u)}{\|u\|_{\mathcal{V}}} \to \infty, \text{ as } \|u\|_{\mathcal{V}} \to \infty.$$

Definition 2.3: An operator A on $\mathcal{H}$ (or on $\mathcal{V}$, respectively) is called *maximal monotone* if it is maximal in the set of monotone operators, where maximality refers to the graphs of the operators.

Remark 2.1: An alternative definition of a maximal monotone operator is the following more useful one:
A is maximal monotone if and only if A is monotone and $Rg(A + \alpha I) = \mathcal{H}$ for all $\alpha > 0$ (cf. [Br], Prop. 2.2, p. 23).

We will often add monotone operators. It is important to know under which circumstances the maximal monotonicity remains intact.

Proposition 2.1: Let A and B be two operators on $\mathcal{H}$. If A is maximal monotone and B is monotone and Lipschitz continuous on $\mathcal{H}$, then $A + B$ is maximal monotone.

Proof: See [Sh], Lemma 2.1, p. 165.

The following not so widely used notions appear in the literature. We will not use them to the same extent as the core concepts mentioned above. We will refer to these facts later on and recommend the reader to return to them when needed. However, the relations will be important to understand the connections between [Ba] and [St].

Definition 2.4: An operator $A : \mathcal{V} \to \mathcal{V}'$ is called *hemicontinuous* if for each $u, v, w \in \mathcal{V}$ the real-valued function $t \mapsto A(u + tw)(v)$ is continuous.

Proposition 2.2: If $A : \mathcal{V} \to \mathcal{V}'$ is monotone and hemicontinuous then A is maximal monotone.

Proof: See [Sh], Proposition 2.2 and Lemma 2.1, p. 38f.

Definition 2.5: An operator $A : \mathcal{V} \to \mathcal{V}'$ is called *pseudo-monotone* if $u_n \rightharpoonup u$ and $\limsup Au_n(u_n - u) \leq 0$ imply $Au(u - v) \leq \liminf Au_n(u_n - v)$ for all $v \in \mathcal{V}$.

Proposition 2.3: If $A : \mathcal{V} \to \mathcal{V}'$ is monotone and hemicontinuous then A is pseudo-monotone.

Proof: See [Sh], Proposition 2.2, p.41.

We considered the different types of operators. To every maximal monotone operator there exist two more objects: Its resolvent and its semigroup.

Definition 2.6: Let A be maximal monotone. The operator defined by $J_\alpha = (I + \alpha A)^{-1}$ on $\mathcal{H}$ is called the *resolvent of* A *on* $\mathcal{H}$.

Proposition 2.4: Each J_α is a contraction on $\mathcal{H}$. The family of resolvents $(J_\alpha)_{\alpha>0}$ satsifies the resolvent equation

$$J_\alpha = J_\beta \circ (\frac{\beta}{\alpha} I + (1 - \frac{\beta}{\alpha}) J_\alpha), \qquad \alpha, \beta > 0.$$

Proof: See [Sh], p.159.

Definition 2.7: Let A be maximal monotone on $\mathcal{H}$. Then the *Yosida* *approximation* of A is the operator

$$A_\alpha = \frac{1}{\alpha}(I - J_\alpha), \qquad \alpha > 0.$$

Proposition 2.5: Let A be a maximal monotone operator on a Hilbert space $\mathcal{H}$. Then the following hold:

i) Each A_α is maximal monotone and Lipschitz continuous with constant $\frac{1}{\alpha}$, $\alpha > 0$.

ii) $(A_\alpha)_\beta = A_{\alpha+\beta}$, $\alpha, \beta > 0$.

iii) For each $u \in D(A)$, $\|A_\alpha u\|$ converges upward to $\|Au\|$, $\lim_{\alpha \to 0} A_\alpha(u) = Au$, and
$$\|A_\alpha u - Au\|^2 \leq \|Au\|^2 - \|A_\alpha u\|^2, \qquad \alpha > 0.$$

iv) For each $u \notin D(A)$, $\|A_\alpha u\|$ is increasing and unbounded as $\alpha \to 0$.

Proof: See [Sh], Theorem IV.1.1, p.161.

Definition 2.8: Let K be a subset of a Hilbert space $\mathcal{H}$ and let $\{S(t)\}_{t \geq 0}$ be a family of mappings from K to K dependent on a parameter t.

Then $S(t)$ is called a *strongly continuous semigroup of nonlinear contractions* (or for convenience only *semigroup*) on K if it satisfies the following properties:

(1) $S(0) = Id$ and $S(t_1) \circ S(t_2) = S(t_1 + t_2)$ for all $t_1, t_2 \geq 0$.

(2) $\lim_{t \to 0} \|S(t)u - u\| = 0$ for all $u \in K$.

(3) $\|S(t)u - S(t)v\| \leq \|u - v\|$ for all $u, v \in K$ and for all $t \geq 0$.

Definition 2.9: We say that a semigroup of contractions is *generated by the operator* $-A : \mathcal{H} \to \mathcal{H}$ if we have for all $u \in D(A)$ that

$$\lim_{t \to 0} \frac{1}{t}(u - S(t)u) = -Au$$

So $S(t)u$ can be regarded as the solution of the Cauchy problem

$$\frac{d}{dt}u = -Au.$$

by identifying $S(t)u_0 = u(t)$ for some initial condition $u(0) = u_0 \in \mathcal{H}$.

To the well-known theorem about the correspondence between linear semi-groups and linear operators by Yosida and Phillips there exists a non-linear counterpart using maximal monotone operators instead.

Theorem 2.1: For every maximal monotone operator A on a Hilbert space $\mathcal{H}$ there exists a unique semigroup $S(t)$ on $\overline{D(A)}$ which is generated by $-A$.

Conversely, let C be a closed, convex subset of $\mathcal{H}$. Then for every semi-group $S(t)$ on C there exists a unique maximal monotone operator A such that $\overline{D(A)} = C$ and $S(t)$ coincides with the semi-group generated by $-A$.

Proof: See [Br], Theorems 3.1/4.1.

Remark 2.2: When speaking of abstract convex sets we recommend the reader to think of a practical example. Typical examples for convex sets in a function space like $L^2(\mathbb{R})$ would be the set of all positive functions

$$L^2_+(\mathbb{R}) := \{f \in L^2(\mathbb{R}) \mid f \geq 0\}$$

or the set of all sub-markovian functions

$$L^2_m(\mathbb{R}) := \{f \in L^2(\mathbb{R}) \mid 0 \leq f \leq 1\}.$$

We shall use the following projection theorem for Hilbert spaces:

Theorem 2.2 (Projection Theorem): For each closed convex non-empty subset C of $\mathcal{H}$ there is a projection operator $P_C : \mathcal{H} \to C$ for which $P_C(u_0)$ is that point of C with minimal distance to $u_0 \in \mathcal{H}$; it is characterized by

$$P_C(u_0) \in \mathcal{H} : (P_C(u_0) - u_0, v - P_C(u_0)) \geq 0, \qquad v \in C.$$

Proof: See [Sh], Cor. I.2.1, p.9.

It follows from this characterization that the function P_C satisfies

$$\|P_C(u_0) - P_C(v_0)\|^2 \leq (P_C(u_0) - P_C(v_0), u_0 - v_0), \quad u_0, v_0 \in \mathcal{H}.$$

From this we see that P_C is a contraction, i.e.,

$$\|P_C(u_0) - P_C(v_0)\| \leq \|u_0 - v_0\|, \quad u_0, v_0 \in \mathcal{H},$$

and that the operator P_C is monotone

$$(P_C(u_0) - P_C(v_0), u_0 - v_0) \geq 0, \quad u_0, v_0 \in \mathcal{H}.$$

Remark 2.3: For our typical examples of a convex set $L^2_+(\mathbb{R})$ and $L^2_m(\mathbb{R})$ we have the fairly obvious projections given by $P_C u := u^+$ and $P_C u := u^+ \wedge 1$, where $u^+ := \max\{u, 0\}$ and $u \wedge v := \min\{u, v\}$.

Proposition 2.6 (Minty-Rockafellar): If $A : \mathcal{H} \to \mathcal{H}$ is maximal monotone and $(J_\alpha)_{\alpha>0}$ its resolvent then $\overline{D(A)}$ is convex and $\lim_{\alpha\to 0} J_\alpha(u) = P_{\overline{D(A)}}(u)$ for each $u \in \mathcal{H}$.

Proof: See [Sh], Prop. IV.1.7, p. 160.

Theorem 2.3 (Brézis): Let A be a maximal monotone operator on the Hilbert space $\mathcal{H}$ and let $S(t)$ be the semi-group generated by $-A$. Let C be a closed convex subset of $\mathcal{H}$, such that $P_{\overline{D(A)}}(C) \subset C$. Then the following properties are equivalent:

i) $(I + \alpha A)^{-1} C \subset C \quad$ for all $\alpha > 0$.

ii) $(Au, u - P_C u) \geq 0 \quad$ for all $u \in D(A)$.

iii) $S(t)(\overline{D(A)} \cap C) \subset C \quad$ for all $t \geq 0$.

Proof: See [Br], Prop. 4.5.

The last Theorem is an important fact that one should keep in mind when we discuss the invariance of convex sets under some resolvent. By the Theorem this invariance property of the resolvent implies directly the invariance of the semigroup, which in turn gives the solutions of the Cauchy problem (see Definition 2.9).
In our typical example $L^2_+(\mathbb{R})$ this means that if we have a positivity preserving resolvent we can conclude immediately from a positive initial data function that the solution of the corresponding Cauchy problem must be positive, too.
Analogously, a sub-markovian input function u_0 would mean that the solution $u(t)$ to the problem will be sub-markovian again.

3 Results

As we will show results in three different settings, this section is subdivided in the corresponding subsections. Of course, these settings are closely related to each other and do not stand apart.

General Setting as in [Br]

Since we can relax some of the assumptions of the last Theorem and since the proof in [Br] is quite indirect, we give a direct proof for the relation between an operator and its resolvent.

Proposition 3.1: Let $\mathcal{H}$ be a real Hilbert space, $C \subset \mathcal{H}$ a closed convex set and P_C the (orthogonal) projection onto C. Let A be an operator on $\mathcal{H}$ and $\alpha > 0$ such that $I + \alpha A : D(A) \to \mathcal{H}$ is one-to-one. Define $J_\alpha := (I + \alpha A)^{-1}$. Assume that $C \subset D(J_\alpha) := Rg(I + \alpha A)$.

Furthermore, let

$$(Au, u - P_C u) \geq 0, \quad \forall u \in D(A). \tag{1}$$

Then we have that $J_\alpha(C) \subset C$.

Proof: Let $u \in C$. Then we have

$$\begin{aligned} & (J_\alpha u - P_C(J_\alpha u), J_\alpha u - P_C(J_\alpha u)) \\ \overset{(1)}{\leq} \; & (J_\alpha u - P_C(J_\alpha u), J_\alpha u + \alpha A J_\alpha u) - (J_\alpha u - P_C(J_\alpha u), P_C(J_\alpha u)) \\ = \; & (J_\alpha u - P_C(J_\alpha u), u) - (J_\alpha u - P_C(J_\alpha u), P_C(J_\alpha u)) \\ = \; & (J_\alpha u - P_C(J_\alpha u), u - P_C(J_\alpha u)) \\ \leq \; & 0 \end{aligned}$$

by the Projection Theorem. Consequently,

$$\|J_\alpha u - P_C(J_\alpha u)\|_{\mathcal{H}}^2 \leq 0.$$

And this implies that $J_\alpha u = P_C(J_\alpha u)$, i.e. $J_\alpha u \in C$.

Note that we needed neither monotonicity of A nor the contraction properties of the resolvent for the proof.

The other direction of the desired equivalence is based on the statement of the Theorem 2.3 (Brézis) that

$$(I + \alpha A)^{-1} C \subset C \text{ for every } \alpha > 0 \Leftrightarrow S(t)(\overline{D(A)} \cap C) \subset C \text{ for every } t \geq 0.$$

We have to use the implication from left to right for our next proposition. For that reason we have to make stronger assumptions on A.

Proposition 3.2: Let A be a maximal monotone operator on a real Hilbert space $\mathcal{H}$ and let $J_\alpha = (I + \alpha A)^{-1}$ be the corresponding resolvent. Let $S(t)$ denote the semigroup generated by $-A$.

Furthermore, let $C \subset \mathcal{H}$ be closed and convex, $J_\alpha(C) \subset C$ for all $\alpha > 0$ and P_C be the orthogonal projection in $\mathcal{H}$ onto C. Let $P_C(D(A)) \subset \overline{D(A)}$. Then we have

$$(Au, u - P_C u) \geq 0 \text{ for all } u \in D(A).$$

Proof: Let $u \in D(A)$. We know from Theorem 2.3 (Brézis) that $S(t)(\overline{D(A)} \cap C) \subset C$ for all $t \geq 0$.

Thus we conclude that $S(t)(P_C u) \in C$ for all $t \geq 0$, since $P_C u \in \overline{D(A)}$ by assumption. With the Projection Theorem we obtain

$$\begin{aligned}(u - S(t)P_C u, u - P_C u) &= (u - P_C u, u - P_C u) + (P_C u - S(t)P_C u, u - P_C u) \\ &\geq \|u - P_C u\|_{\mathcal{H}}^2.\end{aligned}$$

This leads to

$$\begin{aligned}(Au, u - P_C u) &= \lim_{t\to 0} \frac{1}{t}(u - S(t)u, u - P_C u) \\ &= \lim_{t\to 0} \frac{1}{t}((S(t)P_C u - S(t)u, u - P_C u) + (u - S(t)P_C u, u - P_C u)) \\ &\geq \limsup_{t\to 0} \frac{1}{t}((S(t)P_C u - S(t)u, u - P_C u) + (u - P_C u, u - P_C u)) \\ &\geq \limsup_{t\to 0} \frac{1}{t}(-\|S(t)u - S(t)P_C u\|\|u - P_C u\| + \|u - P_C u\|^2) \\ &= \limsup_{t\to 0} \frac{1}{t}(\|u - P_C u\|(\|u - P_C u\| - \|S(t)u - S(t)P_C u\|)) \\ &\geq 0,\end{aligned}$$

since $S(t)$ is a contraction on $\mathcal{H}$, i.e. $\|S(t)u - S(t)P_C u\| \leq \|u - P_C u\|$.

Setting as in [Ba]

We will now move to a more special framework based on the work and the notation of [Ba].

It is defined there a form $a(\cdot\,,\cdot)$ which is monotone in the first variable and linear in the second one. While the work of [Ba] concentrates more on the relations of this operator to its semigroup we are more interested in its relation to its resolvent. We investigate which conditions are sufficient and necessary for the resolvent to be invariant on convex sets.

Let now $\mathcal{V}$ be a real reflexive Banach space densely and continuously embedded in $\mathcal{H}$ and let $a : \mathcal{V} \times \mathcal{V} \to \mathbb{R}$ be an application satisfying the following properties:

i) $a(u, \cdot\,) \in \mathcal{V}'$ for all $u \in \mathcal{V}$.

ii) $a(u, u - \hat{u}) \geq a(\hat{u}, u - \hat{u})$ for all $u, \hat{u} \in \mathcal{V}$ (monotonicity).

iii)
$$\lim_{t \to 0} a(u + tv, w) = a(u, w) \text{ for all } u, v, w \in \mathcal{V} \text{ (hemicontinuity).}$$

iv) For all $u_0 \in \mathcal{V}$ we have that
$$\lim_{\|u\|_{\mathcal{V}} \to \infty} \frac{a(u, u - u_0) + \|u\|_{\mathcal{H}}^2}{\|u\|_{\mathcal{V}}} = \infty \text{ (coercivity)}$$

To give the reader a better imagination of this setting we recommend to think always of the typical example $L^p(G) \hookrightarrow L^2(G) \hookrightarrow L^{p'}(G)$, where $G \subset \mathbb{R}^n$ bounded and $p \geq 2$.

Remark 3.1: Note that in the setting of [Ba] an operator $(A, D(A))$ on $\mathcal{H}$ associated to a is defined as follows:

$$u \in D(A) \Leftrightarrow u \in \mathcal{V} \text{ and there is some } Au \in \mathcal{H} \text{ satisfying } (Au, w) = a(u, w)$$

for all $w \in \mathcal{V}$. This is the definition of A by the Riesz isometry identifying $D(A)$ with the following set:

$$D(A) := \{u \in \mathcal{V} |\, \mathcal{V} \ni w \mapsto a(u, w) \text{ is continuous on } \mathcal{H}\}$$

It is known that this operator A is maximal monotone on $\mathcal{H}$ (cf. [Br], Exemple 2.3.7, p.26). This follows from the hemicontinuity and the monotonicity of a (cf. Proposition 2.2).

We will now establish the above mentioned equivalence with the following two propositions. A very similar and much more general equivalence is proven in [Ba] between the semigroup and the operator. Our results fit in nicely with those of [Ba].

Proposition 3.3: Let C be a closed convex subset of $\mathcal{H}$, P_C the projection from $\mathcal{H}$ onto C. Let $J_\alpha := (I + \alpha A)^{-1}$, $\alpha > 0$, be the resolvent of A on $\mathcal{H}$.

Suppose that $u \in \mathcal{V} \Rightarrow P_C u \in \mathcal{V}$ and $a(u, u - P_C u) \geq 0$.
Then we have

$$J_\alpha(C) \subset C \text{ for all } \alpha > 0.$$

Proof: Let $u \in D(A)$. Then by assumption $u \in \mathcal{V}$ and also by assumption $P_C u \in \mathcal{V}$. Thus,

$$(Au, u - P_C u) = a(u, u - P_C u) \geq 0.$$

Then the claim follows from Proposition 3.1.

As for the first two propositions the other direction turns out to be harder to prove.

Proposition 3.4: Let C, P_C and J_α, $\alpha > 0$, be as in Proposition 3.3 and suppose that $J_\alpha(C) \subset C$ for all $\alpha > 0$. Then we have

$$u \in \mathcal{V} \Rightarrow P_C u \in \mathcal{V} \text{ and } a(P_C u, u - P_C u) \geq 0.$$

Remark 3.2: This implies also that $a(u, u - P_C u) \geq 0$, since $a(u, u - P_C u) \geq a(P_C u, u - P_C u) \geq 0$ by the monotonicity of $a(\cdot,\cdot)$ (property ii)).

Proof: Let $u \in \mathcal{V}, \alpha > 0$. From the properties of the resolvent we conclude that $J_\alpha(P_C u) \in D(A) \subset \mathcal{V}$. Furthermore, using the definition of J_α we have that

$$\begin{aligned}
& a(J_\alpha P_C u, J_\alpha P_C u - u) \\
= \; & (AJ_\alpha P_C u, J_\alpha P_C u - u) \\
= \; & \frac{1}{\alpha}(P_C u - J_\alpha P_C u, J_\alpha P_C u - u) \\
= \; & -\frac{1}{\alpha}(P_C u - J_\alpha P_C u, P_C u - J_\alpha P_C u) + \frac{1}{\alpha}(P_C u - J_\alpha P_C u, P_C u - u) \\
= \; & -\frac{1}{\alpha}\|P_C u - J_\alpha P_C u\|^2 + \frac{1}{\alpha}(J_\alpha P_C u - P_C u, u - P_C u) \\
\leq \; & 0
\end{aligned}$$

by the Projection Theorem since $J_\alpha P_C u \in C$ by assumption.
This implies that

$$a(J_\alpha P_C u, J_\alpha P_C u - u) + \|J_\alpha P_C u\|^2 \leq \|J_\alpha P_C u\|^2 \leq c\|P_C u\| \, \|J_\alpha P_C u\|_{\mathcal{V}}$$

for some constant $c > 0$, since $\|\cdot\| \leq c\|\cdot\|_{\mathcal{V}}$.
By the coercivity in iv) we conclude that $(J_\alpha P_C u)_{\alpha>0}$ is bounded in $\mathcal{V}$. Since $J_\alpha P_C u \overset{\alpha \to 0}{\longrightarrow} P_C u$ in $\mathcal{H}$ by Lemma 3.1 stated and proven below, e.g. by

[MR], Lemma 2.12, we also know that $P_C u \in \mathcal{V}$. So, applying Lemma 3.1 with $P_C u$ replacing u we can conclude that

$$a(P_C u, P_C u - u) = \lim_{\alpha \to 0} a(J_\alpha P_C u, J_\alpha P_C u - u) \leq 0.$$

Hence,

$$a(P_C u, u - P_C u) \geq 0.$$

The lemma we used for the proof above is part of Lemma 1.8 from [Ba]. Some more general results are proven there, but we only quote the statements needed for our work. As the proof of this part of Lemma 1.8 in [Ba] is not very detailed we give a more explicit proof here.

Lemma 3.1: Let A be maximal monotone, let $J_\alpha = (I + \alpha A)^{-1}$ be the resolvent of A and let $u \in \mathcal{V}$.

As $\alpha \to 0$ we have that $J_\alpha u \to u$ in $\mathcal{H}$, $J_\alpha u \rightharpoonup u$ in $\mathcal{V}$, $a(J_\alpha u, w) \to a(u, w)$ for every $w \in \mathcal{V}$ and $a(J_\alpha u, J_\alpha u) \to a(u, u)$. In particular, $\overline{D(A)} = \mathcal{H}$.

Proof: Let $u \in \mathcal{V}$ and $u_\alpha := J_\alpha u$.
We know by the result of Minty-Rockafellar (cf. Proposition 2.6) that $u_\alpha \to P_{\overline{D(A)}} u$ in $\mathcal{H}$ as $\alpha \to 0$.

By definition of J_α we have that

$$a(u_\alpha, u_\alpha - u) = \frac{1}{\alpha}(u - u_\alpha, u_\alpha - u) = -\frac{1}{\alpha}\|u - u_\alpha\|^2 \leq 0. \tag{2}$$

So by the coercivity we conclude that $\{u_\alpha\}_{\alpha > 0}$ is bounded in $\mathcal{V}$, hence by monotonicity (2) implies that

$$0 = \lim_{\alpha \to 0} \alpha\, a(u, u_\alpha - u) \leq \limsup_{\alpha \to 0} \alpha\, a(u_\alpha, u_\alpha - u) = -\|u - P_{\overline{D(A)}} u\|^2.$$

So, $u = P_{\overline{D(A)}} u$ and therefore $u_\alpha \to u$ in $\mathcal{H}$ as $\alpha \to 0$ and hence e.g. by [MR], Lemma 2.12, it follows that $u_\alpha \rightharpoonup u$ in $\mathcal{V}$.

Let $w \in \mathcal{V}$, $t > 0$. Then first using (2) and then monotonicity we obtain

$$\begin{aligned} \liminf_{\alpha \to 0} a(u_\alpha, w) &\geq \frac{1}{t} \liminf_{\alpha \to 0} a(u_\alpha, u_\alpha - u + tw) \\ &\geq \frac{1}{t} \liminf_{\alpha \to 0} a(u - tw, u_\alpha - u + tw) \\ &= a(u - tw, w), \end{aligned}$$

where we used in the last step that $u_\alpha \rightharpoonup u$ in $\mathcal{V}$. Letting $t \to 0$ by hemicontinuity it follows that

$$\liminf_{\alpha \to 0} a(u_\alpha, w) \geq a(u, w).$$

Replacing w by $-w$ we obtain $\limsup_{\alpha\to 0} a(u_\alpha, w) \le a(u, w)$, so

$$\lim_{\alpha\to 0} a(u_\alpha, w) = a(u, w) \tag{3}$$

for all $w \in \mathcal{V}$. Furthermore,

$$a(u_\alpha, u_\alpha) - a(u, u) = a(u_\alpha, u_\alpha - u) + a(u_\alpha, u) - a(u, u). \tag{4}$$

But by (2) and monotonicity

$$a(u, u_\alpha - u) \le a(u_\alpha, u_\alpha - u) \le 0,$$

hence by (3)

$$\lim_{\alpha\to 0} a(u_\alpha, u_\alpha - u) = 0$$

and thus by (4)

$$\lim_{\alpha\to 0} a(u_\alpha, u_\alpha) = a(u, u).$$

Since $u_\alpha = J_\alpha u \in D(A)$, it follows from the first assertion that $u \in \overline{D(A)}$. Since $u \in \mathcal{V}$ was arbitrary, it follows that $\mathcal{V} \subset \overline{D(A)}$. But $\overline{\mathcal{V}} = \mathcal{H}$ by assumption.

Setting as in [St]

Following the framework of [St], we now consider a maximal monotone (nonlinear) operator M, perturbed by an unbounded, linear operator Λ. In general, the operator obtained by adding these two operators is not maximal monotone on $\mathcal{H}$. We show firstly, that nonetheless the resolvent of this operator has the usual properties and then give some criteria for which the resolvent is invariant on convex sets.

We generalize his results to reflexive Banach spaces $\mathcal{V}$ and $\mathcal{V}'$.

Let $A := M - \Lambda : \mathcal{F} \mapsto \mathcal{V}'$, where M, Λ and $\mathcal{F}$ are defined as follows.

$M : \mathcal{V} \to \mathcal{V}'$ satisfies the properties

(M1) M is hemicontinuous.

(M2) $\langle Mu - Mv, u - v\rangle \ge 0$ for all $u, v \in \mathcal{V}$ with equality only if $u = v$ (strict monotonicity).

(M3) $\frac{\langle Mu, u-u_0\rangle}{\|u\|_\mathcal{V}} \to \infty$ as $\|u\|_\mathcal{V} \to \infty$ for all $u_0 \in \mathcal{V}$ (coercivity).

Remark 3.3: Since M is defined on all of $\mathcal{V}$, it follows by a result of Browder and Rockafellar that its monotonicity implies its local boundedness, i.e., $M(B)$ is a bounded set in $\mathcal{V}'$ whenever B is a bounded set in $\mathcal{V}$ (cf. [R]). We shall use this below without further notice.

Note also that our assumptions are more general than the ones made in [St]. Still, the results carry over and we will show this below.

Let $\Lambda : D(\Lambda, \mathcal{H}) \to \mathcal{H}'$ be a linear operator generating a C_0-semigroup $(U_t)_{t\geq 0}$. We assume that $(U_t)_{t\geq 0}$ can be restricted to a C_0-semigroup in $\mathcal{V}$. Then this is the corresponding semigroup to the part of Λ on $\mathcal{V}$. The corresponding semigroup can be extended to a semigroup on $\mathcal{V}'$ and its generator is the dual operator of Λ (cf. [St] and [Pa]).

Let $(V_\alpha)_{\alpha>0}$ denote the resolvent corresponding to $(\Lambda, D(\Lambda, \mathcal{H}))$ and let $(\hat{V}_\alpha)_{\alpha>0}$ denote the dual resolvent corresponding to the dual operator $(\hat{\Lambda}, D(\hat{\Lambda}, \mathcal{H}'))$ of Λ.

Let $(\Lambda, \mathcal{F})$ denote the closure of the operator $\Lambda : D(\Lambda, \mathcal{H}) \cap \mathcal{V} \to \mathcal{V}'$. Then $\mathcal{F}$ is a real Banach space with the norm

$$\|u\|_{\mathcal{F}}^2 = \|u\|_{\mathcal{V}}^2 + \|\Lambda u\|_{\mathcal{V}'}^2.$$

Note that $(A, \mathcal{F})$ as an operator from $\mathcal{V}$ to $\mathcal{V}'$ is monotone and $\mathcal{F}$ is dense in $\mathcal{V}$. For $u \in \mathcal{F}, v \in \mathcal{V}$ we define

$$a(u, v) := \langle Mu, v\rangle \text{ and } e(u, v) := a(u, v) - \langle \Lambda u, v\rangle$$

and for $\alpha > 0$

$$e_\alpha(u, v) := \alpha\, e(u, v) + (u, v)_{\mathcal{H}}.$$

Note that the definitions vary a bit from the ones used in [St]. This difference has its origin in the different definitions of the resolvent. In the linear case like in [St] one considers the resolvent G_α, where αG_α is a contraction for all $\alpha > 0$ and the strong continuity is assumed for $\alpha \to \infty$. In the nonlinear case however, the resolvent J_α is a contraction right away and strong continuity holds for $\alpha \to 0$, so α always has to be thought of as being 'small'. As in [St], we will show that the resolvent is given by the inverse mapping to e_α.

Stannat showed the existence of a resolvent for the operator A in the case that M is linear and the underlying space $\mathcal{V}$ a real Hilbert space. We now show that this is even the case when M is nonlinear and $\mathcal{V}$ a reflexive Banach space.

Lemma 3.2: Let M satisfy assumptions $(M1) - (M3)$. Then M is a bijection.

Proof: By [Sh], Corollary II.2.2, M is surjective since it is hemicontinuous, monotone, bounded and coercive on a reflexive Banach space. The injectivity is obvious by $(M2)$.

Proposition 3.5: Let M satisfy assumptions $(M1)-(M3)$ and $f \in \mathcal{V}'$. Then there exists one and only one solution $u \in \mathcal{F}$ to the equation $Mu - \Lambda u = f$.

Remark 3.4: Note that we consider here the case where $A = M - \Lambda : \mathcal{F} \supset \mathcal{V} \to \mathcal{V}'$ is monotone. So A is defined only on a subset of $\mathcal{V}$. This proposition corresponds to [St], Proposition I.3.2. The proof is very similar, but since it has not been done before in this setting, we repeat it here.

Proof: Firstly, we will show the existence. We proceed in three steps.

Existence:

For $\alpha > 0$ let the Yosida-approximations $\Lambda_\alpha : \mathcal{V} \to \mathcal{V}'$ be defined by $\langle \Lambda_\alpha u, \cdot \rangle := \frac{1}{\alpha}(V_\alpha u - u, \cdot)_{\mathcal{H}}$.

Step 1: "Approximation" of the equation $Mu - \Lambda u = f$ through the equation $Mu - \Lambda_\alpha u = f$.

Since $\|V_\alpha\|_{L(\mathcal{H})} \leq 1$ we obtain that $\langle \Lambda_\alpha u, u \rangle \leq 0$ for all $u \in \mathcal{V}$. Λ_α is linear and bounded (hence continuous) since

$$|\langle \Lambda_\alpha u, v \rangle| \leq \frac{2}{\alpha} \|u\|_{\mathcal{H}} \|v\|_{\mathcal{H}} \leq \frac{2}{\alpha} \|u\|_{\mathcal{V}} \|v\|_{\mathcal{V}}$$

which implies $\|\Lambda_\alpha u\|_{\mathcal{V}'} \leq \frac{2}{\alpha}\|u\|_{\mathcal{V}}$. Therefore, Λ_α is continuous on $\mathcal{V}$ and thus $M - \Lambda_\alpha$ satisfies assumption $(M1)$.

Since

$$\langle (M - \Lambda_\alpha)u - (M - \Lambda_\alpha)v, u - v \rangle \geq \langle Mu - Mv, u - v \rangle$$

$(M2)$ is obvious and $(M3)$ is satisfied since $\langle (M - \Lambda_\alpha)u, u \rangle \geq \langle Mu, u \rangle$. By Lemma 3.2 there exists some element $u_\alpha \in \mathcal{V}$ such that $Mu_\alpha - \Lambda_\alpha u_\alpha = f$.

Step 2: Since $\langle Mu_\alpha, u_\alpha \rangle \leq \langle Mu_\alpha - \Lambda_\alpha u_\alpha, u_\alpha \rangle = \langle f, u_\alpha \rangle \leq \|f\|_{\mathcal{V}'} \|u_\alpha\|_{\mathcal{V}}$ we obtain that $\sup_{\alpha>0} \|u_\alpha\|_{\mathcal{V}} < \infty$ by $(M3)$ and consequently, $\sup_{\alpha>0} \|Mu_\alpha\|_{\mathcal{V}'} < \infty$ by the boundedness of M. Hence for $v \in \mathcal{V}$

$$\begin{aligned} \langle \Lambda_\alpha u_\alpha, v \rangle &= -\langle Mu_\alpha - \Lambda_\alpha u_\alpha, v \rangle + \langle Mu_\alpha, v \rangle \\ &= -\langle f, v \rangle + \langle Mu_\alpha, v \rangle \\ &\leq (\|f\|_{\mathcal{V}'} + \|Mu_\alpha\|_{\mathcal{V}'}) \|v\|_{\mathcal{V}} \end{aligned}$$

and therefore $\sup_{\alpha>0} \|\Lambda_\alpha u_\alpha\|_{\mathcal{V}'} < \infty$.

Hence there exists some subsequence $(\alpha_n)_{n\geq 1}$ such that $\lim_{n\to\infty} \alpha_n = 0$, $u_{\alpha_n} \rightharpoonup u$ in $\mathcal{V}$ for some $u \in \mathcal{V}$, $Mu_{\alpha_n} \rightharpoonup h$ in $\mathcal{V}'$ for some $h \in \mathcal{V}'$, and $\Lambda_{\alpha_n} u_{\alpha_n} \rightharpoonup g$ in $\mathcal{V}'$ for some $g \in \mathcal{V}'$ (cf. [Sh], Theorem II.1.1).

By the strong continuity of the dual resolvent $(\hat{V}_\alpha)_{\alpha>0}$ in $\mathcal{V}'$ we obtain that

$$\lim_{n\to\infty} \langle v, V_{\alpha_n} u_{\alpha_n} \rangle = \lim_{n\to\infty} \langle \hat{V}_{\alpha_n} v, u_{\alpha_n} \rangle = \langle v, u \rangle$$

for all $v \in \mathcal{V}'$ and therefore, $V_{\alpha_n} u_{\alpha_n} \rightharpoonup u$ in $\mathcal{V}$.

Since $\|V_{\alpha_n} u_{\alpha_n}\|_{\mathcal{V}} \leq \|u_{\alpha_n}\|_{\mathcal{V}}$ and $\|\Lambda V_{\alpha_n} u_{\alpha_n}\|_{\mathcal{V}'} = \|\Lambda_{\alpha_n} u_{\alpha_n}\|_{\mathcal{V}'}$ we conclude that $\sup_{n\geq 1} \|V_{\alpha_n} u_{\alpha_n}\|_{\mathcal{F}} < \infty$ which implies that $u \in \mathcal{F}$ and $\Lambda u = g$.

Step 3: In this last part of the proof, [St] uses a stronger assumption than $(M2)$, namely strong monotonicity. We will show that this is not needed - strict monotonicity is enough.

The claim is to show that $h = Mu$. By monotonicity we have for all $v \in D(\Lambda, \mathcal{V})$ that

$$\begin{aligned}
\limsup_{n\to\infty} \langle \Lambda_{\alpha_n} u_{\alpha_n}, u_{\alpha_n}\rangle &= \limsup_{n\to\infty} \langle \Lambda_{\alpha_n} u_{\alpha_n}, u_{\alpha_n} - v\rangle + \langle \Lambda_{\alpha_n} u_{\alpha_n}, v\rangle \\
&\leq \limsup_{n\to\infty} \langle \Lambda_{\alpha_n} v, u_{\alpha_n} - v\rangle + \langle \Lambda_{\alpha_n} u_{\alpha_n}, v\rangle \\
&= \langle \Lambda v, u - v\rangle + \langle \Lambda u, v\rangle.
\end{aligned}$$

Since $D(\Lambda, \mathcal{V}) \subset \mathcal{F}$ dense this inequality holds for all $u \in \mathcal{F}$, in particular, for $u = v$ which implies $\limsup_{n\to\infty} \langle \Lambda_{\alpha_n} u_{\alpha_n}, u_{\alpha_n}\rangle \leq \langle \Lambda u, u\rangle$. Again by monotonicity we then obtain

$$\begin{aligned}
0 &\leq \limsup_{n\to\infty} \langle M u_{\alpha_n} - Mu, u_{\alpha_n} - u\rangle \\
&= \limsup_{n\to\infty} (\langle f - Mu, u_{\alpha_n} - u\rangle + \langle \Lambda_{\alpha_n} u_{\alpha_n}, u_{\alpha_n} - u\rangle) \\
&\leq \langle \Lambda u, u\rangle - \langle \Lambda u, u\rangle = 0.
\end{aligned}$$

Thus we have that $\lim_{n\to\infty} \langle M u_{\alpha_n}, u_{\alpha_n}\rangle = \langle h, u\rangle$.

It follows that for all $v \in \mathcal{V}$

$$\langle h - Mv, u - v\rangle = \lim_{n\to\infty} \langle M u_{\alpha_n} - Mv, u_{\alpha_n} - v\rangle \geq 0.$$

Choosing $v := u - \lambda w, \lambda > 0, w \in \mathcal{V}$ we get $\lambda\langle h - M(u - \lambda w), w\rangle \geq 0$, so $\langle h - M(u - \lambda w), w\rangle \geq 0$, so by the hemicontinuity of M for $\lambda \to 0$ we get $\langle h - Mu, w\rangle \geq 0$ for all $w \in \mathcal{V}$, hence $Mu = h$.

Uniqueness: $Mu - \Lambda u = Mv - \Lambda v$ implies $0 = \langle Mu - Mv, u - v\rangle - \langle \Lambda(u - v), u - v\rangle \geq \langle Mu - Mv, u - v\rangle$ and hence $u = v$ by $(M2)$.

Thus, the proof is complete.

Define now $M_\alpha : \mathcal{V} \to \mathcal{V}'$ by $\langle M_\alpha u, \cdot\,\rangle := \alpha\, a(u, \cdot\,) + (u, \cdot\,)_{\mathcal{H}}$. Then the conditions $(M1) - (M3)$ hold for M_α, so the above proposition applies to M_α.

We show now that the resolvent – which Stannat only needed for the linear case – also exists in the nonlinear case, but with the nonlinear resolvent equation.

Proposition 3.6: For all $\alpha > 0$ there exists a bijection $W_\alpha : \mathcal{V}' \to \mathcal{F}$ which is monotone as a map from $\mathcal{V}'$ to $\mathcal{V}(= \mathcal{V}'')$ such that

$$e_\alpha(W_\alpha f, v) = \langle f, v\rangle \text{ for all } f \in \mathcal{V}', v \in \mathcal{V}.$$

$(W_\alpha)_{\alpha>0}$ satisfies the resolvent equation

$$W_\alpha = W_\beta \circ (\frac{\beta}{\alpha} I + (1 - \frac{\beta}{\alpha})W_\alpha), \qquad \alpha, \beta > 0.$$

In particular, $Rg(W_\alpha)$ is independent of $\alpha > 0$.

Proof: If $\alpha > 0$ and $f \in \mathcal{V}'$ there exists a unique $W_\alpha f \in \mathcal{F}$ such that

$$M_\alpha(W_\alpha f) - \Lambda(W_\alpha f) = f$$

by the proposition before and therefore,

$$e_\alpha(W_\alpha f, v) = \langle f, v\rangle \text{ for all } v \in \mathcal{V}.$$

We have that

$$\begin{aligned} \langle f - g, W_\alpha f - W_\alpha g\rangle &= \langle (M_\alpha - \Lambda)(W_\alpha f) - (M_\alpha - \Lambda)(W_\alpha g), W_\alpha f - W_\alpha g\rangle \\ &\geq 0. \end{aligned}$$

Thus the mapping $f \mapsto W_\alpha f$ is monotone from $\mathcal{V}'$ to $\mathcal{V}(= \mathcal{V}'')$. $W_\alpha : \mathcal{V}' \to \mathcal{F}$ is bijective by construction as we mentioned above.

Let $f \in \mathcal{V}', v \in \mathcal{V}$. We have

$$\begin{aligned} e_\beta(W_\beta(\frac{\beta}{\alpha} f + W_\alpha f - \frac{\beta}{\alpha} W_\alpha f), v) &= \frac{\beta}{\alpha}\langle f, v\rangle + (W_\alpha f, v) - \frac{\beta}{\alpha}(W_\alpha f, v) \\ &= \frac{\beta}{\alpha} e_\alpha(W_\alpha f, v) + (W_\alpha f, v) - \frac{\beta}{\alpha}(W_\alpha f, v) \\ &= \frac{\beta}{\alpha}(e_\alpha(W_\alpha f, v) - (W_\alpha f, v)) + (W_\alpha f, v) \\ &= \frac{\beta}{\alpha}(\alpha e(W_\alpha f, v)) + (W_\alpha f, v) \\ &= \beta\, e(W_\alpha f, v) + (W_\alpha f, v) \\ &= e_\beta(W_\alpha f, v). \end{aligned}$$

Hence from the uniqueness part in the preceding proposition we conclude that $W_\alpha f = W_\beta(\frac{\beta}{\alpha} f + (1 - \frac{\beta}{\alpha})W_\alpha f)$.

By restricting the operator W_α to $\mathcal{H}$ we obtain an operator $G_\alpha : \mathcal{H} \to \mathcal{H}$ for all $\alpha > 0$ since $\mathcal{F} \subset \mathcal{H}$.

Proposition 3.7: $(G_\alpha)_{\alpha>0}$ as defined above defines a resolvent of monotone contractions on $\mathcal{H}$. $Rg(G_\alpha)$ is independent of $\alpha > 0$ and for all $f \in Rg(G_1)$

$$\lim_{\alpha\to 0} G_\alpha f = f \text{ in } \mathcal{H}.$$

Proof: Clearly Proposition 3.6 implies that G_α satisfies the resolvent equation for all $\alpha > 0$, hence $Rg(G_\alpha)$ is independent of $\alpha > 0$. For all $f, g \in \mathcal{H}$ by the monotonicity of e we obtain that

$$\begin{aligned}
\|G_\alpha f - G_\alpha g\|^2 &= (G_\alpha f, G_\alpha f - G_\alpha g) - (G_\alpha g, G_\alpha f - G_\alpha g)\\
&\le (G_\alpha f, G_\alpha f - G_\alpha g) - (G_\alpha g, G_\alpha f - G_\alpha g)\\
&\quad +\alpha\, e(G_\alpha f, G_\alpha f - G_\alpha g) - \alpha\, e(G_\alpha g, G_\alpha f - G_\alpha g)\\
&= e_\alpha(G_\alpha f, G_\alpha f - G_\alpha g) - e_\alpha(G_\alpha g, G_\alpha f - G_\alpha g)\\
&= (f, G_\alpha f - G_\alpha g) - (g, G_\alpha f - G_\alpha g)\\
&\le \|f-g\|\|G_\alpha f - G_\alpha g\|.
\end{aligned}$$

So we obtain that

$$(G_\alpha f - G_\alpha g, f-g) \ge \|G_\alpha f - G_\alpha g\|^2 \ge 0$$

and

$$\|G_\alpha f - G_\alpha g\| \le \|f-g\| \text{ for all } f, g \in \mathcal{H}.$$

Let $f := G_1 h, h \in \mathcal{H}$. Then for all $\alpha > 0$ by the monotonicity of e

$$\begin{aligned}
\frac{1}{\alpha}\|G_\alpha f - f\|^2 &\le \frac{1}{\alpha}((G_\alpha f, G_\alpha f - f) - (f, G_\alpha f - f)\\
&\quad + \alpha e(G_\alpha f, G_\alpha f - f) - \alpha e(f, G_\alpha f - f))\\
&= \frac{1}{\alpha}(e_\alpha(G_\alpha f, G_\alpha f - f) - e_\alpha(f, G_\alpha f - f))\\
&= \frac{1}{\alpha}(f, G_\alpha f - f) - e(f, G_\alpha f - f) - \frac{1}{\alpha}(f, G_\alpha f - f)\\
&= -\, e_1(G_1 h, G_\alpha f - f) + (f, G_\alpha f - f)\\
&= (f-h, G_\alpha f - f)\\
&\le \|f-h\|\,\|G_\alpha f - f\|.
\end{aligned}$$

Now consider the restriction of $(A, \mathcal{F})$ to $Rg(G_1)$ and denote it by $A_\mathcal{H}$. So, $D(A_\mathcal{H}) = Rg(G_1)$. Clearly, $A_\mathcal{H}$ is then monotone on $\mathcal{H}$ and $(I + \alpha A_\mathcal{H})^{-1} = G_\alpha$ as operators on $\mathcal{H}$.

In particular, $A_\mathcal{H}$ is maximal monotone on $\mathcal{H}$, so by Theorem 2.1 it generates a semigroup $S(t)$ on $\overline{D(A_\mathcal{H})}$. We know that $(I+\frac{t}{n}A_\mathcal{H})^{-n}u \to S(t)u$ for $n \to \infty$ (cf. [Br], Cor. 4.4, p. 126).

Remark 3.5: Since $(A_{\mathcal{H}}, D(A_{\mathcal{H}}))$ is maximal monotone on $\mathcal{H}$ and $(G_\alpha)_{\alpha>0}$ is its associated resolvent, we know by Proposition 2.6 (Minty-Rockafellar) that for all $f \in \mathcal{H}$ as $\alpha > 0$ we have $G_\alpha f \to P_{\overline{D(A_{\mathcal{H}})}} f$ in $\mathcal{H}$ which is even stronger than the last part of the preceding proposition.

We will now give some conditions under which the resolvent G_α is invariant on convex sets and try to split up the conditions required on the corresponding operator $A_{\mathcal{H}}$. I.e., we try to elaborate the distinct assumptions one has to make on the operators M and Λ such that the resolvent is invariant as desired.

Lemma 3.3: Let $u \in \mathcal{F}$. Then $G_\alpha u \rightharpoonup u$ in $\mathcal{V}$ as $\alpha \to 0$. In particular, $\overline{D(A_{\mathcal{H}})} = \mathcal{H}$.

Proof: We have for $\alpha > 0$

$$\begin{aligned} a(G_\alpha u, G_\alpha u - u) &= e(G_\alpha u, G_\alpha u - u) + \langle \Lambda G_\alpha u, G_\alpha u - u \rangle \\ &= \frac{1}{\alpha}(u - G_\alpha u, G_\alpha u - u) + \langle \Lambda G_\alpha u, G_\alpha u - u \rangle \qquad (5) \\ &\leq -\frac{1}{\alpha}\|u - G_\alpha u\|^2 + \langle \Lambda u, G_\alpha u - u \rangle \end{aligned}$$

since $\langle \Lambda v, v \rangle \leq 0$ for all $v \in \mathcal{F}$ by [St], Lemma 2.5.

Hence

$$a(G_\alpha u, G_\alpha u - u) \leq \|\Lambda u\|_{\mathcal{V}'}(\|G_\alpha u\|_{\mathcal{V}} + \|u\|_{\mathcal{V}}),$$

so by $(M3)$ $(G_\alpha u)_{\alpha>0}$ is bounded in $\mathcal{V}$. It then follows by (5) and the monotonicity of e that

$$\begin{aligned} 0 &= \lim_{\alpha\to 0} \alpha\, e(u, G_\alpha u - u) \leq \limsup_{\alpha\to 0} \alpha\, e(G_\alpha u, G_\alpha u - u) \\ &= -\|u - P_{\overline{D(A_{\mathcal{H}})}} u\|^2. \end{aligned}$$

Hence $u = P_{\overline{D(A_{\mathcal{H}})}} u$, so $u \in \overline{D(A_{\mathcal{H}})}$, i.e. $\mathcal{F} \subset \overline{D(A_{\mathcal{H}})}$.
In particular, we conclude that $G_\alpha u \to u$ in $\mathcal{H}$ as $\alpha \to 0$, so $u \in \mathcal{V}$ and $G_\alpha u \rightharpoonup u$ in $\mathcal{V}$ as $\alpha \to 0$. Furthermore, since $\mathcal{F}$ is dense in $\mathcal{V}$ and $\mathcal{V}$ is dense in $\mathcal{H}$, the last assertion also follows.

We will now state our main theorems. Note that we always use the projection mapping only on the 'middle' space $\mathcal{H}$, which is a Hilbert space. Thus we can relax our assumptions on $\mathcal{V}$ and work with reflexive Banach spaces.

Theorem 3.2: Let C be a closed convex subset of $\mathcal{H}$, P_C the orthogonal projection onto $\mathcal{H}$.

Suppose we have that

a) $u \in \mathcal{F} \Rightarrow P_C u \in \mathcal{V}$, $\|P_C u\|_{\mathcal{V}} \leq c\,\|u\|_{\mathcal{V}}$ for some $c \in (0,\infty)$ and $a(u, u - P_C u) \geq 0$.

b) $u \in D(\Lambda, \mathcal{H}) \cap \mathcal{V} \Rightarrow (-\Lambda u, u - P_C u) \geq 0$.

Then we have that $G_\alpha(C) \subset C$ for all $\alpha > 0$.

Proof: We have that $D(A_{\mathcal{H}}) \subset \mathcal{F}$. It is then sufficient to show that for all $u \in \mathcal{F}$ we have that

$$e(u, u - P_C u) = a(u, u - P_C u) - \langle \Lambda u, u - P_C u \rangle \geq 0,$$

for then we conclude from Proposition 3.1 that $G_\alpha(C) \subset C$.

Let $u \in \mathcal{F}$. By construction $D(\Lambda, \mathcal{H}) \cap \mathcal{V}$ is dense in $\mathcal{F}$, so there exists a sequence $(u_n)_{n\geq 0} \subset D(\Lambda, \mathcal{H}) \cap \mathcal{V}$ with $u_n \to u$ in $\mathcal{V}$ and $\Lambda u_n \to \Lambda u$ in $\mathcal{V}'$.

Moreover, we know that $P_C u_n \to P_C u$ in $\mathcal{H}$ due to the continuity of the projection.

Since $(P_C u_n)_{n\geq 0}$ is bounded in $\mathcal{V}$, we have that $P_C u_n \rightharpoonup P_C u$ in $\mathcal{V}$.

Furthermore, using *b*) we have

$$-\langle \Lambda u, u - P_C u \rangle = -\lim_{n\to\infty} \langle \Lambda u_n, u_n - P_C u_n \rangle \geq 0.$$

So by *a*), $(Au, u - P_C u) \geq 0$ and the claim follows by Proposition 3.1.

It is not clear if the opposite direction can be shown. We will now relax the conditions a bit and make some stronger assumptions in order to be able to prove an equivalence.

Theorem 3.3: Let the objects be defined as in Theorem 3.2. Then the following are equivalent:

i) $G_\alpha(C) \subset C$.

ii) $u \in D(A_{\mathcal{H}}) \Rightarrow P_C u \in \mathcal{V}$ and $e(u, u - P_C u) \geq 0$.

Proof:
ii) $\Rightarrow$ *i*) :
Let $u \in D(A_{\mathcal{H}})$. Then

$$(A_{\mathcal{H}} u, u - P_C u) = e(u, u - P_C u) \geq 0,$$

and the claim follows by Proposition 3.1.

i) $\Rightarrow$ *ii*):

We proceed similarly as in the proof of Proposition 3.4, but with the form $e(\cdot\,,\cdot)$ instead of $a(\cdot\,,\cdot)$.

Let $u \in D(A_{\mathcal{H}})$. Then $G_\alpha(P_C u) \in D(A_{\mathcal{H}}) \cap C \subset \mathcal{F} \cap C$ and

$$\begin{aligned} a(G_\alpha P_C u, G_\alpha P_C u - u) &= e(G_\alpha P_C u, G_\alpha P_C u - u) + \langle \Lambda G_\alpha P_C u, G_\alpha P_C u - u \rangle \\ &\le 0 + \langle \Lambda u, G_\alpha P_C u - u \rangle \\ &\le \|\Lambda u\|_{\mathcal{V}'}(\|G_\alpha P_C u\|_{\mathcal{V}} + \|u\|_{\mathcal{V}}), \end{aligned}$$

where the second step follows as in the proof of Proposition 3.4 and since $\langle \Lambda v, v \rangle \le 0$ for all $v \in \mathcal{F}$ (cf. [St], Lemma 2.5).
Hence,

$$\frac{a(G_\alpha P_C u, G_\alpha P_C u - u)}{\|G_\alpha P_C u\|_{\mathcal{V}}} \le \|\Lambda u\|_{\mathcal{V}'} + \frac{\|\Lambda u\|_{\mathcal{V}'}\|u\|_{\mathcal{V}}}{\|G_\alpha P_C u\|_{\mathcal{V}}}.$$

This implies by the coercivity of $a(\cdot\,,\cdot\,)$ that $(G_\alpha P_C u)_{\alpha>0} \subset \mathcal{V}$ is bounded in $\mathcal{V}$. Since $G_\alpha P_C u \to P_C u$ in $\mathcal{H}$ as $\alpha \to 0$ by Remark 3.5 and Lemma 3.3, it follows that $P_C u \in \mathcal{V}$. Since $\overline{D(A_{\mathcal{H}})} = \mathcal{H}$ by Lemma 3.3 we can now apply Proposition 3.2 to complete the proof.

4 Examples

We will now give two examples for our theory.

We will relate to the porous medium equation as posed in [Sh], p. 142. We use the results proven there and show that we can apply our results to the solutions of this equation. We achieve this by translating the problem as posed in [Sh] to the more useful setting of [St]. We solve the problem there and try to show that this is equivalent to solving it in the original setting. However, this example has to remain incomplete as we are not able to show that the projection on H^{-1} divides the support of a function u into a positive and a negative part in the same way as the projection on L^2.

In the second example, we show that our theory is applicable to the *p-Laplacian.* We show that convex sets in L^2 are invariant under the resolvent of the p-Laplacian, perturbed by an unbounded linear operator Λ.

Porous Medium Equation

Firstly, we repeat the setting of the equation.

Let G be a bounded domain in $\mathbb{R}^n$ and let $T > 0$ be fixed. Suppose that we are given a function $d : [0, T] \times \mathbb{R} \to \mathbb{R}$ such that

$d(t, \xi)$ is measurable in t and continuous in ξ.

$|d(t, \xi)| \leq c|\xi|$, for some $c > 0$ and all $\xi \in \mathbb{R}, 0 \leq t \leq T$.

$(d(t, \xi) - d(t, \eta))(\xi - \eta) \geq 0$, for $\xi, \eta \in \mathbb{R}$, with equality only for $\xi = \eta$.

$d(t, \xi)\xi \geq \alpha|\xi|^2$, for some $\alpha > 0$ and all $\xi \in \mathbb{R}, 0 \leq t \leq T$.

Remark 4.1: We have to set the function $k(t)$ used in [Sh], p. 142, to zero. Furthermore, instead of permitting $L^p(0, T; \mathcal{H})$ for all $2n/(n + 2) \leq p < \infty$, we only consider the Hilbert space case where $p = 2$. Note that in this case the condition on p is automatically fulfilled for all $n \in \mathbb{N}$.

We will now give a brief summary of how the solution to the general problem can be found (cf. [Sh] for more details). Then we will introduce the notions of propagators and time-space-shifts needed for the results which we will show in the last part of this section.

Consider the semilinear *porous medium equation* with Dirichlet boundary conditions,

$$\begin{aligned} \frac{\partial u(t,x)}{\partial t} - \Delta d(t, u(t,x)) &= 0, \qquad x \text{ in } G, \\ d(t, u(t,\xi)) &= 0, \qquad \xi \text{ on } \partial G, \end{aligned}$$

for $t \in (0, T]$, where the second equation is meant in the sense that $d(t, u(t, \cdot)) \in H_0^1(G)$ for all $t \in (0, T]$.

Remark 4.2: We set the outer force $f(t)$ as used in [Sh], p. 142, to zero. Despite these simplifications, the zero function is not necessarily the only solution as an initial condition $u_0 \in H^{-1}(G)$ as in [Sh] would only mean one boundary condition with respect to the time direction, and this initial condition need not be zero.

It is reasonable to search for this solution in the space $H^{-1}(G)$, the dual space of the Sobolev space $H_0^1(G)$. We identify $H^{-1}(G)$ with $\mathcal{H}$ from our theory above.

The Riesz-identification of these Hilbert spaces $\mathcal{R} : H_0^1(G) \to H^{-1}(G)$ is the isomorphism defined by $\mathcal{R}\varphi(\psi) = (\varphi, \psi)_{H_0^1}$, so we have that $\mathcal{R} = -\Delta$.

The scalar product on $H^{-1}(G)$ is given by

$$(f, g)_{H^{-1}} = (\mathcal{R}^{-1} f, \mathcal{R}^{-1} g)_{H_0^1}, \qquad f, g \in H^{-1}$$

and it satisfies the identities

$$(f, g)_{H^{-1}} = {}_{H^{-1}}\langle f, \mathcal{R}^{-1} g\rangle_{H_0^1} = {}_{H_0^1}\langle \mathcal{R}^{-1} f, g\rangle_{H^{-1}}, \qquad f, g \in H^{-1}$$

If we take the scalar product of the equation in the space $H^{-1}(G)$ and restrict it to $L^2(G)$ we obtain the following expression.

$$(u_t, g)_{H^{-1}} + (d(t, u), g)_{L^2} = 0, \qquad g \in L^2(G)$$

since

$$(-\Delta d(t, u), g)_{H^{-1}} = (d(t, u), g)_{L^2}, \qquad g \in L^2(G)$$

Obviously, the function $d(t, \cdot)$ is now strictly monotone in $L^2(G)$ for every fixed $t \in [0, T]$. So from now on we identify $L^2(G) = \mathcal{V}$ from our theory above.

Thus, we will apply our theoretical setting

$$\mathcal{V} \hookrightarrow \mathcal{H} \cong \mathcal{H}' \hookrightarrow \mathcal{V}'$$

to the relation

$$L^2(G) \hookrightarrow H^{-1}(G) \cong H_0^1(G) \hookrightarrow (L^2(G))'$$

and then apply Theorem 3.1 in order to obtain some invariance results for the resolvent.

Remark 4.3: Note that as we assume that the outer force $f(t) = 0$, we know by [Sh], p. 143, that given some initial condition $u_0 \in L^2(G)$ there is a unique solution $u \in L^2(0, T; L^2(G))$ for the problem satisfying

$$\int_G (\frac{\partial u}{\partial t})\varphi \, dx + \int_G d(t, u)(-\Delta)\varphi \, dx = 0, \quad \varphi \in H_0^1 : \Delta\varphi \in L^2,$$

$$\lim_{t \to 0} (-\Delta)^{-1} u(t) = 0 \text{ in } L^2(G).$$

Especially, for almost every $t \in [0, T]$, $u(t)$ is a function in $L^2(G)$.

The main obstacle we are confronted with now is the time-invariant formulation of the problem. If we identified now $\Lambda = -\frac{\partial}{\partial t}$ and $M = \Delta d(t, \cdot)$ we would easily see that $-\Lambda$ is not positive definite as we do not integrate over the time variable.

This brings yet another problem as now we cannot apply the theory of [St]. It is not guaranteed that in this space the operator $M - \Lambda$ generates a semigroup.

For this reason, we will shift the problem into an environment more suitable for our theory. Instead of looking at time as an exterior variable we integrate it in our equations and consider the change of the time-space. As $M - \Lambda$ in general does not generate a semigroup in our actual setting in $H^{-1}(G)$, we define a similar object – the propagator – on this space and then define a semigroup on the space $L^2(0, T; H^{-1}(G))$. We show that these objects have similar properties operating on convex sets.

Definition 4.1: We call an operator-valued function $U(\cdot\,,\cdot) : D \to \mathcal{B}(X)$, where

$$D := \{(s, t) \in [0, T] \times [0, T] | 0 \le s \le t\},$$

X a Banach space and $\mathcal{B}(X)$ the set of all bounded operators on X, a *propagator (of class C_0)* if the following conditions are satisfied.

i) The function $U(\cdot\,,\cdot)$ is strongly continuous as a function of two variables in the region D.

ii) For every $t \in (0, T]$ the relation $U(t, t) = I$ holds and for every point $(t, r, s) \in (0, T]^3, t \ge r \ge s > 0$, the equality

$$U(s, r)U(r, t) = U(s, t)$$

is valid.

iii) The estimation

$$\sup_{(s,t) \in D} \|U(s, t)\|_{\mathcal{B}(X)} < \infty$$

holds.

In our notation $U(s,t)$ will correspond to $T_{s,t}$, which we shall define now.

Definition 4.2: For $(s,t) \in D$ define the operator $T_{s,t} : H^{-1}(G) \to H^{-1}(G)$ by

$$T_{s,t}u = u_s(t,\cdot) \qquad u \in H^{-1}(G)$$

where u_s denotes the unique solution of the porous medium equation with initial condition u at time s, which is given by [Sh], p. 144. Defined in this way, $T_{s,t}u$ is the propagator of u.

We now claim that with this propagator on the space $H^{-1}(G)$ we can construct a semigroup on the space $L^2(0,T;H^{-1}(G))$.

Definition 4.3: For all $t \in [0,T]$ and $v \in L^2([0,T],H^{-1}(G))$ let

$$\overline{T}_t v(s,\cdot) := \begin{cases} T_{s,s+t}(v(s+t,\cdot)), & \text{if } s+t \leq T \\ 0, & \text{else} \end{cases}$$

in $H^{-1}(G)$.

Proposition 4.1: Let $\overline{T}_t$ be defined as in the definition above. Then $\overline{T}_t$ defines a semigroup on $L^2(0,T;H^{-1}(G))$.

The proof is not difficult and will be left out here. The following result is helpful and will also be used later in this work:

Proposition 4.2: Let the Banach space $\mathcal{V}$ be dense and continuously embedded in the Hilbert space $\mathcal{H}$; identify $\mathcal{H} = \mathcal{H}'$ so that $\mathcal{V} \hookrightarrow \mathcal{H} \hookrightarrow \mathcal{V}'$. The Banach space $W_p(0,T) = \{u \in L^p(0,T;\mathcal{V}) : \frac{du}{dt} \in L^p(0,T;\mathcal{V}')\}$ is contained in $C([0,T],\mathcal{H})$. Moreover, if $u \in W_p(0,T)$ then $|u(\cdot)|^2_{\mathcal{H}}$ is absolutely continuous on $[0,T]$,

$$\frac{d}{dt}|u(t)|^2_{\mathcal{H}} = 2u'(t)(u(t)) \qquad \text{a.e. } t \in [0,T].$$

Proof: See [Sh], Prop. III.1.2, p. 106.

So we have that $(\overline{T}_t)_{t\geq 0}$ defines a C_0-semigroup of contractions on $W_2(0,T) \subset L^2(0,T;H^{-1}(G))$. By construction this is the semigroup generated by $-(M-\Lambda)$.

We can now prove some basic relations between the two settings.

Proposition 4.3: Let $C \subset H^{-1}(G)$ be closed and convex. Let

$$\overline{C} := \{f \in L^2(0,T;H^{-1}(G)) : f(t,\cdot) \in C \,\forall\, t\}$$

Then the following are equivalent:

i) $(\overline{T}_t)_{0\le t\le T}$ is $\overline{C}$-invariant.

ii) $(T_{s,s+t})_{0\le s\le s+t\le T}$ is C-invariant.

Proof: $ii) \Rightarrow i)$:
Let $f \in \overline{C}$. Then we have that $f(s+t,\cdot\,) \in C$. So by assumption $T_{s,s+t}f(s+t,\cdot\,) \in C$ which in turn implies that $(\overline{T}_t f)(s,\cdot\,) \in C$ for all $s \in [0,T], t \in [0,T-s]$.

$i) \Rightarrow ii)$:
Let $g \in C$. Then $g \in \overline{C}$ and by assumption $\overline{T}_t g \in \overline{C}$. By definition we have that $(\overline{T}_t g)(s,\cdot) \in C$ for all $s \in [0,T]$ and the latter is equal to $T_{s,s+t}g(s,\cdot)$ which implies the assertion.

This leads to an easy conclusion for our porous medium equation example.

Corollary 4.1: Let $u \in H^{-1}(G)$, $C \subset \mathcal{H}$ be closed and convex, P_C the usual projection on C.
Assume $M(t)(P_C u)(u - P_C u) \ge 0$ for all $t \in [0,T]$.
Then the propagator $(T_{s,t})_{0\le s\le t\le T}$ corresponding to $(M(t))_{0\le t\le T}$ is C-invariant.

Proof: From Theorem 3.3 we have that under these assumptions the resolvent of $(M(t))_{0\le t\le T}$ as an operator on $L^2(0,T;H^{-1}(G))$ is $\overline{C}$-invariant. By Theorem 2.3 this implies that the semigroup generated by $(-M(t))_{0\le t\le T}$ is $\overline{C}$-invariant. Since the semigroup generated by $(-M(t))_{0\le t\le T}$ is equal to the semigroup defined in Definition 4.3 by uniqueness we conclude by Proposition 4.3 that the propagator $(T_{s,t})_{0\le s\le t\le T}$ is C-invariant.

We would like to show now that these theoretical results are applicable to the porous medium equation.

Let
$$M(t)(u) = -\Delta d(t,u(t)), \qquad \Lambda = -\frac{\partial}{\partial t}.$$
Let C be the closure of $\{f \in L^2(G) | f \ge 0\}$ in $H^{-1}(G)$. Then C is closed and convex in $H^{-1}(G)$. Then
$$\overline{C} = \{f \in L^2(0,T;H^{-1}(G)) | f(t,\cdot\,) \in C\}.$$

Remark 4.4: Note that although we defined convex sets in $H^{-1}(G)$ we will always talk about functions in $L^2(G)$. If we assume that the propagator is C-invariant, this follows from the preceding results. We obtain for $s, t \geq 0$ that $u \in C \Rightarrow T_{s,s+t}u \in C$. Consequently, $T_{s,s+t}u \geq 0$ and by Proposition 4.3 $\overline{T}_t u_s \in \overline{C}$, where u_s is defined as in Definition 4.3. By definition, we have that $\overline{T}_t u(s, \cdot) \in C$, i.e. $\overline{T}_t u(s, \cdot) \geq 0$. By Remark 4.3 we know that $\overline{T}_t u_s$ as a solution of the porous medium equation is a function, i.e. $\overline{T}_t u_s \in L^2(0, T; L^2(G))$. Since $\overline{T}_t u_s = T_{s,s+t}u$, we have that $T_{s,s+t}u \in L^2(G)$, too.

If we could show now the C-invariance of the propagator, we were done. If we assume this, then by Corollary 4.1 $M(t) - \Lambda$ is C-invariant. Thus by Proposition 4.3 the corresponding semigroup is $\overline{C}$-invariant. Theorem 2.3 implies that the corresponding resolvent is also $\overline{C}$-invariant, which was what we wanted.

However, we are faced with problems when checking the conditions required for Theorem 2.3.

For example, we have to check that

$$(-\Lambda u, u - P_C u)_{L^2(0,T;\, H^{-1}(G))} \geq 0.$$

Doing this calculus in $L^2(0, T; L^2(G))$ with the standard projection $P_C u = u^+$ and assuming appropriate boundary conditions is fairly easy. But how does the projection $P_C u$ in H^{-1} look like? Does it have the same support as u^+? In the calculus we use the strict separation of the supports of u^+ and u^-, but it is not clear whether this separation is the same when applying the H^{-1}–projection to u.

So, although there would be many useful applications of such a result (cf. [Sh], p.243), we have to leave this problem open.

The p-Laplacian

Let throughout this subsection G be a bounded open set of $\mathbb{R}^n$ and $p \geq 2$.

Van Beusekom already proved that the p-Laplacian is a Dirichlet form on the Sobolev space $H_0^{1,p}(G)$. Since we need a slightly different setting, we will repeat the proofs important for us here.

Definition 4.4: For $u \in H_0^{1,p}(G)$ define the *p-Laplacian* $\Delta_p u$ as follows:

$$\Delta_p u = -\mathrm{div}(|\nabla u|^{p-2}\nabla u) \in H^{-1,\frac{p}{p-1}}(G) = (H_0^{1,p}(G))'.$$

Generalised solutions of the variational equation

$$\int_G v \Delta_p u \, dx = 0$$

are found in the first order Sobolev space $H^{1,p}(G)$. Usually, we will carry out the integration by parts and write for some $v \in C^\infty(G)$

$$\int_G |\nabla u|^{p-2} (\nabla u, \nabla v) \, dx$$

instead of $\int_G v \Delta_p u \, dx$.

Consider now the space $[0,T] \times G$ for some $T > 0$, where the first variable represents the time component t. Let the p-Laplacian operate on functions of the second variable x.

Proposition 4.5: The p-Laplacian is a strictly monotone operator on $L^p(0,T;H_0^{1,p}(G))$.

Proof: Since $L^p(0,T;C_0^\infty(G))$, where $C_0^\infty(G)$ denotes the space of infinitely differentiable functions with compact support, is dense in $L^p(0,T;H_0^{1,p}(G))$, it is sufficient to show monotonicity on this subset.

Let $u, v \in L^p(0,T;C_0^\infty(G))$. Then using the Cauchy-Schwarz inequality we obtain

$$\begin{aligned}
& \langle \Delta_p u - \Delta_p v, u - v \rangle \\
= & \int_0^T \int_G (\Delta_p u - \Delta_p v, u - v) \, dx \, dt \\
= & \int_0^T \int_G |\nabla u|^{p-2} (\nabla u, \nabla u) + |\nabla v|^{p-2} (\nabla v, \nabla v) \\
& -(|\nabla u|^{p-2} + |\nabla v|^{p-2})(\nabla u, \nabla v) \, dx \, dt \\
\geq & \int_0^T \int_G (|\nabla u|^p + |\nabla v|^p - |\nabla u|^{p-1} |\nabla v| \\
& -|\nabla v|^{p-1} |\nabla u|) \, dx \, dt \\
= & \int_0^T \int_G (|\nabla u|^{p-1} - |\nabla v|^{p-1})(|\nabla u| - |\nabla v|) \, dx \, dt. \qquad (6)
\end{aligned}$$

Both $|\nabla u(t,x)|$ and $|\nabla v(t,x)|$ are nonnegative for any point $(t,x) \in [0,T] \times G$, and raising to the power $p-1$ is a monotone function on $\mathbb{R}$, so

$$(|\nabla u|^{p-1} - |\nabla v|^{p-1})(|\nabla u| - |\nabla v|) \geq 0 \qquad (7)$$

and hence

$$\int_0^T \int_G (|\nabla u|^{p-1} - |\nabla v|^{p-1})(|\nabla u| - |\nabla v|) \, dx \, dt \geq 0.$$

To prove strict monotonicity on $L^p(0,T;H_0^{1,p}(G))$, assume $u,v \in L^p(0,T;C_0^\infty(G))$ and $\int_0^T\int_G(\Delta_p u - \Delta_p v, u-v)\,dx\,dt = 0$. Then (6) and (7) imply $|\nabla u(t,x)| = |\nabla v(t,x)|\ dt\otimes dx$ – a.e.

Let $x \in G, t\in[0,T]$ be fixed. If $|\nabla u(t,x)| = |\nabla v(t,x)| = 0$ we have $\nabla u(t,x) = \nabla v(t,x) = 0$. Thus $u(t,x)$ and $v(t,x)$ are a.e. constant, hence they are zero by Lemma 4.1 below.

Now let us assume $|\nabla u(t,x)| = |\nabla v(t,x)| \neq 0$. In this case, we can write

$$\begin{aligned}&\langle \Delta_p u(t,x) - \Delta_p v(t,x), u(t,x)-v(t,x)\rangle\\ =\ &\int_0^T\int_G |\nabla u(t,x)|^{p-2}|\nabla u(t,x)-\nabla v(t,x)|^2\,dx\,dt = 0.\end{aligned}$$

This yields $\nabla u(t,x) = \nabla v(t,x)\ dt\otimes dx$ – a.e., and we can apply Lemma 4.1 stated below to conclude $u = v\ dt\otimes dx$ – a.e. Resuming, we have

$$\int_0^T\int_G(\Delta_p u - \Delta_p v, u-v)\,dx\,dt = 0 \Rightarrow u = v,$$

and hence we have strict monotonicity on $L^p(0,T;H_0^{1,p}(G))$ by a density argument.

Lemma 4.1: If $u\in L^p(0,T;H_0^{1,p}(G))$ and $\nabla u(t,x) = 0\ dt\otimes dx$ – a.e., then $u=0$ in $L^p(0,T;H_0^{1,p}(G))$.

Proof: By [HKM], Lemma 1.17, we have that $u(t,\cdot\,) = 0$ for every fixed $t\in[0,T]$. This implies that $u=0$ on $[0,T]\times G$.

Proposition 4.6: The p-Laplacian is hemicontinuous on $L^p(0,T;H_0^{1,p}(G))$.

Proof: Let $u,v,w\in L^p(0,T;H_0^{1,p}(G))$. We have to show that

$$\lim_{\varepsilon\to 0}\int_0^T\int_G |\nabla(u+\varepsilon w)|^{p-2}(\nabla(u+\varepsilon w),\nabla v)\,dx\,dt = \int_0^T\int_G |\nabla u|^{p-2}(\nabla u,\nabla v).$$

For $\varepsilon\le 1$ the integrand is dominated by $2^p(|\nabla u|^{p-1}+|\nabla w|^{p-1})|v|$, which is obviously in $L^p(0,T;L^p(G))$.

Proposition 4.7: The p-Laplacian is coercive on $L^p(0,T;H_0^{1,p}(G))$.

Proof: Let $u\in L^p(0,T;C_0^\infty(G))$. We know there exists a positive constant α such that

$$\int_G |u|^p\,dx \le \alpha\int_G |\nabla u|^p\,dx.$$

This is the Poincaré inequality. So we have

$$\|u\|_{1,p}^p \le (1+\alpha)\int_G |\nabla u|^p\,dx,$$

and thus for some $u_0 \in L^p(0,T;H_0^{1,p}(G))$

$$\begin{aligned}
&\int_0^T \int_G |\nabla u|^p - |\nabla u|^{p-2}(\nabla u, \nabla u_0)\,dx\,dt \\
\geq\;& \int_0^T \frac{1}{1+\alpha}\|u\|_{1,p}^p\,dt - \int_0^T \int_G |\nabla u|^{p-1}|\nabla u_0|\,dx\,dt \\
\geq\;& \frac{1}{1+\alpha}\|u\|^p_{L^p(0,T;H_0^{1,p}(G))} - \int_0^T \|\nabla u\|_p^{p-1}\|\nabla u_0\|_p\,dt \\
\geq\;& \frac{1}{1+\alpha}\|u\|^p_{L^p(0,T;H_0^{1,p}(G))} - \frac{1}{1+\alpha}\|u\|^{p-1}_{L^p(0,T;H_0^{1,p}(G))}\|u_0\|_{L^p(0,T;H_0^{1,p}(G))} \\
=\;& \frac{1}{1+\alpha}\|u\|^p_{L^p(0,T;H_0^{1,p}(G))}\left(1 - \frac{\|u_0\|_{L^p(0,T;H_0^{1,p}(G))}}{\|u\|_{L^p(0,T;H_0^{1,p}(G))}}\right),
\end{aligned}$$

and for $\|u\|_{L^p(0,T;H_0^{1,p}(G))} \to \infty$ the last term in brackets vanishes.

Now let $\Lambda = -\frac{\partial}{\partial t}$ be an unbounded linear operator defined as in the setting of [St] (see the corresponding subsection of Chapter 3 above). We want to apply our theory to the operator $\Delta_p + \frac{\partial}{\partial t}$ and consider the convex set C of all positive functions in $\mathcal{H} := L^2(0,T;L^2(G))$. Thus, our setting

$$\mathcal{V} \hookrightarrow \mathcal{H} \hookrightarrow \mathcal{V}'$$

translates to

$$L^p(0,T;H_0^{1,p}(G)) \hookrightarrow L^2(0,T;L^2(G)) \hookrightarrow (L^p(0,T;H_0^{1,p}(G)))'.$$

As we will see later on, we have to make some boundary assumption on Λ in order to be able to apply Theorem 3.2. We may consider one of the following spaces:

$$\mathcal{V}_1 := \{u \in L^p(0,T;H_0^{1,p}(G)) | u(0,x) = u(T,x) \forall x \in G\}.$$

$$\mathcal{V}_2 := \{u \in L^p(0,T;H_0^{1,p}(G)) | u(0,x) = 0 \forall x \in G\}.$$

So $\mathcal{V}_1$ would represent the periodic functions in the interval $[0,T]$, while $\mathcal{V}_2$ corresponds to those functions starting in zero for time zero. Defined like this $\mathcal{V}_1$ and $\mathcal{V}_2$ are closed subspaces of a reflexive Banach space, and as such reflexive Banach spaces themselves with the same norm (cf. [A], Lemma 5.6). Define $u^+ := \max\{u,0\}$ and $u^- := \max\{-u,0\}$. Thus we have that $u = u^+ - u^-$.

We now have to check the necessary conditions for Theorem 3.2:

a) Obviously, $u \in \mathcal{F}$ implies that $P_C u = u^+ \in L^p(0,T;H_0^{1,p}(G))$, since $\|P_C u\|_{\mathcal{V}} = \|u^+\|_{\mathcal{V}} \leq \|u\|_{\mathcal{V}}$, so $c = 1$.

Finally, we have to check that $(\Delta_p u, u - P_C u)_{\mathcal{H}} \geq 0$:

$$\begin{aligned}
(\Delta_p u, u - P_C u)_{L^2(0,T;L^2(G))} &= \int_0^T \int_G \Delta_p u \cdot (-u^-)\, dx\, dt \\
&= -\int_0^T \int_G |\nabla u|^{p-2} \nabla u \cdot \nabla u^- \, dx\, dt \\
&= \int_0^T \int_G |\nabla u^-|^{p-2} \nabla u^- \cdot \nabla u^- \, dx\, dt \\
&= \int_0^T \int_G |\nabla u^-|^p \, dx\, dt \\
&= \int_0^T \|\nabla u^-(t)\|^p_{L^p(G)}\, dt \\
&\geq 0.
\end{aligned}$$

b) Let $u \in D(-\frac{\partial}{\partial t}, L^2(0,T;L^2(G))) \cap L^p(0,T;H_0^{1,p}(G))$. Then we have by Proposition 4.2 that

$$\begin{aligned}
(-\Lambda u, u - P_C u)_{L^2(0,T;L^2(G))} &= \int_0^T \int_G \frac{\partial}{\partial t} u \cdot -u^- \, dx\, dt \\
&= \int_0^T \int_G \frac{\partial u^-}{\partial t} \cdot u^- \, dx\, dt \\
&= \int_0^T \int_G \frac{1}{2} \frac{\partial}{\partial t} |u^-(t,x)|^2 \, dx\, dt \\
&= \frac{1}{2} \int_0^T \frac{\partial}{\partial t} \|u^-(t)\|^2_{L^2(G)}\, dt \\
&= \frac{1}{2}(\|u^-(T)\|_{L^2(G)} - \|u^-(0)\|_{L^2(G)}),
\end{aligned}$$

Now, in case of the periodic functions $\mathcal{V}_1$ we have that this is equal to zero, while for the functions 'starting in the origin' $\mathcal{V}_2$ we only have that it is greater or equal than zero. In both cases, the necessary requirements are fulfilled and we may apply Theorem 3.2.

Note that this does not mean that $u^- = 0$. As an example, consider $u(t,x) = \sin(\frac{2\pi}{T} t) \cdot x$, where $x \in G \subset \mathbb{R}$.

So, we can apply Theorem 3.2 to the operator $\Delta_p + \frac{\partial}{\partial t}$ and obtain that for all $u \in L^p(0,T;H_0^{1,p}(G))$ all convex sets in $L^2(0,T;L^2(G))$ are invariant under the corresponding resolvent. For example, we conclude that for every positive input function u_0 its resolvent remains positive, too. By Theorem 2.3 this implies that the corresponding semigroup $S(\tau)$ is positive for every $\tau > 0$, too. So we know that $S(\tau)u_0 = u(\tau) = u^+(\tau)$. This means that the unique solution of the Cauchy problem is a positive function for every τ.

Since C is not fixed, we can conclude that the solutions of the p-Laplacian belong to any convex set which contains the initial condition u_0. We could construct or find the most suitable convex set for our purpose and find that all solutions will also be contained in the same set. Since there exist many convex sets far more complex than the ones we considered here, this result opens a large field of possible application.

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