Theory of Gibbs measures with unbounded spins: probabilistic and analytical aspects

Habilitation Thesis

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Chapter 1

Introduction

The most repeating key word in this manuscript will be, of course, ***Gibbs measures*** (also called ***Gibbs distributions*** or ***states***). The mathematical notion of a Gibbs distribution, as an equilibrium state for an infinitely large system, was introduced in the pioneering papers of R. Dobrushin [90, 91] and O. E. Lanford and D. Ruelle [179, 251] dated back to 1968–70. Although historically the Gibbs measures appeared for the first time in the framework of classical statistical physics, this notion provided a strong stimulating basis for the systematic development of the theory of random Markov fields (see the monographs [122, 233]). The fundamental idea is that the set of Gibbs measures $\mu \in \mathcal{G}$, which corresponds in statistical mechanics to any given (classical or quantum, respectively lattice or continuous) system, can be defined in terms of its *local specification* $\Pi = \{\pi_\Lambda\}_{\Lambda \Subset \mathbb{L}}$, namely as a set of solutions to the *DLR* (*Dobrushin-Lanford-Ruelle*) equation $\mu\pi_\Lambda = \mu$, $\Lambda \Subset \mathbb{L}$. In other words, these are probability measures on the space $\Omega \ni x$ of infinite volume configurations, which have prescribed conditional probabilities $\mu_\Lambda(\mathrm{d}x|y)$ with respect to the boundary conditions y fixed outside finite regions Λ. The initial object in this construction is the formal *Hamiltonian* $H(x)$ describing the interaction between the whole number (tending to the infinity) of individual constituents of the complex system. This interaction rigorously determines the *local energies* $H_\Lambda(x|y)$ of the corresponding subsystems in Λ, subject to the proper "tempered" boundary conditions y. The *conditional distributions* $\mu_\Lambda(\mathrm{d}x|y)$ are standardly given by the Boltzmann factor, the exponential of the *inverse temperature* $\beta = 1/T$ times the local energy $H_\Lambda(x|y)$. Knowing the Gibbs measures would allow one in principle to compute the equilibrium expectations and spatial correlation functions, and hence to explain the macroscopic behavior of the system in terms of the microscopic characteristics of their constituents. In the realization of the DLR approach there occur, however, serious complications if the spin space for every single component (or particle) is *noncompact* or moreover *infinite dimensional* (as for example, $\mathbb{R}^\nu$ in the case of classical or continuous systems, or respectively spaces of paths or loops from $C(\mathbb{R} \to \mathbb{R}^\nu)$ in the case of quantum lattice systems).

(i) Subject and aims

This habilitation thesis deals with three classes of infinite particle models from equilibrium statistical mechanics:

— *Classical lattice systems with linear spin spaces* $\mathbb{R}^\nu$;

— *Systems of quantum anharmonic oscillators at finite inverse temperature* β;

— *Interacting spin systems on graphs.*

Our aim is to develop a consistent mathematical theory of the corresponding *Gibbs states*, which could universally cover the above types of models. The binding element is a common technique to be used from *probability theory* and *infinite dimensional analysis*, as is pointed out in the title of the thesis.

We shall concentrate on the following fundamental problems concerning the associated Gibbs measures:

— *Existence;*

— *A-priori estimates and support properties;*

— *Uniqueness and decay of correlations;*

— *Phase transitions;*

— *Spectral properties of the corresponding Dirichlet operators;*

— *Ergodicity of the stochastic dynamics.*

To handle these problems, in the habilitation thesis three conceptually different approaches to the study of Gibbs measures will be taken and compared:

— *Traditional DLR or Markov field formalism;*

— *Analytical approach based on the characterization of the Gibbs measures via partial integration and quasi-invariance;*

— *Method of stochastic dynamics, in particular the study of spectral properties of the corresponding generators (self-adjointness, spectral gap estimates, Poincaré and log-Sobolev inequalities).*

Developing the last two alternative (to the usual *DLR* one) approaches, we would like to demonstrate how tools from *stochastic analysis,* in particular the theory of *Dirichlet forms and operators* and associated with them *Markov processes*, can be successfully be applied to this topic.

When compared with the traditional setting, the peculiarities of the above models are as follows:

— *The single spin spaces are noncompact in the case of classical systems and, respectively, infinite dimensional in the quantum case;*

— *The underlying indexing set where the system leaves, is no longer a regular lattice* $\mathbb{Z}^d$, *but can be chosen as a countable set* $\mathbb{L} \subset \mathbb{R}^d$ *or, more generally, as some infinite graph* $\mathbb{G} = (\mathbb{V}, \mathbb{E})$.

In the quantum case we shall take the *Euclidean* (or *path integral*) *approach* to Gibbs states, which is conceptually analogous to the well-known Euclidean strategy in quantum field theory. In this language, our model will be the system of interacting *"temperature loops"* ω_ℓ, $\ell \in \mathbb{L}$, i.e., continuous paths $\omega_\ell : [0, \beta] \to \mathbb{R}^\nu$, $\omega_\ell(0) = \omega_\ell(\beta)$. This leads to an additional and non-trivial single spin analysis in the corresponding (e.g. Hölder, Sobolev, or Lebesgue) loop spaces. On the other hand, the choice of *graphs* as indexing sets reflects a general tendency in modelling complex systems. It raises an important question of how the *geometry* of the underlying graph can influence their macroscopic physical characteristics.

(ii) The models

Let us briefly introduce both the classical and the quantum models of our interest.

In *Chapter 2* we start (in the historically correct order) with a *system of classical anharmonic oscillators* which is described by the heuristic potential energy functional

$$H(x) := H_{\mathrm{cl}}(x) = \sum_{\ell} V_\ell(x_\ell) + \frac{1}{2} \sum_{\ell, \ell'} W_{\ell\ell'}(x_\ell, x_{\ell'}), \qquad x = (x_\ell)_{\ell \in \mathbb{L},} \tag{1.1}$$

where the sums run through a countable set $\mathbb{L} \subset \mathbb{R}^d$ and the displacement q_ℓ is a ν-dimensional vector. The anharmonic continuous potentials V_ℓ, $W_{\ell\ell'}$, which may vary from site to site, are supposed to obey certain uniform bounds responsible for the stability of the whole system. In general, we do not assume that the interaction is translation invariance or has finite range. Therefore, our model describes also systems with long-range interactions and with spacial irregularities, e.g. caused by impurities or random components. In the typical case of $\mathbb{L}$ being a lattice, say $\mathbb{Z}^d$, the model is called the *classical anharmonic crystal.* For fixed inverse temperature β, the associated Gibbs states

$$\mu(\mathrm{d}x) := \frac{1}{Z_\beta} \exp\left\{-\beta H(x)\right\} \times_{\ell \in \mathbb{L}} \mathrm{d}x_\ell \tag{1.2}$$

are rigorously defined as those probability measures on the configuration space $\Omega := \Omega^{\mathrm{cl}} := \times_{\ell \in \mathbb{L}} \mathbb{R}^\nu$, which satisfy the *DLR* equation

$$\mu \pi_\Lambda = \mu, \quad \Lambda \Subset \mathbb{L}. \tag{1.3}$$

The corresponding Gibbs specification $\{\pi_\Lambda(\mathrm{d}x|y),\ y \in \Omega,\ \Lambda \Subset \mathbb{L}\}$ is constructed by means of the local Hamiltonians $H_\Lambda(x|y)$. As is usual for unbounded spin systems, one confines themselves to a proper subset of tempered configurations Ω^{t} and respectively to the tempered Gibbs measures $\mu \in \mathcal{G}^{\mathrm{t}}$ supported by this subset. Since 70-ies there appeared an enormous literature on the classical lattice systems like (1.1); some key references will be mentioned in the connection with concrete problems later. The mostly studied are scalar ferromagnetic $P(\varphi)_d$-models on $\mathbb{Z}^d$, with the harmonic nearest-neigbor interaction $W(x_\ell, x_{\ell'}) := J(x_\ell - x_{\ell'})^2 \geq 0$ and with the polynomial self-interaction $V(x_\ell) := \sum_{1\leq \mathrm{s}\leq \mathrm{p}} b_\ell^{(\mathrm{s})} x_\ell^{2\mathrm{s}}$ where $b_\ell^{(\mathrm{p})} > 0$ and $\mathrm{p} \geq 2$.

Keeping the previous assumptions on the interaction, a *system of quantum anharmonic oscillators* (of mass $m > 0$ and rigidity $a > 0$) is given by the formal Hamiltonian

$$H := H_{\mathrm{q}} = \sum_\ell \left[\frac{1}{2m}|p_\ell|^2 + \frac{a}{2}|q_\ell|^2 + V_\ell(q_\ell)\right] + \frac{1}{2}\sum_{\ell,\ell'} W_{\ell\ell'}(q_\ell, q_{\ell'}). \tag{1.4}$$

This will be the subject of our study in *Chapter 3*. As was mentioned above, we shall follow the Euclidean approach, which relies on the presentation for the quantum Gibbs states via path space integrals. The thermodynamical properties of the quantum system (1.4) at the inverse temperature β then can be described by certain Gibbs measures $\mu \in \mathcal{G}^{\mathrm{t}}$ defined on the *"temperature loop space"*

$$\mathit{\Omega} := \mathit{\Omega}^{\mathrm{q}} := \times_{\ell\in\mathbb{L}} C_\beta := \{\omega = (\omega_\ell)_{\ell\in\mathbb{L}} \mid \omega_\ell \in C_\beta\}, \tag{1.5}$$

where $C_\beta := C_\beta(S_\beta \to \mathbb{R}^\nu)$ is the single spin space of periodic paths on the circle $S_\beta \cong [0,\beta]$. Heuristically, these measures may be written down as

$$\begin{aligned}\mu(\mathrm{d}\omega) \ :\ &= \frac{1}{Z_\beta}\exp\left\{-\sum_\ell \int_0^\beta V_\ell(\omega_\ell(\tau))\mathrm{d}\tau - \right. \qquad (1.6)\\ &\left. -\frac{1}{2}\sum_{\ell,\ell'}\int_0^\beta W_{\ell\ell'}(\omega_\ell(\tau), \omega_{\ell'}(\tau))\mathrm{d}\tau\right\} \times_{\ell\in\mathbb{L}} \chi(\mathrm{d}\omega_\ell),\end{aligned}$$

where $\chi = \mathcal{N}(0, A^{-1})$ is the normal distribution on C_β and $A := -m\mathrm{d}^2/\mathrm{d}\tau^2 + a^2\mathbf{1}$ is the Laplace-Beltrami operator in the Hilbert space $L^2_\beta := L^2_\beta(S_\beta \to \mathbb{R}^\nu)$. Using the *DLR* formalism, the above measures can be rigorously defined as random fields on $\mathbb{L}$ with the prescribed local specification $\{\pi_\Lambda\}_{\Lambda\Subset\mathbb{L}}$. Note that in the large-mass limit $m \to \infty$ of the model (1.4) one gets the infinite system of classical particles (1.1), see Remark 3.5. However, as compared with classical lattice systems, the situation with Euclidean Gibbs measures is more complicated, since now the spin (i.e., loop) spaces themselves are *infinite dimensional* and hence their topological features are much richer.

Lattice systems of the above type (classical and quantum) are commonly viewed in statistical physics as mathematical models of a crystalline substance (for more physical background, see e.g. [7, 100, 117, 129, 164, 214]). The study of such systems is especially motivated by the reason, that they provide a *mathematically rigorous* as well as *physically realistic* description for the important phenomenon of phase transitions (i.e., non-uniqueness of Gibbs states).

In *Chapter 4* we return to the classical spin systems, but now we considerably enrich the situation by substituting the lattice $\mathbb{L}$ by an infinite *graph* $\mathbb{G}(\mathbb{V}, \mathbb{E})$. This means that the particles marked by vertices $v, v' \in \mathbb{V}$ would interact through the potential $W_{\ell\ell'}$ if the corresponding edge $e = [v, v']$ belongs to $\mathbb{E}$. A principal restriction imposed here on the graph $\mathbb{G}(\mathbb{V}, \mathbb{E})$ is that it has *uniformly bounded degree.* As is commonly recognized, the statistical mechanics on graphs requires principally new concepts and techniques over the well known case of lattices. In particular, the lack of translational invariance and the absence of a natural definition of dimension, makes even the statement of many fundamental classical results rather difficult.

Our main interest here will concern the stochastic dynamics. We shall consider the following infinite system of locally interacting Itô's diffusions

$$\mathrm{d}x_v(t) = \frac{1}{2} b_v(x(t))\mathrm{d}t + \mathrm{d}w_v(t), \quad t > 0, \quad v \in \mathbb{V}, \tag{1.7}$$

where the drift term $b = (b_v)_{v \in \mathbb{V}}$ has a gradient form and its components coincide with the *partial logarithmic derivatives* of the measure μ,

$$b_v(x) := -\beta \left[V_v'(x_v) + \sum_{v' \sim v} \partial_{x_v} W_{vv'}(x_v, x_{v'}) \right] \in \mathbb{R}^\nu. \tag{1.8}$$

The stochastic equation (4.36) is usually called the *Glauber dynamics* associated with the Hamiltonian (4.10). During the last three decades such *SDE*'s in infinite dimensions have been extensively studied in the literature, see the survey in Section 4.2

(iii) Problems and methods

Next, we present a short account of the *main problems, basic methods,* and *known results* in the study of Gibbs measures

I. Existence problem. As is typical for systems with noncompact (in our case, also infinite-dimensional) spin spaces, even the initial question of whether the set $\mathcal{G}^{\mathrm{t}}$ is not empty is far from trivial. A useful observation in this respect is that, under natural assumptions on the interaction, any accumulating point of the family π_Λ, $\Lambda \Subset \mathbb{L}$, is certainly Gibbs. Depending on the specific class of quantum lattice models one deals with, the required convergence $\pi_{\Lambda^{(N)}} \to \mu$, $\Lambda^{(N)} \nearrow \mathbb{L}$, and thus the existence of $\mu \in \mathcal{G}^{\mathrm{t}}$ are proved by the following basic methods listed below:

(a) General Dobrushin's criterion for existence of Gibbs distributions. A standard tool for proving existence of the Gibbs measures is the celebrated Dobrushin criterion, see Theorem 1 in [91]. The validity of sufficient conditions of the Dobrushin existence theorem for some lattice systems (1.1) of scalar spins $x_\ell \in \mathbb{R}$ has been verified e.g. in [42, 71, 259]. In those papers the specific properties of the models, such as *attractiveness* and *translation invariance*, were crucial. The direct extension of this scheme to multi-dimensional spins seems to be impossible. Contrary to the classical case, the same problem for the quantum systems (1.4) was not covered at all by any previous work. More precisely, in order to apply the Dobrushin criterion in the latter case, one should estimate in a proper way the expectations $\mathbf{E}_{\pi_\Lambda(\mathrm{d}\omega|\xi)}\left(|\omega_\ell|_{C^\sigma_\beta}\right)$, $\ell \in \Lambda \Subset \mathbb{L}$, of the norm-function in the Hölder spaces C^σ_β, $\sigma \in (0, 1/2)$, compactly embedded in C_β. Because of the non-trivial topology properties of the spaces $C^\sigma_\beta := C^\sigma_\beta(S_\beta \to \mathbb{R})$, so far no technical means were available to get such moment bounds.

(b) Ruelle's technique of superstability estimates (see the original papers [184, 251] and respectively [224] for its modification to the quantum case). This technique in particular requires that the interaction is *translation invariant* and the many-particle potentials have *at most quadratic growth* . As was shown in those papers, for the subclass of boundary conditions $y \in \Omega^{\mathrm{st}} \subset \Omega^{\mathrm{t}}$ the family of probability kernels $\pi_\Lambda(\mathrm{d}x|y)$, $\Lambda \nearrow \mathbb{L}$, is tight and has at least one accumulation point μ from the subset of *superstable* Gibbs measures $\mathcal{G}^{\mathrm{st}} \subseteq \mathcal{G}^{\mathrm{t}}$. This technique also ensures that any $\mu \in \mathcal{G}^{\mathrm{st}}$ is à-priori of sub-Gaussian growth.

(c) Cluster expansions is one of the most powerful methods for the study of Gibbs fields, but it works only in a *perturbative regime*, i.e., when an effective parameter of the interaction is small. In particular, various versions of this technique imply both existence and also uniqueness (but in some weaker than the *DLR* sense) of the associated infinite volume Gibbs distributions (see e.g. [4, 68, 129, 200, 214, 223, 225] and references therein).

(d) Method of correlations inequalities involves more detailed information about the structure of the interaction (for instance, whether they are ferromagnetic or convex). Starting from a number of correlations inequalities (such as *FKG, GKS, Lebowitz, Brascamp-Lieb* etc.) commonly known for classical lattice systems, by a lattice approximation technique (similar to the one used in the Euclidean field theory) one can also extend them to the quantum case (cf. e.g. [7, 129, 255]).

(e) Method of reflection positivity (as a part of **(d)**) applies to the *translation invariant* systems of scalar spins on a lattice $\mathbb{L} := \mathbb{Z}^d$ with *nearest-neighbors pair interactions* (i.e., when $V_\ell := V$ and $W_{\ell\ell'} := W$ if $|\ell - \ell'| = 1$). For a general description of the method and its applications to classical lattice systems we refer e.g. to [262]. The proper modification of this technique for the quantum systems like (1.4) gives the existence of so-called *periodic* Gibbs states μ^{per} (see e.g. [37, 162]). Moreover, the reflection positivity method can also be used to study *phase transitions* in such models with the double-well anharmonicity V. This has been implemented under certain conditions (e.g. in the dimension $d \geq 3$ and for large enough $m, \beta \gg 1$) in [37, 38, 100, 102, 140, 228].

(f) Analytical approach is based on a characterization of Gibbs measures via their *Radon–Nikodym derivatives* or via *integration by parts.* Such alternative descriptions of Gibbs measures have long been known for a number of specific models in statistical mechanics and field theory (see, e.g., [109]–[119], [144], [247]). Both for the classical and for the quantum lattice systems, a complete characterization of $\mu \in \mathcal{G}^{\mathrm{t}}$ in terms of their Radon–Nikodym derivatives has first been proved in [17, 18]. Assuming that the interaction potentials $V_\ell, W_{\ell\ell'}$ are differentiable, it was further shown in [10]–[13] that the later description of $\mu \in \mathcal{G}^{\mathrm{t}}$ is equivalent to their characterization as differentiable measures satisfying integration by parts formulas. The most progress achieved in the analytical approach is related with the problems of *existence* and *à-priori estimates* for the Gibbs measures, see corresponding results in the last-named group of references.

(g) Method of stochastic dynamics (also referred to in quantum physics as "*stochastic quantization*"; see, e.g. [85, 98, 114, 119, 145, 187, 188, 248]). In this method the Gibbs measures are directly treated as *invariant* (more precise, reversible) distributions for the so-called *Glauber* or *Langevin stochastic dynamics*. However, it requires additional technical assumptions on the interaction (among them *at most quadratic growth* of the pair potentials $W_{\ell\ell'}(x_\ell, x_{\ell'})$) to ensure the solvability of the corresponding stochastic evolution equations in infinite dimensions (not to mention the extremely difficult ergodicity problem for them). This method has been first applied in [24] to prove existence of Euclidean Gibbs states for the quantum models like (1.4).

II. A priori estimates for measures in $\mathcal{G}^{\mathrm{t}}$. The next step is to get a *qualitive* description of the set $\mathcal{G}^{\mathrm{t}}$ of all tempered Gibbs measures. First of all, there is an abstract Dynkin-Föllmer representation theory for the set $\mathcal{G}^{\mathrm{t}}$ in terms of its extreme points, which applies in each model (see e.g. [233] for more details). It would be useful to prove the *compactness* of the set $\mathcal{G}^{\mathrm{t}}$ in proper topologies, which again could be nontrivial if the single spin spaces are noncompact. The compactness is closely related with the another problem of getting *uniform estimates* on correlation functionals of the Gibbs measures in terms of the model parameters. This problem was initially posed for the classical lattice systems in [42, 71]; see also [23] for recent developments. There are very few results in the literature about à-priori integrability properties of the tempered Gibbs measures on loop or path spaces (see [119, 138, 153, 220] in the case of Euclidean $P(\varphi)_1$-fields and respectively [10]–[13], [24] in the case of quantum anharmonic crystals). All of them are based either on the analytical method or on the method of stochastic dynamics just mentioned above. Besides, information about the moments of $\mu \in \mathcal{G}^{\mathrm{t}}$ is also useful for studying the Gibbs measures by means of the associated *Dirichlet operators* $\mathbb{H}_\mu$ in the spaces $L^p(\mu)$, $p \geq 1$, (this is known as the Holley–Stroock approach, see Subsection 2.3.5 (ii)).

III. Uniqueness problem. The validity of sufficient conditions of *Dobrushin's uniqueness criterion*, see Theorem 4 in [91], for the quantum lattice system (1.4) with *convex pair interactions of at most quadratic growth* has been first verified in [19]–[21]. In doing so, the coefficients of Dobrushin's matrix were estimated by means of log-Sobolev inequalities proved on the single loop spaces L^2_β. The uniqueness of $\mu \in \mathcal{G}^{\mathrm{t}}$ was established for small values of the inverse temperature $\beta \in (0, \beta_0)$, but under conditions *independent of the particle mass* $m > 0$. Thus, the results still hold in the quasiclassical regime and, being applied to the classical lattice system (1.1), generalize the previous contributions [71, 247]. For a *special (EMN) class* of quantum ferromagnetic models with the polynomial self-interaction, these results have been essentially improved in the series of papers [6]–[9]. The strongest theorem of such type, obtained in [9], establishes the uniqueness of $\mu \in \mathcal{G}^{\mathrm{t}}$ in the small-mass domain $m \in (0, m_0)$ *uniformly at all values of* $\beta > 0$. On the other hand, for small $m \ll 1$ and uniformly for all $\beta \leq +\infty$, the convergence of cluster expansions (independently of the boundary condition) has been proved in [105, 214]. The *low temperature uniqueness* in classical lattice systems with a unique ground state was also studied by means of special cluster expansions which constitute the Pirogov-Sinai theory of phase diagrams, see [182, 205, 259, 292]. The high and low temperature uniqueness of $\mu \in \mathcal{G}^{\mathrm{t}}$ in the (both, classical and quantum) systems with interactions having *superquadratic growth* is an essentially open problem, which will be the subject of Subsections 2.3.2, 2.3.3, and 3.2.5.

IV. Decay of correlations /spectral gaps. The exponential decay of spin correlations for the Gibbs measures is a standard application of the *Dobrushin contraction technique* (cf. e.g. [91, 108, 176]). Another analytical approach to this property is based on the *spectral gap estimates* for the corresponding Dirichlet operator $\mathbb{H}_\mu$; see [53, 186, 288, 291, 295] for its realization for the classical spin systems (1.1). However, in the quantum case, the corresponding *Poincaré* and *log-Sobolev inequalities* for $\mathbb{H}_\mu$ (or uniformly for all Dirichlet operators of the local measures $\pi_\Lambda(\mathrm{d}\omega|\xi)$) have not yet been studied in the literature, except the trivial situation of strictly convex interaction potentials (cf. [191]). On the other hand, for quantum ferromagnets with the polynomial self-interactions like (1.4), the exponential decay of the so-called *Duhamel two-point function* has been used in [9] as a crucial step for proving uniqueness for $\mu \in \mathcal{G}^{\mathrm{t}}$. Note that Poincaré and log-Sobolev inequalities are an important tool to describe the relaxation of stochastic dynamics to its equilibrium (e.g. Gibbs) measures (see [16, 24, 291, 295] in the classical case).

V. Phase transitions. There are basically two powerful techniques for proving occurrence of phase transitions (non-uniqueness of $\mu \in \mathcal{G}^{\mathrm{t}}$) at *low temperatures* β^{-1}, namely, the *reflection positivity* (for $d \geq 3$) and the *energy-entropy (Peierls-type) method* (for $d \geq 2$). However, in practice their applications to quantum lattice systems have been limited so far to the ferromagnetic $P(\varphi)$-models, see e.g. [14, 37, 38, 100, 102, 116, 118, 129, 130, 140, 162, 228]. We shall refer in Subsection 3.3.7 to the first method (already mentioned in Item **I (e)**), which would allow us to prove the so-called *infrared (Gaussian) bounds* on two-point correlation functions calculated for the periodic local states $\pi_{\mathrm{per},\Lambda}$ (cf. [100, 102, 107, 258]). The second technique, in its full generality also known as the *Pirogov-Sinai contour method* [259, 293], has originally been discovered as the so-called Peierls argument for the Ising model and further developed to apply to various spin systems. Its quantum modification was implemented in [130, 130] to studying phase transition in the $(\varphi^4)_2$-model of the Euclidean field theory and respectively in [14, 100, 116, 265] to its lattice approximations like (1.4).

(iv) Overview of the habilitation thesis

We give a summary of Chapters 2–4 with a particular emphasis on the results obtained by the author.

Chapter 2. We shall start with the classical spin model (1.1), which then will be enriched step by step in the subsequent chapters. Section 2.1 is devoted to the general setup, including the definitions of Hamiltonians, potentials, configurations, etc. In particular, in Subsection 2.1.1 we specify our basic Assumptions (**W**), (**J**), and (**V**) on the interaction potentials, which will remain the *same* throughout the whole manuscript. The traditional *DLR* (Dobrushin-Lanford-Ruelle) route, which is used to define the Gibbs measures through their local specification, will be described in all details in Subsection 2.1.2.

In Section 2.2 we shall develop an *elementary new approach* to the existence problem for Gibbs measures, which relies on certain exponential estimates for the one-point kernels $\pi_\ell(\mathrm{d}x|y)$ of the local Gibbs specification (see Lemma 2.9). Note that these estimates are *slightly stronger* than those required in the fundamental *Dobrushin existence criterion*, the formulation of which we recall in Subsection 2.2.1. The key exponential estimate (2.51) for $\pi_\ell(\mathrm{d}x|y)$ as well as the (resulting from it) uniform estimates for all $\pi_\Lambda(\mathrm{d}x|y)$, $\Lambda \Subset \mathbb{L}$, will be obtained in Subsection 2.2.2. As is shown in Subsection 2.2.3, such estimates imply not only the *existence* of at least one $\mu \in \mathcal{G}^{\mathrm{t}}$ (Theorem 2.14), but also yield the *à-priori bounds* on all points of the set $\mathcal{G}^{\mathrm{t}}$ (Theorem 2.15) and its *compactness* in proper topologies (Corollary 2.16). Other important sequel is detailed information about the *support properties* of each $\mu \in \mathcal{G}^{\mathrm{t}}$ to be obtained in Subsection 2.2.4. A conceptual difference from all previous schemes used to verify Dobrushin's criterion (cf. [42, 71, 237, 259, 291, 295]) is that a choice of the compact function $h(x_\ell)$ participating in such estimates now *depends explicitly* on the growth of the Hamiltonian $H_\ell(x_\ell|y)$. As will be demonstrated in Subsection 2.2.5, the method obviously extends to general N-particle interactions or spin systems on graphs, and hence essentially improves all related existence results (see the discussion in Subsection 2.2.1). The extension to the Euclidean Gibbs measures on loop spaces will be performed in Subsection 3.2.2.

In Section 2.3 we concentrate on the uniqueness problem. We shall consider the cases of *high* ($\beta \ll 1$) and *law* ($\beta \gg 1$) *temperatures* in Subsections 2.3.2 and 2.3.3 respectively. A new issue as compared with the previous uniqueness results (cf. e.g. [20, 71, 291]), is that we include the inter-particle interactions of possibly *superquadratic growth*. Our approach will be based on the *Dobrushin-Pechersky uniqueness criterion*, which we formulate in Subsection 2.3.1. This is a modification of the well-known Dobrushin uniqueness criterion especially suited for non-compact spin spaces. In Subsection 2.3.4 we shall also revisit the original *Dobrushin uniqueness criterion* and examine to what extent it can be applied to the interactions obeying superquadratic growth or infinite range. Finally, in Subsection 2.3.5 we present a systematic account on the analytical properties, such as e.g. the *decay of correlations* for the Gibbs measures $\mu \in \mathcal{G}^{\mathrm{t}}$ and the *spectral gaps* for the associated Dirichlet operators $\mathbb{H}_\mu$, which typically occur in the uniqueness regime.

In Section 2.4 we look in more detail at the properties of the local Gibbs specification. In particular, in Subsection 2.4.1 we give a substantial improvement of the famous exponential bound obtained by J. Bellissard and R. Høegh-Krohn in [42]. In Section 2.4.2 we establish the bounds on correlations functions calculated with respect to $\pi_\Lambda(\mathrm{d}x|y)$, which will be *uniform* in all volumes $\Lambda \Subset \mathbb{L}$ and boundary conditions $y \in \Omega^{\mathrm{t}}$. Finally, in Section 2.4.3, we analyze the behavior of the constants in Dobrushin's Compactness Condition as $\beta \to +0/+\infty$. The later results can be used to describe the concentration properties of the measures $\pi_\Lambda(\mathrm{d}x|y)$.

The publications on the results in this chapter are in preparation.

Chapter 3. Here we develop a consistent rigorous theory of the equilibrium thermodynamic properties of quantum models like (1.4), based on a path measure representation of local Gibbs states. In this theory, the model is interpreted as a system of infinite-dimensional spins; its global properties are described by the Euclidean Gibbs measures (1.6) constructed with the help of the *DLR* equation (1.3). As the spins are infinite dimensional, the methods employed are more involved than those used for classical models. Additional complications arise from the fact that we study a general case, where the model has no spacial regularity and the interaction is of infinite range. In view of the latter possibility, the only way to develop the theory is to impose à-priori restrictions on the support of the Gibbs measures, which was done by means of the weights obeying the conditions (3.55)–(3.59).

Section 3.1 is devoted to general aspects of the theory of Euclidean Gibbs measures. In Subsection 3.1.1 we introduce the object of our study in the form of a system of interacting quantum oscillators (3.1), (3.2) and give some physical motivations. The transform from quantum Gibbs states to Euclidean Gibbs measures is described in Subsection 3.1.2. To this end, in Subsection 3.1.3 we introduce the spaces of temperature loops Ω_Λ and the local Gibbs measures μ_Λ on these spaces, which give a canonical realization for the β-periodic stochastic processes generated by the corresponding Schrödinger operators H_Λ, $\Lambda \Subset \mathbb{L}$. The notion of *temperedness*, which is important in all systems with interactions of infinite range, is discussed in Subsection 3.1.4. Finally, in Subsection 3.1.5 we describe in detail the corresponding Gibbsian formalism and define the set $\mathcal{G}^{\mathrm{t}}$ of all tempered Euclidean Gibbs measures μ on the *loop space* $\Omega := [C_\beta(S_\beta)]^{\mathbb{L}}$, see (1.5).

In Section 3.2 we perform the study of the set $\mathcal{G}^{\mathrm{t}}$ in the *general* case, where $W_{\ell\ell'}$ and V_ℓ satisfy natural stability conditions only. In Subsection 3.2.1 we formulate our main results. In particular, we state that: $\mathcal{G}^{\mathrm{t}}$ is non-void and compact (Theorem 3.18); the elements of $\mathcal{G}^{\mathrm{t}}$ obey certain exponential moment estimates (Theorem 3.19) and have a Lebowitz-Presutti type support (Proposition 3.145); $\mathcal{G}^{\mathrm{t}}$ is a singleton if the interaction is not too strong (Theorems 3.22 and 3.23). The proof of these properties relies essentially on the moment estimates in the spaces of Hölder continuous loops $\omega_\ell \in C_\beta^\sigma$, $\sigma \in (0, 1/2)$, which will be established for the probability kernels $\pi_\Lambda(\mathrm{d}\omega|\xi)$ of the local specification $\Pi = \{\pi_\Lambda\}_{\Lambda \Subset \mathbb{L}}$ in Subsection 3.2.2. Supporting results about the weak convergence of measures on loop spaces are included in Subsection 3.2.3. Afterwards, in Subsections 3.2.4 and 3.2.5 we give the proofs of the existence and, respectively, uniqueness results mentioned above. To this end, we apply both Dobrushin's existence and uniqueness criteria as well as the Dobrushin-Pechersky uniqueness theorem. In Section 3.2.6 we discuss some important generalizations of the initial model.

In Section 3.3 we consider in more detail quantum *scalar ferromagnetic* systems, with the emphasis placed on the study of their *critical behavior.* A qualitative theory of phase transitions and quantum effects in such models, which interprets most important experimental data, will be presented in Subsection 3.3.1. Employing the *FKG* order, in Subsection 3.3.2 we show that the set $\mathcal{G}^{\mathrm{t}}$ has maximal and minimal elements (Theorem 3.41). If the model is translation invariant, in Subsection 3.3.3 we prove that the limiting pressure exists and is the same in all states (Theorem 3.43). Note that the study of our model will be crucially based on the *correlation inequalities*, which we collect in Subsection 3.3.4. Of particular interest is the new *comparison criterion* (Proposition 3.64) stated in Subsection 3.3.5. It allows to compare the initial model with a certain (so-called *reference*) model, for which the property desired can be established directly. Subsection 3.3.6 contains important estimates on the pair correlation functions, which will be used to establish uniqueness or phase transitions. Then, in Subsection 3.3.7 we prove (Theorem 3.45) that under natural additional conditions on V_ℓ, the model undergoes a phase transition (for $d \geq 3$) at law temperatures, i.e. large β. On the other hand, as will be shown in Subsection 3.3.8, the set $\mathcal{G}^{\mathrm{t}}$ is a singleton at all temperatures if a quantum stabilization condition is satisfied (Theorem 3.46). Finally, in Subsection 3.3.9 we describe a class of anharmonic potentials V_ℓ, for which $\mathcal{G}^{\mathrm{t}}$ is a singleton at a non-zero external field (Theorem 3.48). Here we use a version of the Lee-Yang theorem, adapted to path measures. All these results are novel both for the quantum model and its classical analogs.

The results of Section 3.2 are partially announced in [227]. The results of Section 3.3 are based on joint work with Yuri Kozitsky (Lublin); part of this has been published in [172]–[174].

Chapter 4. In the last chapter of the thesis we return to the classical spin systems (1.1), but now the particles will live on an *infinite graph* $\mathbb{G}(\mathbb{V}, \mathbb{E})$. The main object will be the Glauber dynamics (1.7), which describes a stochastic evolution of the system towards its thermal equilibrium. In Section 4.1 we fix a standard *DLR* framework for the Gibbs measures on graphs. A reasonable class of infinite graphs having *uniformly bounded degree* is introduced in Subsection 4.1.1, whereas hypotheses on the interaction potentials are listed in Subsection 4.1.2. In Subsection 4.1.3 we define the set of tempered Gibbs measures $\mu \in \mathcal{G}^{\mathrm{t}}$ and adapt to them the basic results of Chapter 2.

In Section 4.2 we present our central result (Theorem 4.9) about the *pointwise ergodicity* of the *nonequilbrium Glauber dynamics* associated with the interacting spin system (1.1). It says that, starting from each initial value $y \in \Omega^{\mathrm{t}}$, the dynamics will *converge exponentially* in the *Wasserstein metric* to the unique Gibbs measure $\mu \in \mathcal{G}^{\mathrm{t}}$. The result is valid under the assumption of *weak dependence*, which typically holds when the strength of the interaction is small or the temperature is high enough. Furthermore, we give *computable bounds* (4.53), (4.54) on the critical values of these parameters and on the speed of the relaxation. To prove the result we shall combine different probabilistic and analytical tools such as the *Lyapunov function* method, *log-Sobolev* and *Talagrand's inequalities*, and *Dobrushin's contraction technique*. We start in Subsection 4.2.1 with an overview of the ergodicity problem. A unique solution to the infinite system of SDE's describing the Glauber dynamics will be constructed in Subsection 4.2.2. The ergodicity result itself will be precisely stated in Subsection 4.2.3. There we shall also write down its formal proof, which involves several steps to be performed in Sections 4.2–4.5. The scheme of proof relies on a proper approximation of the infinite volume solution $x(t, y) \in \Omega^{\mathrm{t}}$ by the solutions $x^{\Lambda}(t, y) \in \Omega^{\mathrm{t}}$ of the cut-off problems. Precise estimate on the L^p-convergence of such approximations, which are based on the so-called *finite propagation* property for locally interacting diffusions, will be established in Subsection 4.2.4. In Subsection 4.2.5 we consider the set $\mathcal{I}^{\mathrm{t}}$ of all *tempered invariant measures* and obtain *à-priori moment bounds* on its elements (similar to those stated for the tempered Gibbs measures $\mu \in \mathcal{G}^{\mathrm{t}}$ in Theorem 2.15). Finally, in Subsection 4.2.6 we briefly discuss a possible generalization of the stochastic dynamics method to the Euclidean Gibbs states (1.6). A new technical issue is that such dynamics will be governed by an infinite number of *parabolic PDE's* perturbed by the *space-time white noise*.

Section 4.3 is dedicated to the study of ergodicity properties of the *finite volume* Glauber dynamics associated with the local Gibbs distributions $\mu_{\Lambda,y}$, $\Lambda \Subset \mathbb{V}$, $y \in \Omega$. This will be realized through such analytical tools as log-Sobolev and Talagrand's inequalities, which are discussed in Subsection 4.3.1. To get explicit bounds on the corresponding log-Sobolev constants $C_{\mathrm{LS}}(\Lambda, y)$, in Subsection 4.3.2 we shall apply a new criterion for the log-Sobolev inequality suggested by F. Otto and M. Reznikoff in [221]. In Subsection 4.3.3 we demonstrate how to estimate the relative entropy for diffusion processes by using Girsanov's transform.

In Section 4.4 we establish *computable* estimates in the Wasserstein distance on the rate of convergence $\mu_{\Lambda,y} \to \mu \in \mathcal{G}^{\mathrm{t}}$ in the thermodynamic limit $\Lambda \nearrow \mathbb{V}$. This will be done in the framework of Dobrushin's uniqueness criterion, which presumes that the local Gibbs specification $\{\pi_\Lambda\}_{\Lambda \Subset \mathbb{V}}$ satisfies the *weak dependence* condition. In Subsection 4.4.1 we give a positive answer to the *measurability problem* for optimal couplings, which arose in the original works of R. Dobrushin and remained open so far. In Subsection 4.4.2 we extend Dobrushin's criterion to unbounded spin systems on *graphs* (Theorems 4.37 and 4.38) and to the lattice systems with interactions of *infinite range* (Theorem 4.46).

Section 4.5 is devoted to the study of Dirichlet operators associated with Gibbs measures, both in the *classical* and in the *quantum* cases. In Subsection 4.5.1 we present an abstract approach to the spectral gap estimates via the *Efron-Stein-Wu inequalities* for weakly dependent Markov fields. In Subsection 4.5.2 we prove the log-Sobolev and Poincaré inequalities for the local Euclidean Gibbs measures $\mu_{\Lambda,\xi}$ on the loop spaces $\Omega_\Lambda := [C_\beta]^\Lambda$, $\Lambda \Subset \mathbb{L}$. We stress that these results are *new* for the quantum anharmonic systems with non-convex interactions. They will be obtained by means of the Efron-Stein-Wu and Otto-Reznikoff criteria applied to proper cylinder approximation of the path measures. In Subsection 4.5.3 we review on the so-called *analytical approach* to the Euclidean Gibbs measures, which was developed in the joint papers [10]–[13]. In Subsection 4.5.4 we study essential self-adjointness of the Dirichlet operators $\mathbb{H}_\mu$ associated with the Gibbs measures $\mu \in \mathcal{G}^{\mathrm{t}}$ in different types of models dealt with in Chapters 2–4.

The preprints on the results in this chapter are in preparation.

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Chapter 2

Classical Spin Systems

2.1 Description of the model

Following traditional lines, we get started with the interacting systems of classical spins. In the subsequent chapters, this model will be enriched step by step.

2.1.1 Hamiltonians, potentials, configurations

We begin by introducing the *standard setup* of the model. In particular, below we specify our basic Assumptions (**W**), (**J**), and (**V**) on the interaction potentials, which will remain the same both in the classical and in the quantum cases.

(i) Formal Hamiltonian

The systems of our interest will live on some countable set $\mathbb{L} \subset \mathbb{R}^d$. To each $\ell \in \mathbb{L}$ let there correspond a multicomponent variable $x_\ell := (x_\ell^i)_{i=1}^\nu$, called *spin*, which takes values in $\mathbb{R}^\nu$. Note that the dimensions $\nu, d \in \mathbb{N}$ may be arbitrarily high and do not need to coincide. Using a standard interpretation from statistical mechanics, one can speak about a system of classical particles performing ν-dimensional oscillations, with the vector displacements x_ℓ, around their nonstable equilibrium positions at the sites of $\mathbb{L}$. The *configuration space* of the system $\Omega := [\mathbb{R}^\nu]^{\mathbb{L}}$ consists of all sequences $x = (x_\ell)_{\ell \in \mathbb{L}}$; the subset of finite sequences will be denoted by $\Omega_{\rm fin}$. The potential energy of the configuration $x \in \Omega$ is given by a *formal Hamiltonian*

$$H(x) = \sum_\ell V_\ell(x_\ell) + \frac{1}{2} \sum_{\ell, \ell'} W_{\ell\ell'}(x_\ell, x_{\ell'}). \tag{2.1}$$

Actually, (2.1) does not make sense itself as a functional on the whole Ω and is represented by the family of local Hamiltonians $H_\Lambda(x|y)$ related to the bounded volumes $\Lambda \subset \mathbb{L}$ and proper boundary conditions $y \in \Omega^{\rm t}$, cf. (2.23), (2.17). To be more specific, in (2.1) we restrict ourselves to the case of of anharmonic self- and two-particles interactions only, which however are *not necessarily translation-invariant* and *may have* *infinite range* (concerning a possible extension to many-particle interactions and more general Hamiltonians see Subsections 2.2.5 and 2.2.5).

The indexing set $\mathbb{L}$ is equipped with the Euclidean distance $|\ell - \ell'|$ inherited from $\mathbb{R}^d$. Moreover, it is supposed to fulfill the following condition of *spatial regularity*:

Assumption $(\mathbf{L}_d)$ *For every* $p > d$

$$\Xi_p := \sup_{\ell} \sum_{\ell'} (1 + |\ell - \ell'|)^{-p} < \infty. \tag{2.2}$$

Furthermore, the convergence in (2.2) is uniform in the following sense: for any $\epsilon > 0$ *one finds* $N(p, \epsilon) \in \mathbb{N}$*, such that for all* $\ell \in \mathbb{L}$ *and* $N \geq N(p, \epsilon)$

$$\sum_{\ell' \in \mathbb{L}:\ |\ell' - \ell| > N} (1 + |\ell - \ell'|)^{-p} < \epsilon. \tag{2.3}$$

In other words, (2.2) means that big amounts of the elements of $\mathbb{L}$ cannot accumulate in the subsets of $\mathbb{R}^d$ of small volume. The extra condition (2.3) is only needed to include the interactions of infinite range, otherwise it can be omitted as we point out in Remark 2.1 (iii). Assumption $(\mathbf{L}_d)$ surely holds if $\mathbb{L}$ is the integer lattice $\mathbb{Z}^d$; in the latter case the model is called the *classical anharmonic crystal.* For convenience, we always suppose that $0 \in \mathbb{L}$.

We use the following *notation* related to the spin/space structure: $(\cdot, \cdot)$ denotes the scalar product and $|\cdot|$ respectively the distance in all Euclidean spaces $\mathbb{R}^d$, $\mathbb{R}^\nu$ etc. The symbols $\sum_\ell$ and $\sum_{\ell,\ell'}$ everywhere mean infinite sums being taken over all $\ell \in \mathbb{L}$ and ordered pairs $(\ell, \ell') \in \mathbb{L}^2$. As usual, $|\Lambda|$ stands for the *cardinality* (i.e., number of points) and Λ^c – for the *complement* of a subset $\Lambda \subseteq \mathbb{L}$; for shorthand we write $\Lambda \Subset \mathbb{L}$ if $1 \leq |\Lambda| < \infty$. For a given $r \geq 1$, by

$$\partial_r \Lambda := \left\{ \ell' \in \Lambda^c \;\middle|\; \operatorname{dist}(\ell'; \Lambda) := \inf_{\ell \in \Lambda} |\ell - \ell'| \ \leq r \right\}, \tag{2.4}$$
$$\partial_r \ell := \{ \ell' \in \mathbb{L} \mid 0 < |\ell - \ell'| \ \leq r \}.$$

we define respectively the *r-boundary* of the set $\Lambda \subseteq \mathbb{L}$ and the *r-vicinity* of the point $\ell \in \mathbb{L}$. A sequence of finite volumes $\mathcal{L} := \{\Lambda_N\}_{N \in \mathbb{N}}$ is called *cofinal* if it is ordered by inclusion and exhausts the whole $\mathbb{L}$. Furthermore, $\Lambda \nearrow \mathbb{L}$ means the limit taken along any unspecified sequence $\mathcal{L}$ of this type.

(ii) Conditions on the interaction potentials

Throughout the *whole* manuscript, the interaction potentials are given by *continuous functions*

$$V_\ell : \mathbb{R}^\nu \to \mathbb{R}, \quad V_\ell(0) = 0, \quad \ell \in \mathbb{L}, \tag{2.5}$$
$$W_{\ell\ell'} = W_{\ell'\ell} : \mathbb{R}^\nu \times \mathbb{R}^\nu \to \mathbb{R}, \quad W_{\ell\ell} \equiv 0, \quad \ell, \ell' \in \mathbb{L},$$

satisfying the following conditions:

Assumption (W) *There exist constants* $R \geq 2$, $C_W \geq 0$ *and a symmetric matrix* $\mathbf{J} = (J_{\ell\ell'})_{\mathbb{L}\times\mathbb{L}}$ *with the non-negative entries and zero diagonal*

$$J_{\ell\ell'} = J_{\ell\ell'} \geq 0, \quad J_{\ell\ell} = 0, \quad \ell, \ell' \in \mathbb{L}, \tag{2.6}$$

such that for all $x_\ell, x_{\ell'} \in \mathbb{R}^\nu$

$$|W_{\ell\ell'}(x_\ell, x_{\ell'})| \leq \frac{1}{2} J_{\ell\ell'}(C_W + |x_\ell|^R + |x_{\ell'}|^R). \tag{2.7}$$

Assumption (J) *The matrix* $\mathbf{J}$ *is fastly decreasing, that is for all* $p \geq 0$

$$||\mathbf{J}||_p \ := \sup_\ell \sum_{\ell'} J_{\ell\ell'}(1 + |\ell - \ell'|)^p < \infty. \tag{2.8}$$

Assumption (V) *There exist a continuous function* $V : \mathbb{R}^\nu \to \mathbb{R}$ *and constants* $P > R$, $A_V > 0$, *and* $B_V \in \mathbb{R}$, *such that for all* $\ell \in \mathbb{L}$ *and* $x_\ell \in \mathbb{R}^\nu$

$$A_V |x_\ell|^P + B_V \leq V_\ell(x_\ell) \leq V(x_\ell). \tag{2.9}$$

Without loss of generality, we suppose that the pair potentials $W_{\ell\ell'}$ vanish at the diagonal and are invariant with respect to all permutations of the coordinates ℓ, ℓ' and variables $x_\ell, x_{\ell'}$. By adding a constant to V_ℓ, one may always agree that $V_\ell(0) = 0$. Assumptions (**W**), (**J**), and (**V**) listed above will be the basic ones (even though repeated in main statements to make partial reading possible), some their stronger versions or modifications will be introduced as needed later.

Remark 2.1 (i) The conditions (2.5)–(2.9) are fulfilled for many classes of interactions of physical relevance. An important example is the *polynomials*

$$V_\ell(x_\ell) := \sum\nolimits_{\mathtt{s}=1}^{\mathtt{p}} b_\ell^{(\mathtt{s})} |x_\ell|^{2\mathtt{s}}, \quad W_{\ell\ell'}(x_\ell, x_{\ell'}) := \sum\nolimits_{\mathtt{s}=1}^{\mathtt{r}} c_{\ell\ell'}^{(\mathtt{s})} |x_\ell - x_{\ell'}|^{2\mathtt{s}}, \tag{2.10}$$

in which $1 \leq \mathtt{r} < \mathtt{p}$ and the coefficients $b_\ell^{(\mathtt{s})}$, $c_{\ell\ell'}^{(\mathtt{s})} \in \mathbb{R}$ are kept uniformly in certain intervals (so that $0 < \inf_\ell b_\ell^{(\mathtt{p})} \leq \sup_{\mathtt{s},\ell} |b_\ell^{(\mathtt{s})}| < \infty$ and $\sup_{\mathtt{s},\ell,\ell'} |c_\ell^{(\mathtt{s})}| < \infty$). In fact, from V_ℓ one always can extract a *quadratic term* $U(x_\ell) := a|x_\ell|^2/2 + (h, x_\ell)$ with an arbitrarily large $a > 0$ and an *external field* $h \in \mathbb{R}^\nu$, so that (2.9) is still true for the potentials $\tilde{V}_\ell := V_\ell - U$ with any positive $\tilde{A}_V < A_V$. Merely speaking, our hypotheses mean that the inter-particle interaction is dominated by the self-potentials, which implies a *lattice stabilization effect.* The case $P = R$ is allowed as well, but it needs a more accurate analysis (which gives rise to Assumption $(\mathbf{V}_1)$ in Subsection 2.2.2 below).

(ii) The condition (2.8) ensures that the (symmetric) matrix $\mathbf{J} = (J_{\ell\ell'})_{\mathbb{L}\times\mathbb{L}}$ generates a linear bounded operator in each of the Banach spaces

$$l_p^1(\mathbb{L}) := \left\{ u = (u_\ell)_{\ell\in\mathbb{L}} \in \mathbb{R}^{\mathbb{L}} \;\middle|\; |u|_{l_p^1} := \sum_\ell (1 + |\ell|)^{-p} |u_\ell| < \infty \right\}, \tag{2.11}$$

$$l_p^\infty(\mathbb{L}) := \left\{ u \in \mathbb{R}^{\mathbb{L}} \;\middle|\; |u|_{l_p^\infty} := \sup_\ell \left\{ (1 + |\ell|)^{-p} |u_\ell| \right\} < \infty \right\}, \quad p \geq 0,$$

and its norm there does not exceed $||\mathbf{J}||_p$. The interaction is of *finite range* if there exists $r > 0$ such that $W_{\ell\ell'} \equiv 0$ whenever $|l - l'| > r$. The model is called *translation invariant* if $\mathbb{L} := \mathbb{Z}^d$ and $V_\ell := V$, $W_{\ell\ell'} := W_{\ell-\ell'}$ for all ℓ, ℓ'.

(iii) We can substitute the condition (2.3) in Assumption ($\mathbf{L}_d$) by the following *supplement* to Assumption ($\mathbf{J}$): for any $\epsilon > 0$ one finds $N(p, \epsilon) \in \mathbb{N}$, such that for all $\ell \in \mathbb{L}$ and $N \geq N(p, \epsilon)$

$$\sum_{\ell' \in \mathbb{L}:\ |\ell' - \ell| > N} J_{\ell\ell'}(1 + |\ell - \ell'|)^{-p} < \epsilon. \tag{2.12}$$

The latter surely holds for all interaction of finite range obeying

$$||\mathbf{J}||_0 \ := \sup_{\ell} \sum_{\ell'} J_{\ell\ell'} < \infty. \tag{2.13}$$

(iii) Spaces of configurations

In a usual way we endow the configuration space Ω with the *product topology* $\mathcal{T}(\Omega)$ which is the weakest topology such that all finite volume projections

$$\Omega \ni x \mapsto \mathbb{P}_\Lambda x := x_\Lambda := (x_\ell)_{\ell \in \Lambda} \in [\mathbb{R}^\nu]^\Lambda := \Omega_\Lambda, \quad \Lambda \Subset \mathbb{L}, \tag{2.14}$$

are continuous, and with the corresponding *Borel σ-algebra* $\mathcal{B}(\Omega)$ which is the smallest σ-algebra containing all open sets. It is a well known fact for product spaces that $\mathcal{B}(\Omega)$ coincides with the σ-algebra generated by *cylinder sets*

$$\{x \in \Omega \ \mid x_\Lambda \in B_\Lambda\}, \quad B_\Lambda \in \mathcal{B}(\Omega_\Lambda), \ \Lambda \Subset \mathbb{L}.$$

By $\mathcal{P}(\Omega)$ and $\mathcal{P}(\Omega_\Lambda)$ we denote the set of all probability measures respectively on $(\Omega, \mathcal{B}(\Omega))$ and $(\Omega_\Lambda, \mathcal{B}(\Omega_\Lambda))$.

As is typical for systems with unbounded spins, we next have to restrict ourselves to certain subsets Ω^{t} of reasonable configurations and, respectively, to the measures $\mu \in \mathcal{P}(\Omega)$ supported by such $x \in \Omega^{\mathrm{t}}$. Their optimal choice is strongly determined by the conditions imposed on the interaction, see Proposition 2.3. To this end we introduce the scale of weighted Banach spaces

$$\Omega_p := \left\{ x \in \Omega \ \middle| \ \|x\|_p := \left[\sum_\ell (1 + |\ell|)^{-p} |x_\ell|^R \right]^{1/R} < \infty \right\}, \quad p > d. \tag{2.15}$$

Here the parameter $R \geq 2$ is fixed and describes the (largest possible) order of polynomial growth allowed for the pair potentials $W_{\ell\ell'}$ by Assumption ($\mathbf{W}$). The restriction $p > d$ is motivated by the inclusion $\mathbf{1} \in \Omega_p$, which we would like to have for technical convenience. In the subsequent we shall crucially use the fact that the embeddings

$$\Omega_p \hookrightarrow \Omega_{p'} \ \textit{ are compact whenever } \ p < p'. \tag{2.16}$$

The latter means that for all $r \in (0,\infty)$ the balls $B_p(r) := \{x \in \Omega \mid \|x\|_p \le r\}$ are (closed) compact sets in $\Omega_{p'}$.

Now we define the subset of (classical) *tempered configurations*

$$\Omega^{\mathrm{t}} := \Omega^{\mathrm{t}}_{\mathrm{cl}} := \bigcup_{p>d} \Omega_p = \{x \in \Omega \mid \exists p = p(x) > 0 : \ \|x\|_p < \infty \ \} \tag{2.17}$$

and, respectively, the subset of *tempered measures*

$$\mathcal{P}^{\mathrm{t}}(\Omega) := \{\mu \in \mathcal{P}(\Omega) \ \mid \exists p = p(\mu) > d : \ \mu(\Omega_p) = 1\}. \tag{2.18}$$

By the above construction

$$\mathcal{P}^{\mathrm{t}}(\Omega) = \bigcup_{p>d} \mathcal{P}(\Omega_p) \subset \mathcal{P}(\Omega^{\mathrm{t}}), \tag{2.19}$$

where $\mathcal{P}(\Omega_p)$ and $\mathcal{P}(\Omega^{\mathrm{t}})$ consist of those $\mu \in \mathcal{P}(\Omega)$ which are supported respectively by Ω_p and Ω^{t}. We consider Ω^{t} as a *locally convex* topological space equipped with the topology of *inductive limit*, that is the strongest one for which all embeddings $\Omega_p \hookrightarrow \Omega^{\mathrm{t}}$ are continuous. The reader is warned that this topology is however not metrizable. Given measurable functions $f, g : \Omega \to \mathbb{R}$, by

$$\begin{gathered}\mathbf{E}_\mu f = \mu(f) := \int_\Omega f \mathrm{d}\mu, \quad \mathbf{Var}_\mu f = \mu(f;f) := \mathbf{E}_\mu(f^2) - (\mathbf{E}_\mu f)^2, \\ \mathbf{Cov}_\mu(f;g) = \mu(f;g) := \mathbf{E}_\mu(fg) - \mathbf{E}_\mu f \cdot \mathbf{E}_\mu g\end{gathered} \tag{2.20}$$

we denote their *expectation*, *variance*, and *covariation* with respect to the measure $\mu \in \mathcal{P}(\Omega)$, provided the integrals in (2.20) make sense.

2.1.2 Tempered Gibbs measures

Let us fix some $\beta := 1/T > 0$ having the meaning of *inverse absolute temperature.* We shall often suppress the parameter β from the notation, so far no clarity is lost. In classical statistical mechanics one looks at the corresponding Gibbs states which are probability measures on $(\Omega, \mathcal{B}(\Omega))$ of a formal appearance

$$\mu(\mathrm{d}x) := \left(\int_\Omega \exp\{-\beta H(x)\} \times_{\ell\in\mathbb{L}} \mathrm{d}x_\ell \right)^{-1} \exp\{-\beta H(x)\} \times_{\ell\in\mathbb{L}} \mathrm{d}x_\ell.$$

To give a rigorous meaning to such μ we traditionally shall follow the *DLR* (*Dobrushin–Lanford–Ruelle*) route. Recall that the standard sources on the *DLR* approach are the monographs [122, 233].

(i) Local Gibbs specification

The Gibbs random fields are determined by means of their *local specification* $\Pi = \{\pi_\Lambda\}_{\Lambda \Subset \mathbb{L}}$. In our contest, this is a family of *stochastic kernels*

$$\mathcal{B}(\Omega) \times \Omega \ni (B, y) \mapsto \pi_\Lambda(\Delta|y) \in [0,1]$$

defined by

$$\pi_\Lambda(B|y) := \begin{cases} Z_\Lambda^{-1}(y) \int\limits_{\Omega_\Lambda} \exp\left\{-\beta H_\Lambda(x_\Lambda|y)\right\} \mathbf{1}_B(x_\Lambda \times y_{\Lambda^c})\, \mathrm{d}x_\Lambda, & y \in \Omega^{\mathrm{t}}, \\ 0, & y \notin \Omega^{\mathrm{t}}, \end{cases} \tag{2.21}$$

where $\mathrm{d}x_\Lambda := \times_{\ell\in\Lambda} \mathrm{d}x_\ell$ and $\mathbf{1}_\Delta$ stands for the indicator on $B \in \mathcal{B}(\Omega)$. Here

$$Z_\Lambda(y) := \int_{\Omega_\Lambda} \exp\left\{-\beta H_\Lambda(x_\Lambda|y)\right\} \mathrm{d}x_\Lambda \tag{2.22}$$

is the normalization factor (so-called *partition function*) and

$$H_\Lambda(x_\Lambda|y) := H_\Lambda(x_\Lambda) + \sum_{\ell\in\Lambda,\ \ell'\in\Lambda^c} W_{\ell\ell'}(x_\ell, y_{\ell'}) \tag{2.23}$$

is the interaction in the finite volume Λ under the boundary condition $y \in \Omega^{\mathrm{t}}$ fixed in the complement Λ^c. Thereby,

$$H_\Lambda(x_\Lambda) := \sum_{\ell\in\Lambda} V_\ell(x_\ell) + \frac{1}{2} \sum_{\ell,\ell'\in\Lambda} W_{\ell\ell'}(x_\ell, x_{\ell'}) \tag{2.24}$$

can be looked upon as the local Hamiltonian related to the *empty boundary condition* $y = \varnothing$. By Proposition 2.3 below, (2.21)–(2.23) make sense for the potentials V_ℓ, $W_{\ell\ell'}$ we deal with. For each $\Lambda \Subset \mathbb{L}$ and $y \in \Omega^{\mathrm{t}}$, the probability measure $\pi_\Lambda(\mathrm{d}x|y)$ is concentrated on configurations of the form $x = (x_\Lambda, y_{\Lambda^c}) \in \Omega^{\mathrm{t}}$. It is reasonable to consider its *finite volume projection*

$$\mu_{\Lambda,y}(\mathrm{d}x_\Lambda) := \pi_\Lambda(\mathrm{d}x|y) \circ \mathbb{P}_\Lambda^{-1} \in \mathcal{P}(\Omega_\Lambda), \tag{2.25}$$

which is called the *local Gibbs distribution* under the boundary condition y. By the above construction, the family $\Pi = \{\pi_\Lambda\}_{\Lambda\Subset\mathbb{L}}$ is a specification in the sense that it obeys the *consistency property*

$$\int_\Omega \pi_\Lambda(B|x)\pi_{\Lambda'}(\mathrm{d}x|y) = \pi_{\Lambda'}(B|y), \quad \Lambda \subseteq \Lambda', \tag{2.26}$$

holding for all $B \in \mathcal{B}(\Omega)$ and $y \in \Omega$. In the subsequent we crucially shall use that

$$\int_\Omega \exp\left\{\lambda \sum_{\ell\in\Lambda} |x_\ell|^R\right\} \pi_\Lambda(\mathrm{d}x|y) < \infty, \quad \forall \lambda > 0, \tag{2.27}$$

which will be clear from the estimate (2.32).

Remark 2.2 (i) In the definition of a local Gibbs specification (cf. page 28 of [122]), one usually claims that each kernel $\pi_\Lambda(\mathrm{d}x|y)$ is a probability measure on Ω. Here we *modify* this definition by including the possibility for $\pi_\Lambda(\mathrm{d}x|y)$ to vanish if y do not belong to the subset Ω^{t}. The following will be clear from Remark 2.5 (i): whatever values of $\pi_\Lambda(\mathrm{d}x|y)$ are taken for $y \notin \Omega^{\mathrm{t}}$, they do not affect the properties of those Gibbs measures which are supported by Ω^{t}.

(ii) For each specification $\{\pi_\Lambda\}_{\Lambda \Subset \mathbb{L}}$ the notions of *weak* and *strong measurability* are equivalent. It is easy to show that the measurability of the family of (real-valued) functions $y \to \pi_\Lambda(B|y)$ for all $B \in \mathcal{B}(\Omega)$ implies the measurability of the mapping $y \to \pi_\Lambda(\mathrm{d}x|y) \in (\mathcal{P}(\Omega), \mathcal{W})$. Here $\mathcal{W}$ is the standard topology of weak convergence for measures $\mu \in \mathcal{P}(\Omega)$. This assertion will be relevant for constructing y-measurable couplings for $\pi_\Lambda(\mathrm{d}x|y)$ in Section 4.4.

(iii) If the interaction has finite range, the probability kernels $\pi_\Lambda(\mathrm{d}x|y)$ are defined by the integral representation in (2.21) for all $y \in \Omega$.

Let $C_{\mathrm{b}}(\Omega_p)$ denote the Banach space of all *bounded continuous functions* $f : \Omega_p \to \mathbb{R}$ endowed with the sup-norm. An important observation is that the specification $\Pi = \{\pi_\Lambda\}_{\Lambda \Subset \mathbb{L}}$ is *regular* in the following sense:

Proposition 2.3 *For each $\Lambda \Subset \mathbb{L}$ and $p > d$, the next properties hold under Assumptions* (**W**), (**J**), *and* (**V**):

(i) For every finite radius ball $B_p(r)$, the mapping $\Omega_p \times \Omega_p \ni (x, y) \mapsto H_\Lambda(x_\Lambda|y)$ is uniformly continuous on $B_p(r) \times B_p(r)$. Moreover,

$$-\infty < \inf_{x\in\Omega,\ y\in B_p(r)} H_\Lambda(x_\Lambda|y) \le \sup_{x,y\in B_p(r)} H_\Lambda(x_\Lambda|y) < \infty. \tag{2.28}$$

(ii) The partition function $\Omega_p \ni y \mapsto Z_\Lambda(y)$ is continuous. Moreover,

$$-\infty < \inf_{y\in B_p(r)} Z_\Lambda(y) \le \sup_{y\in B_p(r)} H_\Lambda(x_\Lambda|y) < \infty. \tag{2.29}$$

(ii) ***Feller Property:*** *For any bounded measurable $f : \Omega \to \mathbb{R}$, let us define*

$$(\pi_\Lambda f)(y) := \int_\Omega f(x)\pi_\Lambda(\mathrm{d}x|y), \quad y \in \Omega. \tag{2.30}$$

Then, $f \mapsto \pi_\Lambda f$ is a contraction operator in $C_{\mathrm{b}}(\Omega_p)$.

Proof. (i) By the definition (2.23), $H_\Lambda(x_\Lambda|y) = \lim_{N\to\infty} H_\Lambda(x_\Lambda|y_{\Lambda_N})$ for any cofinal sequence $\mathcal{L} = \{\Lambda_N\}_{N\in\mathbb{N}}$. By the construction, the mappings $\Omega_p \times \Omega_p \ni (x, y) \mapsto H_\Lambda(x_\Lambda|y_{\Lambda_N^c})$ are bounded and uniformly continuous on the sets $B_p(r) \times B_p(r)$ for all $r > 0$ and $p > d$. Next we observe that

$$\sum_{\ell'\in\Lambda^c} |W_{\ell\ell'}(x_\ell, y_{\ell'})| \le \frac{1}{2}\left[||\mathbf{J}||_0\left(|x_\ell|^R + C_W\right) + ||\mathbf{J}||_p \|y_{\Lambda^c}\|_p^R (1 + |\ell|)^p\right], \tag{2.31}$$

and hence the series in (2.23) converges uniformly on $B_p(r) \times B_p(r)$. The limit mapping $(x, y) \mapsto H_\Lambda(x_\Lambda|y)$ is thus uniformly continuous and bounded on $B_p(r) \times B_p(r)$. Furthermore,

$$\begin{aligned} H_\Lambda(x_\Lambda|y) \ &\geq \ (B_V - C_W||\mathbf{J}||_0)|\Lambda| + A_V \sum_{\ell\in\Lambda} |x_\ell|^P - \frac{1}{2}||\mathbf{J}||_0 \sum_{\ell\in\Lambda} |x_\ell|^R \\ &\quad -\frac{1}{2}||\mathbf{J}||_p ||y_{\Lambda^c}||_p^R \sum_{\ell\in\Lambda} (1+|\ell|)^p, \end{aligned} \tag{2.32}$$

which justifies the lower bound in (2.28).

(ii) The required properties of the partition function $Z_\Lambda(y)$ follow by Lebesgue's dominated convergence theorem applied to the right-hand side in (2.22).

(iii) For $f \in C_{\mathrm{b}}(\Omega_p)$ and $y \in \Omega_p$, we can write

$$(\pi_\Lambda f)(y) = \int_{\Omega_\Lambda} F_\Lambda(x_\Lambda|y) \mathrm{d}x_\Lambda,$$

where, according to the claim (i), the integrand

$$(x, y) \mapsto F_\Lambda(x_\Lambda|y) \ := \ f(x_\Lambda \times y_{\Lambda^c}) \exp\left\{-H_\Lambda(x_\Lambda|y)\right\} / Z_\Lambda(y)$$

is continuous on $\Omega_p \times \Omega_p$. Moreover, by (2.28) the map

$$\Omega_p \ni y \mapsto \sup_{x\in\Omega} |F_\Lambda(x_\Lambda|y)|$$

is locally bounded. This allows us to apply Lebesgue's dominated convergence theorem and obtain the continuity of $\Omega_p \ni y \mapsto (\pi_\Lambda f)(y)$. Obviously,

$$\sup_{y\in\Omega_p} |\pi_\Lambda f(y)| =: ||\pi_\Lambda f||_{C_{\mathrm{b}}(\Omega_p)} \leq ||f||_{C_{\mathrm{b}}(\Omega_p)}, \tag{2.33}$$

which completes the proof of (iii). ■

(ii) DLR equation and its solutions $\mu \in \mathcal{G}^{\mathrm{t}}$

Now we recall the general definition of Gibbs random fields and discuss its peculiarities for the model (2.1).

Definition 2.4 *A probability measure $\mu \in \mathcal{P}(\Omega)$ is called a **Gibbs measure** (or **state**) for the local specification $\Pi = \{\pi_\Lambda\}_{\Lambda \Subset \mathbb{L}}$ if it satisfies the DLR equilibrium equation*

$$(\pi_\Lambda \mu)(B) := \int_\Omega \pi_\Lambda(B|y)\mu(\mathrm{d}y) = \mu(B), \tag{2.34}$$

*for all $\Lambda \Subset \mathbb{L}$ and $B \in \mathcal{B}(\Omega)$. Fixed an inverse temperature β, the associated set of all (classical) Gibbs states for the system (2.1) will be denoted by $\mathcal{G} := \mathcal{G}_{\mathrm{cl}}$. We mostly shall be concerned with the subset of **tempered Gibbs measures***

$$\begin{aligned} \mathcal{G}^{\mathrm{t}} := \mathcal{G}^{\mathrm{t}}_{\mathrm{cl}} &:= \mathcal{G} \cap \mathcal{P}^{\mathrm{t}}(\Omega) \\ &= \left\{\mu \in \mathcal{G} \mid \exists p = p(\mu) > d: \ \mu(\Omega_p) = 1\right\}. \end{aligned} \tag{2.35}$$

Remark 2.5 (i) Definition 2.4 ensures that each of $\mu \in \mathcal{G}$, provided such exist, is *à-priori* supported by $\Omega^{\rm t}$. Indeed, by (2.21) it holds $\pi_\Lambda(\Omega \setminus \Omega^{\rm t}|y) = 0$ for every $\Lambda \Subset \mathbb{L}$ and $y \in \Omega$. Hence, by (2.34) one readily has that $\mu(\Omega \setminus \Omega^{\rm t}) = 0$. In turn, by (2.35) each of $\mu \in \mathcal{G}^{\rm t}$ has to be supported by a certain Banach space $\Omega_p \subset \Omega^{\rm t}$. This additional restriction $\mu(\Omega_p) = 1$ will appear for technical reasons, once we proceed to getting uniform moment estimates on the Gibbs measures (cf. the proof of Theorem 2.15 below). Having got such estimates, we then *à-posteriori* may conclude (cf. Propositions 2.17 and 2.19) that all $\mu \in \mathcal{G}^{\rm t}$, as the solutions to the DLR equation (2.34), have some *universal support* $\Omega^{\rm supp} \subset \bigcap_{p>d} \Omega_p$ which consists of $x \in \Omega$ obeying the asymptotic bound $\limsup_{|\ell|\to\infty} \left\{ |x_\ell|^R / \log(1 + |\ell|) \right\} = 0$.

(ii) Our starting notion of temperedness (2.35) is more extended as those used in the earlier papers. So, $\mathcal{G}^{\rm t}$ surely contains all Gibbs measures satisfying the following condition in terms of their moment sequence

$$\exists p = p(\mu) > 0 : \ \sum_{\ell} (1 + |\ell|)^{-p} \mu(|x_\ell|) < \infty,$$

which was imposed in [42, 71, 291]. Thus, $\mathcal{G}^{\rm t}$ includes the (even smaller) class $\mathcal{G}^{\rm st}$ of so-called *"superstable"* Gibbs states, which for lattice models was introduced in [184, 254], see also Remark 2.21 below.

(iii) According to (2.25) and (2.34), the *finite volume projections* $\mu_\Lambda := \mu \circ \mathbb{P}_\Lambda^{-1}$ of $\mu \in \mathcal{G}$ are given by

$$\mu_\Lambda({\rm d}x_\Lambda) := \int_\Omega \mu_{\Lambda,y}({\rm d}x_\Lambda)\mu({\rm d}y) \tag{2.36}$$

and so far are not known explicitly. Unlike Kolmogorov's theorem, a problem (2.34) of reconstructing a measure through its conditional distributions also may admit an *infinite number* of solutions or *none* of them.

(iv) Actually, $\mu \in \mathcal{P}(\Omega)$ is a Gibbs distribution for the local specification $\{\pi_\Lambda\}_{\Lambda \Subset \mathbb{L}}$ if it satisfies the DLR equation (2.34) for all *one-point sets* $\Lambda := \{\ell\}$ (see Theorem 1.33 in [122] and Theorem 8.1 in [234]). So, all information about the measures $\mu \in \mathcal{G}^{\rm t}$ could be derived from the family of their *one-point conditional distributions*

$$\mu_{\ell,y}({\rm d}x_\ell) := \pi_\ell({\rm d}x|y) \circ \mathbb{P}_\ell^{-1}, \quad \text{for } \ell \in \mathbb{L},\ y \in \Omega^{\rm t}.$$

They have the explicit form (cf. (2.21), (2.23))

$$\mu_{\ell,y}({\rm d}x_\ell) := Z_\ell^{-1}(y) \exp\left\{-\beta H_\ell(x_\ell|y)\right\} {\rm d}x_\ell, \tag{2.37}$$

where

$$H_\ell(x_\ell|y) := V_\ell(x_\ell) + \sum\nolimits_{\ell'(\neq \ell)} W_{\ell,\ell'}(x_\ell, y_{\ell'}), \tag{2.38}$$

$$Z_\ell(y) := \int_{\mathbb{R}^\nu} \exp\left\{-\beta\left[V_\ell(x_\ell) + \sum\nolimits_{\ell'(\neq\ell)} W_{\ell,\ell'}(x_\ell, y_{\ell'})\right]\right\} {\rm d}x_\ell.$$

Abusing notation we write $H_\ell(x_\ell|y)$, $\pi_\ell({\rm d}x|y)$, and the like for the corresponding objects indexed by $\{\ell\}$.

Example 2.6 Another typical condition on the interaction, which is stronger than (2.8), is as follows:

Assumption $(\mathbf{J}_\delta)$ *The matrix J is decaying exponentially quickly, that is for some $\delta > 0$*

$$||\mathbf{J}||_\delta := \sup_\ell \sum_{\ell'} J_{\ell\ell'} \exp\{\delta|\ell - \ell'|\} < \infty. \tag{2.39}$$

Now it would be naturally to define the subsets of (exponentially) tempered configurations and, respectively, Gibbs measures by

$$\Omega^{(\mathrm{e})\mathrm{t}} := \left\{ x \in \Omega \;\middle|\; \forall \delta > 0 : \; \|x\|_\delta := \left[\sum_\ell |x_\ell|^R \exp\{-\delta|\ell - \ell'|\} \right]^{1/R} < \infty \right\},$$
$$\mathcal{G}^{(\mathrm{e})\mathrm{t}} := \left\{ \mu \in \mathcal{G} \;\middle|\; \mu(\Omega^{(\mathrm{e})\mathrm{t}}) = 1 \right\}, \tag{2.40}$$

so that the inclusions $\Omega^{\mathrm{t}} \subseteq \Omega^{(\exp)\mathrm{t}}$ and $\mathcal{G}^{\mathrm{t}} \subseteq \mathcal{G}^{(\exp)\mathrm{t}}$ hold. Thus, more restrictive assumptions on the interaction allows us to consider more extended classes of tempered Gibbs measures, and vice versa.

By $\mathcal{W}_p$ we denote the *weak topology* on the set $\mathcal{P}(\Omega_p)$ of all probability measures supported by Ω_p. It is generated by the local base

$$\mathcal{U}^{\varepsilon}_{f_1,\ldots,f_N}(\mu) = \left\{ \nu \in \mathcal{P}(\Omega_p) \; \left|\mu(f_i) - \nu_(f_i)\right| < \varepsilon, \;\; 1 \le i \le N \right\}, \tag{2.41}$$

with all possible choices of $\varepsilon > 0$, $N \in \mathbb{N}$, and functions $f_i \in C_{\mathrm{b}}(\Omega_p)$. Note that $C_{\mathrm{b}}(\Omega_p)$ is *a measure determining class* for $\mu \in \mathcal{P}(\Omega_p)$. With this topology the set $\mathcal{P}(\Omega_p)$ becomes a *Polish* (i.e., complete separable metrizable) *space*, cf. Theorem 6.5, page 46 of [226]. Analogously, by substituting $C_{\mathrm{b}}(\Omega)$ for $C_{\mathrm{b}}(\Omega_p)$ in (2.41), one gets the weak topology $\mathcal{W}$ on the set $\mathcal{P}(\Omega)$. Having in mind applications to $\mu \in \mathcal{G}^{\mathrm{t}}$, it is more convenient to use the topology $\mathcal{W}_p$ which is stronger than the one induced on $\mathcal{P}(\Omega_p)$ by $\mathcal{W}$. The next statement, which is an immediate sequel of the regularity property (2.30), suggests us a standard way of constructing $\mu \in \mathcal{G}^{\mathrm{t}}$.

Proposition 2.7 *For each $p > d$, every $\mathcal{W}_p$-accumulation point of the family $\{\pi_\Lambda(\cdot|y) \mid \Lambda \Subset \mathbb{L},\, y \in \Omega_p\}$, as $\Lambda \nearrow \mathbb{L}$, is the tempered Gibbs measure.*

Proof. A measure $\mu \in \mathcal{P}(\Omega_p)$ solves (2.34) iff for all $f \in C_{\mathrm{b}}(\Omega_p)$ and $\Lambda \Subset \mathbb{L}$

$$\int_\Omega f(x)\mu(\mathrm{d}x) = \int_\Omega (\pi_\Lambda f)(x)\mu(\mathrm{d}x). \tag{2.42}$$

Let $\mathcal{L} := \{\Lambda_N\}_{N\in\mathbb{N}}$ be a cofinal sequence such that $\{\pi_{\Lambda_N}(\cdot|y_N)\}_{N\in\mathbb{N}}$ converges in $\mathcal{W}_p$ to some μ. By (2.26) one has for every $\Lambda_N \supseteq \Lambda$

$$\int_\Omega f(x)\pi_{\Lambda_N}(\mathrm{d}x|y_N) = \int_\Omega \pi_\Lambda f(x)\,\pi_{\Lambda_N}(\mathrm{d}x|y_N).$$

Now, employing Proposition 2.3 we can pass here to the limit $N \to \infty$ and thus get (2.42). ■

Remark 2.8 Suppose for simplicity that the interaction has *finite range* $r \in (0, \infty)$. Another class of local distributions, whose $\mathcal{W}_p$-accumulation points surely belong to $\mathcal{G}^{\mathrm{t}}$, are those with *empty boundary conditions* $y = \varnothing$ (see the proof of Lemma 3.53). They are defined by means of the local Hamiltonians (2.24) as $\pi_\Lambda(\mathrm{d}x|\varnothing) := Z_\Lambda^{-1} \exp\{-H_\Lambda(x_\Lambda)\}\, \mathrm{d}x_\Lambda \times \delta_{0_{\Lambda^c}}(\mathrm{d}x_{\Lambda^c})$ and are consistent with the Gibbs specification (2.21) in the sense that

$$\int \pi_\Lambda(B|y)\pi_{\Lambda'}(dy|\varnothing) = \pi_{\Lambda'}(B|\varnothing) \quad \text{for all } B \in B(\Omega),\ \ \Lambda' \supseteq \Lambda \cup \partial_r \Lambda. \tag{2.43}$$

If the interaction is *translation invariant*, one also considers the so-called local Gibbs distributions $\pi_\Lambda^{\mathrm{per}}(\mathrm{d}x)$ with *periodic boundary conditions* (see Subsection 3.3.7 below). They satisfy the consistency property similar to (2.43) and thus any their $\mathcal{W}_p$-accumulation point is the *translation invariant* measure from $\mathcal{G}^{\mathrm{t}}$ (cf. Lemma 3.70).

2.2 Existence problem

The initial step in any study of Gibbs measures is to verify whether $\mathcal{G}^{\mathrm{t}} \neq \varnothing$, which is however *not evident* for systems with noncompact spins. In this section we develop an *elementary new approach* to the existence problem which relies on certain exponential estimates for the one-point kernels $\pi_\ell(\mathrm{d}x|y)$ of the local Gibbs specification (see Lemma 2.9). We note that these estimates are *stronger* than those required in Dobrushin's existence criterion (see Subsection 2.2.1). As a result they imply not only the *existence* of at least one $\mu \in \mathcal{G}^{\mathrm{t}}$ (Theorem 2.14), but also yield the *uniform bounds* on all points of the set $\mathcal{G}^{\mathrm{t}}$ (Theorem 2.15) and its *compactness* in proper topologies (Corollary 2.16). Other important consequence is detailed information about the *support properties* of each $\mu \in \mathcal{G}^{\mathrm{t}}$ to be obtained in Subsection 2.2.4. A conceptual difference from all previous schemes used to verify Dobrushin's criterion (cf. [42, 71, 237, 259, 291, 295]) is that a choice of the compact function $h(x_\ell)$ participating in such estimates now *depends explicitly* on the growth of the Hamiltonian $H_\ell(x_\ell|y)$. The method obviously extends to general N-particle interactions or spin systems on graphs (see Subsection 2.2.5) and essentially improves all related existence results (see the discussion in Subsection 2.2.1). We stress that its simplicity is astonishing if one compares the short proof of our main technical result, Lemma 2.9, with the whole chapters in [42, 237, 259] devoted to verifying Dobrushin's compactness condition in some particular (e.g. $P(\varphi)$-) models.

2.2.1 Dobrushin's existence criterion

The existence problem goes back to the pioneering papers of R. Dobrushin from 1968–70, where the *general existence criterion* for Gibbs random fields was first given (Theorem 1 in [91]; see also Theorem 1.3 in [259] and Theorem in [122]). Given a specification $\mathit{\Pi} = \{\pi_\Lambda\}_{\Lambda \Subset \mathbb{L}}$, the main assumption of this criterion requires that the one-point probability kernels $\pi_\ell(\mathrm{d}x|y)$ satisfy the following:

Compactness Condition $(\mathbf{D}_1)$ *There exist a compact function* $h : \mathbb{R}^\nu \to \mathbb{R}_+ \cup \{+\infty\}$ *and nonnegative constants* $\mathcal{C}$ *and* $I_{\ell\ell'}$, $\ell \neq \ell'$, *such that* *(i) The matrix* $\mathbf{I} = (I_{\ell\ell'})_{\mathbb{L}\times\mathbb{L}}$ *is strictly contractive in* $l^\infty(\mathbb{L})$ *(or its transposition* $\mathbf{I}^{\mathrm{t}}$ *respectively in* $l^1(\mathbb{L})$*), that is*

$$||\mathbf{I}||_0 := ||\mathbf{I}||_{\mathcal{L}(l^\infty(\mathbb{L}))} = ||\mathbf{I}^{\mathrm{t}}||_{\mathcal{L}(l^1(\mathbb{L}))} = \sup_\ell \sum_{\ell'(\neq\ell)} I_{\ell\ell'} < 1. \tag{2.44}$$

(ii) For all $\ell \in \mathbb{L}$ *and* $y \in \Omega$

$$\int_\Omega h(x_\ell)\pi_\ell(\mathrm{d}x|y) \leq \mathcal{C} + \sum_{\ell'(\neq\ell)} I_{\ell\ell'} h(y_{\ell'}). \tag{2.45}$$

Recall that the function h is called *compact* if all its level sets $\{x_\ell | h(x_\ell) \leq c < \infty\}$ are relatively compact in $\mathbb{R}^\nu$. The Dobrushin criterion yields that for any boundary condition $y \in \Omega$ such that $\sup_\ell h(y_\ell) < \infty$, the family $\{\pi_\Lambda(\cdot|y)\}_{\Lambda \Subset \mathbb{L}}$ is $\mathcal{W}$-relatively compact (but not yet in the topology $\mathcal{W}_p$ as was needed in Proposition 2.7). To ensure that each its limit point $\mu \in \mathcal{P}(\Omega)$ is Gibbs, Condition ($\mathbf{D}_1$) must be supplemented by additional continuity and quasilocality assumptions on the specification $\{\pi_\Lambda\}_{\Lambda\Subset\mathbb{L}}$ (which, of course, is superfluous when the interaction has finite radius and thus $I_{\ell\ell'} = 0$ if $|\ell - \ell'| > r$). Moreover, any measure $\mu \in \mathcal{G}$ constructed in such a way obeys the à-priori bound $\sup_\ell \mathbf{E}_\mu[h(x_\ell)] < \infty$. If $h(x_\ell) \geq |x_\ell|^R$, this implies that $\mu \in \mathcal{P}(\Omega_p)$ for all $p > d$ and hence $\mu \in \mathcal{G}^{\mathrm{t}} (\neq \varnothing)$.

So, one faces here the problem (which was commonly believed to be non-trivial one) *how to check the compactness properties* like (2.45) for systems with unbounded spins. By now, there were only few papers [42, 71, 289, 290, 291, 295] and monographs [237, 259] mostly dealing with the *translation invariant* lattice systems of *scalar* spins $x_\ell \in \mathbb{R}$ interacting via *attractive* pair potentials just as $J_{\ell-\ell'}(x_\ell - x_{\ell'})^2 \geq 0$. The *strongest result* of such type, which was obtained by J. Bellissard and R. Høegh-Krohn (cf. Proposition III.1 and Theorem III.2 in [42]), establishes the estimate

$$\int_\Omega \exp\{\kappa|x_\ell|\}\pi_\ell(\mathrm{d}x|y) \leq \exp\left\{\mathcal{K} + 2\mathcal{L}\kappa + \mathcal{M}\kappa^2 + \kappa \sum_{\ell'(\neq\ell)} I_{\ell\ell'}|y_\ell|\right\}. \tag{2.46}$$

For fixed $\beta = 1$, it is valid with some universal constants $\mathcal{K}, \mathcal{L}, \mathcal{M} > 0$, which depend neither on $\ell \in \mathbb{L}$, $y \in \mathbb{R}^{\mathbb{L}}$ nor on $\kappa > 0$. By Jensen's inequality (saying that $\exp(\mathbf{E}_\mu f) \leq \mathbf{E}_\mu(\exp f)$ for any $\mu \in \mathcal{P}(\Omega)$, $f \in L^1(\mu)$) this readily implies the Dobrushin bound (2.7) with the compact function $h(x_\ell) = |x_\ell|$. However, the tools used in the quoted papers were designed for the concrete models, so that their extension to multi (or infinite) dimensional spin spaces and general interactions seems to be impossible. Note that the exponential in the right-hand-side in (2.46) is *linear* with respect to $|y_\ell|$, whereas by Assumption ($\mathbf{W}$) we have *nonlinear* (i.e., polynomial of order $R \geq 2$) dependence on $y \in \Omega$ in $W_{\ell\ell'}(x_\ell, y_{\ell'})$. For this reason alone, in [42, 237, 259] the advanced *asymptotic methods* were needed to prove the desired estimates (2.45) and (2.46) directly with such choice of $h(x_\ell)$.

As was already mentioned, one of our goals is to present *an elementary new approach* to getting Dobrushin's compactness estimates, that seems to have been overlooked before. Indeed, being much motivated by (2.46), instead of the Dobrushin bound (2.45) we shall prove the *stronger exponential bound*

$$\int_{\Omega} \exp\{h(x_\ell)\}\, \pi_\ell(\mathrm{d}x|y) \leq \exp\left\{ \mathcal{C} + \sum_{\ell'(\neq \ell)} I_{\ell\ell'} h(y_{\ell'}) \right\}, \tag{2.47}$$

which nevertheless can be derived much easily in view of the additive structure of the Hamiltonian (2.38). The second important moment is that (unlike all previous papers) we give an *explicit* construction of the proper compact function $h(x_\ell)$, which *nonlinearly* depends on $|x_\ell|$ and precisely takes into account a possible growth of the pair potentials. One more peculiarity of our scheme is that it is based on the compactness argument in the (*stronger* than $\mathcal{W}$) topologies $\mathcal{W}_p$ which allows to handle the interactions of infinite range. Without principal changes the method extends to much more general interactions (e.g. of unbounded order and infinite range) given by many-particle potentials of superquadratic growth (cf. Subsections 2.2.5 and 2.2.5 below). Moreover, after obvious modifications these arguments apply to the quantum lattice systems with infinite dimensional spin (e.g. path or loop) spaces, which will be demonstrated in Subsections 3.2.2 and 3.2.4. Recent developments show that the method also applies to the interacting particle systems in continuum (which is beyond our scope here and will be done in a separate paper), so that to certain extent it can be viewed as an *alternative* to the superstability estimates due to D. Ruelle [254].

2.2.2 Moment estimates for the local Gibbs specification

In this subsection we establish the integrability properties of the kernels $\pi_\Lambda(\mathrm{d}\omega|y)$ needed later for the existence result of Theorem 2.14.

Conventionally this will be done in a slightly *more general framework* than was described in Subsection 2.1.1. Keeping the former Assumptions (**W**) and (**J**) on the pair interaction, we substitute Assumption (**V**) by the weaker one, which allows us to consider the potentials V_ℓ and $W_{\ell\ell'}$ having the *same* order (i.e., $P = R$) of polynomial growth:

Assumption ($\mathbf{V}_1$) *There exist a continuous function $V : \mathbb{R}^\nu \to \mathbb{R}$ and constants $A_1 > 0$, $B_1 \in \mathbb{R}$, such that for all $\ell \in \mathbb{L}$ and $x_\ell \in \mathbb{R}^\nu$*

$$A_1 |x_\ell|^R + B_1 \leq V_\ell(x_\ell) \leq V(x_\ell). \tag{2.48}$$

Moreover, the constant A_1 can be chosen large enough, so that the following relation holds:

$$\frac{2}{3} A_1 > ||\mathbf{J}||_0 := \sup_{\ell} \sum_{\ell'(\neq \ell)} J_{\ell\ell'}. \tag{2.49}$$

The initial Assumption (**V**) imposed in Subsection 2.1.1 implies the validity of ($\mathbf{V}_1$) with arbitrary large A_1, so that the condition (2.49) always holds if $P > R$. The *stability* properties of the system will be described by the positive parameter

$$\varDelta_1 := A_1 - \frac{1}{2}||\mathbf{J}||_0, \tag{2.50}$$

which by (2.49) fulfills $\varDelta_1 > ||\mathbf{J}||_0$. Actually, $\varDelta_1 > 0$ guarantees by itself that the specification $\{\pi_\Lambda\}_{\Lambda \Subset \mathbb{L}}$ is well-defined and possesses the exponential integrability (2.27) with any $\kappa < \beta \varDelta_1$. By (2.49) we assume a stronger hypothesis, which will suffice for the existence of $\mu \in \mathcal{G}^{\mathrm{t}}$ (cf. Corollary 2.11).

The *key technical result* is the following *exponential bound* for the one-point kernels $\pi_\ell(\mathrm{d}x|y)$ subject to the fixed boundary condition $y \in \Omega^{\mathrm{t}}$.

Lemma 2.9 *Suppose that Assumptions* (**W**), (**J**), *and* ($\mathbf{V}_1$) *hold. Then, for every positive* $\kappa < \varDelta_1$, *there exists a corresponding* $\varUpsilon := \varUpsilon(\beta, \kappa) > 0$ *such that for all* $\ell \in \mathbb{L}$ *and* $y \in \Omega^{\mathrm{t}}$

$$\int_\Omega \exp\left\{\beta\kappa|x_\ell|^R\right\} \pi_\ell(\mathrm{d}x|y) \leq \exp\left\{\beta\left(\varUpsilon + \sum\nolimits_{\ell'(\neq \ell)} J_{\ell\ell'}|y_{\ell'}|^R\right)\right\}. \tag{2.51}$$

Proof. By Assumptions (**W**), (**J**) one has that for all $x, y \in \Omega^{\mathrm{t}}$

$$\sum_{\ell' \neq \ell} |W_{\ell\ell'}(x_\ell, y_{\ell'})| \leq \frac{||\mathbf{J}||_0}{2}|x_\ell|^R + \frac{1}{2}\sum_{\ell'(\neq \ell)} J_{\ell\ell'}\left(C_W + |y_{\ell'}|^R\right). \tag{2.52}$$

By this estimate and the definition (2.37) of $\mu_{\ell,y}(\mathrm{d}x_\ell) := \pi_\ell(\mathrm{d}x|y) \circ \mathbb{P}_\ell^{-1}$

$$\begin{aligned} &\int_\Omega \exp\left\{\beta\kappa|x_\ell|^R\right\} \pi_\ell(\mathrm{d}x|y) \\ &\leq (X_\ell/Y_\ell) \cdot \exp\left\{\beta\left(C_W||\mathbf{J}||_0 + \sum\nolimits_{\ell'(\neq \ell)} J_{\ell\ell'}|y_{\ell'}|^R\right)\right\}, \end{aligned} \tag{2.53}$$

where

$$X_\ell := \int_{\mathbb{R}^\nu} \exp\left\{-\beta\left[V_\ell(x_\ell) - \left(\kappa + \frac{||\mathbf{J}||_0}{2}\right)|x_\ell|^R\right]\right\} \mathrm{d}x_\ell \tag{2.54}$$

$$Y_\ell := \int_{\mathbb{R}^\nu} \exp\left\{-\beta\left[V_\ell(x_\ell) + \frac{||\mathbf{J}||_0}{2}|x_\ell|^R\right]\right\} \mathrm{d}x_\ell. \tag{2.55}$$

Using the upper and lower bounds in (2.48), one observes that

$$X := \sup_\ell X_\ell \leq \exp\{-\beta B_1\} \int_{\mathbb{R}^\nu} \exp\left\{-\beta\left(\varDelta_1 - \kappa\right)|x_\ell|^R\right\} \mathrm{d}x_\ell < \infty, \tag{2.56}$$

$$Y := \inf_\ell Y_\ell \geq \int_{\mathbb{R}^\nu} \exp\left\{-\beta\left(\frac{||\mathbf{J}||_0}{2}|x_\ell|^R + V(x_\ell)\right)\right\} \mathrm{d}x_\ell > 0. \tag{2.57}$$

This yields the required estimate (2.51) with the constant

$$\varUpsilon := \varUpsilon(\beta, \kappa) := \beta^{-1}\log\left(X/Y\right) + C_W||\mathbf{J}||_0 < \infty. \tag{2.58}$$

■

Remark 2.10 (i) The integral in (2.56) can be calculated explicitly. So, passing for $f : \mathbb{R}_+ \to \mathbb{R}_+$ to the polar coordinates by the formula

$$\int_{\mathbb{R}^\nu} f(|x_\ell|)\mathrm{d}x_\ell = \frac{2\pi^{\nu/2}}{\Gamma(\nu/2)} \int_0^\infty r^{\nu-1} f(r)\mathrm{d}r$$

and using the properties of the Γ-function (see e.g. Section VII in [70])

$$\Gamma(z) := \int_0^\infty r^{z-1} e^{-r} \mathrm{d}r, \quad z \in \mathbb{C}, \quad \Im(z) > 0, \tag{2.59}$$

one finds that for any $\lambda > 0$

$$\int_{\mathbb{R}^\nu} \exp\left\{-\lambda |x_\ell|^R\right\} \mathrm{d}x_\ell = (2/R) \cdot \frac{\pi^{\nu/2}\Gamma(\nu/R)}{\lambda^{\nu/R}\Gamma(\nu/2)}. \tag{2.60}$$

(ii) It is important to know *asymptotics* of $\Upsilon(\beta, \kappa)$ as $\beta \to +0$ or $\beta \to \infty$ (cf. Subsection 2.4.3). In the Gaussian case, where $V_\ell(x_\ell) := A|x_\ell|^2$ and $W_{\ell\ell'}(x_\ell, x_{\ell'}) := J_{\ell\ell'}(x_\ell, x_{\ell'})_{\mathbb{R}^\nu}$, one has by (2.60) that for all $\kappa < A$

$$\int_\Omega \exp\left\{\beta\kappa |x_\ell|^2\right\} \pi_\ell(\mathrm{d}x|0) = (1 - \kappa/A)^{-\nu/2}.$$

Thus $\Upsilon(\beta, \kappa)$ is growing not slowly than $\mathcal{O}(\beta^{-1})$ as $\beta \to +0$.

A subsequent application of Jensen's inequality to the both sides in (2.51) gives us the following *Dobrushin-type estimates:*

Corollary 2.11 *(i) Let us pick some $\kappa \in (||\mathbf{J}||_0, \varDelta_1)$ in the statement of Lemma 2.9. Then, for all $y \in \Omega^{\mathrm{t}}$, the kernels $\pi_\ell(\mathrm{d}x|y)$ obey Dobrushin's bound $(\mathbf{D}_1)$ with the compact function*

$$\mathbb{R}^\nu \ni x_\ell \mapsto h(x_\ell) := |x_\ell|^R, \tag{2.61}$$

constant $\mathcal{C} := \Upsilon/\kappa$, and contractive matrix

$$\mathbf{I} = (I_{\ell\ell'})_{\mathbb{L}\times\mathbb{L}}, \quad I_{\ell\ell'} := J_{\ell\ell'}/\kappa, \quad ||\mathbf{I}||_0 < 1. \tag{2.62}$$

(ii) In addition, suppose that the following relation holds

$$\varDelta_1^{1/R} > ||\mathbf{J}^{1/R}||_0 := \sup_\ell \sum_{\ell(\neq\ell)'} J_{\ell\ell'}^{1/R}, \tag{2.63}$$

which allows us to choose some $\kappa \in (||\mathbf{J}^{1/R}||_0^R, \varDelta_1)$. Then $(\mathbf{D}_1)$ is also satisfied with the norm-function $h(x_\ell) := |x_\ell|$ and coefficients

$$\mathcal{C} := (\Upsilon/\kappa)^{1/R}, \quad I_{\ell\ell'} := (J_{\ell\ell'}/\kappa)^{1/R}.$$

The next step will be to get the similar to (2.51) moment estimates for all $\pi_\Lambda(\mathrm{d}x|y)$ *uniformly in volumes* $\Lambda \Subset \mathbb{L}$. We define

$$n_\ell(\Lambda|y) := \log\left\{\int_\Omega \exp\left\{\beta\kappa|x_\ell|^R\right\}\pi_\Lambda(\mathrm{d}x|y)\right\}, \tag{2.64}$$

which are nonnegative and finite for all $\kappa < \varDelta_1$.

Lemma 2.12 *Let everything be as in the statement of Lemma 2.9. Then, for any* $p > d$*, there exists a finite* $\Upsilon_p := \Upsilon_p(\beta,\kappa) > 0$ *such that for all* $\ell_0 \in \mathbb{L}$ *and* $y \in \Omega_p$

$$\limsup_{\Lambda\nearrow\mathbb{L}}\left[\sum_{\ell\in\Lambda} n_\ell(\Lambda|y)\cdot(1+|\ell-\ell_0|)^{-p}\right] \le \beta\,\Upsilon_p. \tag{2.65}$$

Herefrom, in particular, for all $\kappa < \varDelta_1$

$$\limsup_{\Lambda\nearrow\mathbb{L}}\int_\Omega \exp\left\{\beta\kappa|x_\ell|^R\right\}\pi_\Lambda(\mathrm{d}x|y) \le \exp\{\beta\,\Upsilon_p\}. \tag{2.66}$$

Proof. A simple trick consists in considering a family of norms on Ω_p,

$$\|x\|_{p,\varepsilon} := \left[\sum_\ell (1+\varepsilon|\ell|)^{-p}|x_\ell|^R\right]^{1/R}, \quad \varepsilon > 0, \tag{2.67}$$

which are equivalent to the initial one $\|x\|_p$. Respectively, we set

$$||\mathbf{J}||_{p,\varepsilon} := \sup_\ell \sum_{\ell'(\neq\ell)} J_{\ell\ell'}(1+\varepsilon|\ell-\ell'|)^p, \quad p > d,\ \varepsilon > 0. \tag{2.68}$$

We claim that, for any given $p > d$ and $\iota > 0$, one finds a small enough $\varepsilon := \varepsilon(p,\iota) > 0$ such that

$$||\mathbf{J}||_{p,\varepsilon} - ||\mathbf{J}||_0 < \iota. \tag{2.69}$$

To this end we take advantage of the fact that the matrix $\mathbf{J}$ is quickly decreasing, that is $||\mathbf{J}||_{p'} < \infty$ for all $p' > d$. Then (2.69) is confirmed by the following computations

$$\begin{aligned} ||\mathbf{J}||_{p,\varepsilon} &\le \sup_\ell \sum_{\ell':\,|\ell'-\ell|\le N} J_{\ell\ell'}(1+\varepsilon|\ell-\ell'|)^p + \sup_\ell \sum_{\ell':\,|\ell'-\ell|>N} J_{\ell\ell'}(1+\varepsilon|\ell-\ell'|)^p \\ &\le (1+\varepsilon N)^p||\mathbf{J}||_0 + ||\mathbf{J}||_{2p}\sup_\ell \sum_{\ell':\,|\ell'-\ell|>N}(1+|\ell-\ell'|)^{-p} \to ||\mathbf{J}||_0, \end{aligned} \tag{2.70}$$

as $N \to \infty$ and then $\varepsilon := \varepsilon(N) \to 0$. Here we have crucially used the condition (2.3) in Assumption ($\mathbf{L}_d$) (or alternatively by Remark 2.1 (iii), the condition (2.12) in Assumption ($\mathbf{J}$)). Thus, by (2.49), (2.50), and (2.68) we may fix some $\varepsilon \in (0,1)$ such that

$$||\mathbf{J}||_0 \le ||\mathbf{J}||_{p,\varepsilon} < \varDelta_1. \tag{2.71}$$

Without loss of generality let us consider only $\kappa \in (||\mathbf{J}||_{p,\varepsilon}, \varDelta_1)$, so that $\kappa^{-1}\sum_{\ell'(\neq \ell)} J_{\ell\ell'} < 1$. Integrating both sides of (2.51) with respect to $\pi_\Lambda(\mathrm{d}x|y)$ with arbitrary $y \in \Omega_p$ and taking into account the consistency property (2.34), we arrive at the following estimate

$$\begin{aligned} n_\ell(\Lambda|y) &\leq \beta\left(\varUpsilon + \sum_{\ell' \in \Lambda^c} J_{\ell\ell'}|y_{\ell'}|^R\right) \\ &\quad + \log\left\{\int_\Omega \exp\left(\sum_{\ell'\in\Lambda} \kappa^{-1} J_{\ell\ell'} \cdot \beta\kappa |x_{\ell'}|^R\right)\pi_\Lambda(\mathrm{d}x|y)\right\} \\ &\leq \beta\left(\varUpsilon + \sum_{\ell' \in \Lambda^c} J_{\ell\ell'}|y_{\ell'}|^R\right) + \kappa^{-1}\sum_{\ell'\in\Lambda} J_{\ell\ell'} n_{\ell'}(\Lambda|y), \end{aligned} \tag{2.72}$$

where the constant $\varUpsilon := \varUpsilon(\beta,\kappa)$ is given by (2.58). Here we have used the multiple Hölder inequality

$$\mathbf{E}_\mu\left(\prod_{j=1}^n f_j^{\alpha_j}\right) \leq \prod_{j=1}^n \left(\mathbf{E}_\mu f_j\right)^{\alpha_j}, \tag{2.73}$$

valid for any probability measure μ, functions $f_j \geq 0$, and numbers $\alpha_j \geq 0$ such that $\sum_{j=1}^n \alpha_j \leq 1$. After summing in (2.72) over $\ell \in \Lambda$ with the weights $(1+\varepsilon|\ell_0 - \ell|)^{-p}$, one gets

$$\begin{aligned} n_{\ell_0}(\Lambda|y) &\leq \sum_{\ell\in\Lambda} n_\ell(\Lambda|y)\cdot(1+\varepsilon|\ell_0-\ell|)^{-p} \\ &\leq \frac{\beta}{1-\kappa^{-1}||\mathbf{J}||_{p,\varepsilon}}\left[\varUpsilon\sum_{\ell\in\Lambda}(1+\varepsilon|\ell_0-\ell|)^{-p} + ||\mathbf{J}||_{p,\varepsilon}\|y_{\Lambda^c}\|_{p,\varepsilon}^R\right]. \end{aligned} \tag{2.74}$$

For $y \in \Omega_p$ the second term in the brackets in the right-hand side in (2.74) tends to zero as $\Lambda \nearrow \mathbb{L}$, whereas by Assumption $(\mathbf{L}_d)$ the first one is uniformly bounded by

$$\mathbf{\Xi}_{p,\varepsilon} := \sup_{\ell_0}\sum_{\ell}(1+\varepsilon|\ell_0-\ell|)^{-p} \leq \varepsilon^{-p}\mathbf{\Xi}_p < \infty. \tag{2.75}$$

We thus have

$$\begin{aligned} \limsup_{\Lambda\nearrow\mathbb{L}} n_{\ell_0}(\Lambda|y) &\leq \limsup_{\Lambda\nearrow\mathbb{L}}\left[\sum_{\ell\in\Lambda} n_\ell(\Lambda|y)\cdot(1+\varepsilon|\ell-\ell_0|)^{-p}\right] \\ &\leq \beta\varUpsilon\frac{\mathbf{\Xi}_{p,\varepsilon}}{1-\kappa^{-1}||\mathbf{J}||_{p,\varepsilon}} =: \beta\varUpsilon_p, \end{aligned} \tag{2.76}$$

which completes the proof of (2.65) and (2.66). ■

Remark 2.13 The above proof actually ignores the sign of $W_{\ell\ell'}$. In fact, the *lower-boundedness* of the interaction could improve the result. Additionally to (**W**) and (**J**), let us suppose that there exists $c_W \in \mathbb{R}$ such that for all ℓ, ℓ'

$$\tilde{W}_{\ell\ell'} := W_{\ell\ell'} - c_W J_{\ell\ell'}/2 \geq 0.$$

Then the models with the potentials $W_{\ell\ell'}$ and $\tilde{W}_{\ell\ell'}$ are equivalent as such related to the same specification (2.21). This allows us to replace the condition (2.49) in Assumption ($\mathbf{V}_1$) by the *weaker* one $\Delta_1 := A_1 - \frac{1}{2}||\mathbf{J}||_0 > 0$. Indeed, tracing the proof of Lemma 2.9, we observe that for all $\kappa \in (0, A_1)$

$$\int_{\Omega} \exp\left\{\beta\kappa|x_\ell|^R\right\} \pi_\ell(\mathrm{d}x|y) \le \left\{\beta\left(\tilde{\Upsilon} + \frac{1}{2}\sum\nolimits_{\ell'(\neq\ell)} J_{\ell\ell'}|y_{\ell'}|^R\right)\right\}. \tag{2.77}$$

The constant in the right-hand side can be chosen as

$$\tilde{\Upsilon} := \beta^{-1}\log\left(\tilde{X}/Y\right) + \frac{1}{2}||\mathbf{J}||_0(C_W - c_W),$$

where Y is the same as in (2.57) and

$$\tilde{X} := \exp\left\{-\beta B_1\right\} \int_{\mathbb{R}^\nu} \exp\left\{-\beta(A_1 - \kappa)|x_\ell|^R\right\} \mathrm{d}x_\ell.$$

Herefrom, picking any $\kappa \in (||\mathbf{J}||_0/2, A_1)$, we get Compactness Condition ($\mathbf{D}_1$) with $h(x_\ell) := |x_\ell|^R$ and $||\mathbf{I}||_0 < 1$.

2.2.3 Existence and à-priori estimates for $\mu \in \mathcal{G}^{\mathrm{t}}$

Here we prove our main Theorems 2.14 and 2.15 describing the set $\mathcal{G}^{\mathrm{t}}$. Once the required bound (2.45) for the one-point kernels $\pi_\ell(\mathrm{d}x|y)$ has been established (cf. Corollary 2.11), one could apply *Dobrushin's criterion* which yields the relatively compactness of the family $\{\pi_\Lambda(\mathrm{d}x|0)\}_{\Lambda\Subset\mathbb{L}}$ in the weak topology $\mathcal{W}$ on $\mathcal{P}(\Omega)$. After an additional technical work, one may further conclude that any limit point of this family indeed belongs to $\mathcal{G}^{\mathrm{t}}$. As already mentioned above, such standard scheme of proving existence of Gibbs measures was realized for some special models in [71, 237, 259]. We however prefer to follow another way which is strongly motivated by the paper of J. Bellissard and R. Høegh-Krohn [42]. The *main idea* is to show that the *uniform bounds* (2.65) for $\pi_\Lambda(\mathrm{d}x|y)$, combined with the *compactness argument* in the topologies $\mathcal{W}_p$ (which are stronger than $\mathcal{W}$), readily imply the existence of $\mu \in \mathcal{G}^{\mathrm{t}}$. On this way, we also get à-priori moment bounds like (2.65) to be valid for all measures $\mu \in \mathcal{G}^{\mathrm{t}}$.

If instead of ($\mathbf{V}_1$) we use the initial Assumption ($\mathbf{V}$) with $P > R$, then the previous Lemma 2.12 and the subsequent Theorems 2.1 and 2.2 will certainly hold *for all* values of $\beta, \kappa > 0$. The reason is that the key relation (2.71) is always fulfilled by choosing large enough values of the parameter A_1 in (2.49). Furthermore, this means that now we can *drop* the additional condition (2.3) in Assumption ($\mathbf{L}_d$) or respectively (2.12) in Assumption ($\mathbf{J}$).

Theorem 2.14 *Let Assumptions* ($\mathbf{V}_1$), ($\mathbf{J}$), *and* ($\mathbf{W}$) *be satisfied. Then, the set of tempered Gibbs measures is not empty, i.e.,* $\mathcal{G}^{\mathrm{t}} \neq \varnothing$. *In particular, it contains all* $\mathcal{W}_{p'}$*-limit points of the family* $\{\pi_\Lambda(\mathrm{d}\omega|y)\}_{\Lambda\nearrow\mathbb{L}}$ *with any* $y \in \Omega_p$ *and* $p' > p$.

Proof. For fixed $\kappa < \varDelta_1$ and $y \in \varOmega_p$, by (2.65) and Jensen's inequality we have that

$$\limsup_{\Lambda \nearrow \mathbb{L}} \int_{\varOmega} ||x_\Lambda||_p^R \pi_\Lambda(\mathrm{d}x|y) \leq \varUpsilon_p/\kappa. \tag{2.78}$$

Hence, one finds a finite $C_p(y) > 0$ such that

$$\sup_{\Lambda \Subset \mathbb{L}} \int_{\varOmega} ||x||_p^R \pi_\Lambda(\mathrm{d}x|y) \leq \mathcal{C}_p(y). \tag{2.79}$$

The embeddings $\varOmega_p \hookrightarrow \varOmega_{p'}$ are compact whenever $p < p'$ (cf. Remark 2.1 (ii)), which by Prokhorov's criterion implies the $\mathcal{W}_{p'}$-relatively compactness of the family $\{\pi_\Lambda(\mathrm{d}\omega|y)\}_{\Lambda \in \mathcal{L}}$ as $\mathcal{L} \nearrow \mathbb{L}$. By Fatou's lemma, each of its limit points $\mu \in \mathcal{P}(\varOmega_{p'})$ satisfies

$$\int_{\varOmega} ||x||_p^R \mu(\mathrm{d}x) \leq \mathcal{C}_p(y), \tag{2.80}$$

and thus is supported by $\varOmega_p$. By Proposition 2.7 every such μ is surely Gibbs. ■

The next important sequel of the bound (2.66) is the *uniform integrability* estimate for all tempered Gibbs measures.

Theorem 2.15 *Under the assumptions of Theorem 2.14, for every $\lambda < \beta\varDelta_1$ there exists a positive $C_{2.81} := C_{2.81}(\beta, \lambda)$ such that for all $\mu \in \mathcal{G}^{\mathrm{t}}$*

$$\sup_{\ell} \int_{\varOmega} \exp\left\{\lambda |x_\ell|^R\right\} \mu(\mathrm{d}x) \leq C_{2.81}. \tag{2.81}$$

Proof. Let us first fix some $p > d$, and consider only those $\mu \in \mathcal{G}^{\mathrm{t}}$ which are supported by the corresponding $\varOmega_p$. Setting $\kappa := \lambda/\beta < \varDelta_1$, by means of (2.34), (2.66), and Fatou's lemma we have the following estimates with arbitrary $N > 0$

$$\begin{aligned} & \int_{\varOmega} \exp\left\{\min\left(\beta\kappa|x_\ell|^R;\ N\right)\right\} \mu(\mathrm{d}x) \\ = \ & \limsup_{\Lambda \nearrow \mathbb{L}} \int_{\varOmega_p} \int_{\varOmega} \exp\left\{\min\left(\beta\kappa|x_\ell|^R;\ N\right)\right\} \pi_\Lambda(\mathrm{d}x|y)\mu(\mathrm{d}y) \\ \leq \ & \int_{\varOmega_p} \left[\limsup_{\Lambda \nearrow \mathbb{L}} \int_{\varOmega} \exp\left\{\beta\kappa|x_\ell|^R\right\} \pi_\Lambda(\mathrm{d}x|y)\right] \mu(\mathrm{d}y) \leq \exp\{\beta\varUpsilon_p\}, \end{aligned} \tag{2.82}$$

where the constant $\varUpsilon_p$ was introduced in (2.76). Applying Fatou's lemma once more, we conclude from (2.82) that for all $\mu \in \mathcal{G}^{\mathrm{t}} \cap \mathcal{P}(\varOmega_p)$

$$\begin{aligned} & \int_{\varOmega} \exp\left\{\beta\kappa|x_\ell|^R\right\} \mu(\mathrm{d}x) \\ & \leq \limsup_{N \to \infty} \int_{\varOmega} \exp\left\{\min\left(\beta\kappa|x_\ell|^R; N\right)\right\} \mu(\mathrm{d}x) \leq \exp\{\beta\varUpsilon_p\}, \end{aligned} \tag{2.83}$$

and hence by Jensen's inequality

$$\sup_{\ell} \int_{\Omega} |x_\ell|^R \mu(\mathrm{d}x) < \Upsilon_p/\kappa. \tag{2.84}$$

The latter implies by Chebyshev's inequality that any $\mu \in \mathcal{G}^{\mathrm{t}}$ is actually supported by $\cap_{p>d}\Omega_p$. Thus, (2.83) yields us the desired estimate (2.81) with the constant $C_{2.81} := \exp\left(\beta \inf_{p>d} \Upsilon_p\right)$, which is the same for all $\mu \in \mathcal{G}^{\mathrm{t}}$. ■

Corollary 2.16 *$\mathcal{G}^{\mathrm{t}}$ is the $\mathcal{W}_p$-compact set in each $\mathcal{P}(\Omega_p)$, $p > d$.*

Proof. Similarly to the proof of Theorem 2.14, by Prokhorov's tightness criterion and the estimate (2.81) we get the relative $\mathcal{W}_p$-compactness of $\mathcal{G}^{\mathrm{t}}$. In view of the Feller property (Lemma 2.7), the set $\mathcal{G}^{\mathrm{t}}$ is closed and hence compact. ■

Finally, we stress that the estimate (2.81) for the measures $\mu \in \mathcal{G}^{\mathrm{t}}$ is *à-priori* in the sense that it in principle can be proven before establishing their existence. The bound in the right-hand side in (2.81) is *uniform* for all $\mu \in \mathcal{G}^{\mathrm{t}}$ and depends on the inverse temperature β and parameters of the model only. The à-priori bound (2.81) plays a crucial role in the theory of the set $\mathcal{G}^{\mathrm{t}}$. As will be seen in Subsections 2.3.5 and 4.5.4, it is also important in the study of the Dirichlet operators $\mathbb{H}_\mu$ and the stochastic dynamics $\exp(-t\mathbb{H}_\mu)_{t\geq 0}$ associated with the measures $\mu \in \mathcal{G}^{\mathrm{t}}$.

2.2.4 Support properties of $\mu \in \mathcal{G}^{\mathrm{t}}$

There are at least two immediate sequels from the à-priori bound (2.81), see Propositions 2.17 and 2.19 below. The first one says that all finite volume projections of $\mu \in \mathcal{G}^{\mathrm{t}}$ are of *sub-Gaussian growth*. This is a (slightly weaker) version of the so-called *regularity property* for Gibbs measures firstly discovered by D. Ruelle within his technique of superstability estimates (cf. Definition 3.2 and Theorem 4.4 in [184]).

Proposition 2.17 *Suppose that Assumptions $(\mathbf{V}_1)$, $(\mathbf{J})$, and $(\mathbf{W})$ hold, and let us take any $\Lambda \Subset \mathbb{L}$ with $|\Lambda| < \Delta_1/||\mathbf{J}||_0$. Then, for each $\mu \in \mathcal{G}^{\mathrm{t}}$, its finite volume projection $\mu_\Lambda := \mu \circ \mathbb{P}_\Lambda^{-1}$ is absolutely continuous with respect to the Lebesgue measure $\mathrm{d}x_\Lambda$ on $\mathbb{R}^{\nu|\Lambda|}$. The corresponding Radon–Nikodym derivative obeys the Ruelle-type bound*

$$\frac{\mathrm{d}\mu_\Lambda(x_\Lambda)}{\mathrm{d}x_\Lambda} =: \rho_{\mu,\Lambda}(x_\Lambda) \leq \exp\left\{-\beta \sum\nolimits_{\ell\in\Lambda} \left(\Delta_1 |x_\ell|^R - \mathcal{K}_\Lambda\right)\right\}, \tag{2.85}$$

with a constant $\mathcal{K}_\Lambda$ being the same for all such μ and depending only on the number of points in Λ.

Proof. From (2.34) and (2.36) it is easily to see that the Radon–Nikodym derivatives, if such exist, should have the form

$$\begin{aligned} \rho_{\mu,\Lambda}(x_\Lambda) = {} & \exp\left\{-\beta H_\Lambda(x_\Lambda)\right\} \\ & \times \int_{\Omega} [1/Z_\Lambda(y)] \exp\left\{-\beta \sum\nolimits_{\ell\in\Lambda,\ \ell'\in\Lambda^c} W_{\ell\ell'}(x_\ell, y_{\ell'})\right\} \mu(\mathrm{d}y), \end{aligned} \tag{2.86}$$

where $Z_\Lambda(y)$ and $H_\Lambda(x_\Lambda)$ are given respectively by (2.38) and (2.24). So, the only thing one needs to check is the validity of the upper bound (2.85), which in turn implies $\rho_{\mu,\Lambda} \in L^1(\mu)$ and hence $\mu_\Lambda(\mathrm{d}x_\Lambda) \ll \mathrm{d}x_\Lambda$. By (2.31), (2.32) and the arguments similar to those used in the proof of Lemma 2.9, we find that

$$\rho_{\mu,\Lambda}(x_\Lambda) \leq (1/Y)^{|\Lambda|} \int_\Omega \exp\left\{\beta \sum\nolimits_{\ell\in\Lambda,\ \ell'\in\Lambda^c} J_{\ell\ell'}|y_{\ell'}|^R\right\} \mu(\mathrm{d}y) \qquad (2.87)$$
$$\times \exp\left\{\beta\left[-\sum\nolimits_{\ell\in\Lambda} V_\ell(x_\ell) + \frac{1}{2}||\mathbf{J}||_0 \sum\nolimits_{\ell\in\Lambda} |x_\ell|^R + C_W||\mathbf{J}||_0|\Lambda|\right]\right\},$$

with the constant $Y > 0$ defined by (2.57). The integral in the first line in (2.87) can be estimated by the Hölder inequality (2.73) and Theorem 2.15. Its value does not exceed some $C_{2.81}$, which corresponds to an arbitrary choice of $\lambda \in (\beta||\mathbf{J}||_0|\Lambda|,\ \beta\varDelta_1)$ in (2.81). Together with the growth conditions (2.48)–(2.50) for V_ℓ, this yields us the required bound on $\rho_{\mu,\Lambda}$ with the constant

$$\mathcal{K}_\Lambda := \beta^{-1}\left[(1/|\Lambda|)\log C_{2.81} - \log Y\right] + C_W||\mathbf{J}||_0 - B_1. \qquad (2.88)$$

For $\Lambda = \{\ell\}$ the result holds with $\mathcal{K}_\ell := \beta^{-1}\log(C_{2.81}/Y) - B_1$. ■

Remark 2.18 If $\mathcal{K} := \sup_{\Lambda\Subset\mathbb{L}} \mathcal{K}_\Lambda < \infty$, this would mean the *Ruelle bound*

$$\rho_{\mu,\Lambda}(x_\Lambda) \leq \exp\left\{-\beta \sum\nolimits_{\ell\in\Lambda} \left(\varDelta_1|x_\ell|^R - \mathcal{K}\right)\right\}, \qquad (2.89)$$

cf. Definition 3.2 in [184]. Assuming that (**V**) holds with some $P > R$, one easily can deduce from (2.87), (2.88) that $\mathcal{K}_\Lambda = \mathcal{O}(|\Lambda|^{R/(P-R)})$. If the interaction has finite range, a further analysis shows that $\mathcal{K}_\mathcal{L} := \sup_{\Lambda\in\mathcal{L}} \mathcal{K}_\Lambda < \infty$ for any cofinal sequence $\mathcal{L}$ such that $\sup_{\Lambda\in\mathcal{L}} \left\{|\partial_r(\Lambda)|^{P/(P-R)}/|\Lambda|\right\} < \infty$. Surely, this is the case if $P > Rd$ and all $\Lambda \in \mathcal{L}$ are cubes in $\mathbb{L} := \mathbb{Z}^d$.

Let us recall that the set of tempered Gibbs measures was introduced by means of the rather moderate restriction (2.35), which roughly coincides with what is needed to define the local specification $\{\pi_\Lambda\}_{\Lambda\Subset\mathbb{L}}$. We now show that all $\mu \in \mathcal{G}^{\mathrm{t}}$ indeed are carried by a much smaller universal subset (2.95); in the case of $R = 2$ the latter is known as the *Lanford–Lebowitz–Presutti support* (cf. Definition 3.2 in [184]). To this end, let us define for $b > 0$

$$\varXi(b,R) = \left\{x \in \Omega \mid (\exists \Lambda_x \Subset \mathbb{L})\ (\forall \ell \in [\Lambda_x]^c) : |x_\ell|^R \leq b\log(1+|\ell|)\right\}, \qquad (2.90)$$

which are Borel subsets of Ω^{t}.

Proposition 2.19 *Given $\lambda > 0$, let us consider all $\mu \in \mathcal{P}(\Omega)$ which fulfill*

$$\sup_\ell \mathbf{E}_\mu\left(\exp\lambda|x_\ell|^R\right) =: \mathcal{C}(\mu,\lambda) < \infty. \qquad (2.91)$$

Then, simultaneously for all such measures (and hence by Theorem 2.15, for all $\mu \in \mathcal{G}^{\mathrm{t}}$), one has

$$\mu(\varXi(b,R)) = 1, \quad \text{with any } b > d/\lambda,$$

where d is the dimension of $\mathbb{R}^d \supset \mathbb{L}$ in Assumption $(\mathbf{L}_d)$.

Proof. We proceed similarly to the proof of Lemma 3.1 in [184]. The complement to the set (2.90) can be written as

$$[\Xi(b,R)]^c = \bigcap_{\Lambda \Subset \mathbb{L}} \bigcup_{\ell \in \Lambda^c} [\Xi_\ell(b,R)]^c, \tag{2.92}$$

where

$$\Xi_\ell(b,R) := \left\{ x \in \Omega \;\middle|\; |x_\ell|^R \le b \log(1+|\ell|) \right\}.$$

By Chebyshev's inequality and the estimate (2.91),

$$\mu\left([\Xi_\ell(b,R)]^c\right) \le \mathcal{C}(\mu,\lambda) \cdot (1+|\ell|)^{-b\lambda}. \tag{2.93}$$

Therefore, by (2.92) and (2.93), for any cofinal sequence $\mathcal{L} \nearrow \mathbb{L}$

$$\mu\left([\Xi(b,R)]^c\right) \le \mathcal{C}(\mu,\lambda) \lim_{\Lambda \in \mathcal{L}} \sum_{\ell \in \Lambda^c} (1+|\ell|)^{-b\lambda}. \tag{2.94}$$

By Assumption $(\mathbf{L}_d)$the series in (2.94) convergent as $b > d/\lambda$, which yields the result $\mu\left([\Xi(b,R)]^c\right) = 0$. ■

Corollary 2.20 *Suppose that the basic Assumptions* $(\mathbf{V})$*,* $(\mathbf{J})$*, and* $(\mathbf{W})$ *hold. Then all* $\mu \in \mathcal{G}^{\mathrm{t}}$ *are carried by the universal subset*

$$\begin{aligned} \Omega^{\mathrm{supp}} &:= \bigcap_{b>0,\ R \in [2,P)} \Xi(b,R) \\ &= \left\{ x \in \Omega \;\middle|\; \limsup_{|\ell| \to \infty} \left\{ |x_\ell|^R / \log(1+|\ell|) \right\} = 0, \quad 2 \le R < P \right\}. \end{aligned} \tag{2.95}$$

Furthermore, their projections μ_Λ *obey the uniform bound (2.85) with any* $R < P$*,* $\Delta_1 > 0$ *and a certain* $\mathcal{K}_\Lambda := \mathcal{K}_\Lambda(\beta, R, \Delta_1) > 0$.

Remark 2.21 According to its definition, $\mathcal{G}^{\mathrm{t}}$ contains a class $\mathcal{G}^{\mathrm{st}}$ of the so-called *Ruelle-type "superstable"* Gibbs measures μ. For *translation invariant* systems on $\mathbb{L} := \mathbb{Z}^d$, they were introduced by the support condition

$$\sup_{N \in \mathbb{N}} \left\{ (1+2N)^{-d} \sum_{|\ell| \le N} |x_\ell|^R \right\} \le C(x) < \infty, \quad \forall x \in \Omega \quad (\mu - \text{a.e.}), \tag{2.96}$$

see Definition 3.3 in [184] related to the *particular* case of $R = 2$. One of main statements within Ruelle's approach is that the regularity bound (2.89) and the support property (2.96) are equivalent for the Gibbs measures, see Theorems 4.4 and 4.5 in [184]. However, it is a still *open question* whether $\mathcal{G}^{\mathrm{t}} = \mathcal{G}^{\mathrm{st}}$. A particular answer was given in Remark 2.18 under the restriction that $P > Rd$. What is clear is that any *translation invariant* measure $\mu \in \mathcal{P}(\Omega)$ obeying the exponential bound (2.81) ought to fulfill

$$\sup_{N \in \mathbb{N}} \left\{ (1+2N)^{-d} \sum_{|\ell| \le N} \exp\left(\lambda |x_\ell|^R\right) \right\} \le C(\lambda, x) < \infty, \quad \forall x \in \Omega \ (\mu - \text{a.e.}), \tag{2.97}$$

which is much stronger than (2.96). The latter assertion follows from the multidimensional *individual ergodic theorem* (cf. Theorem 14.A8 in [122]) applied to the stationary process x_ℓ, $\ell \in \mathbb{Z}^d$, defined on the probability space $(\Omega, \mathcal{B}(\Omega), \mu)$. To this end, assuming that the interaction is translation invariant and has finite range, let us construct the so-called *periodic Gibbs states* $\mu^{\mathrm{per}} \in \mathcal{G}^{\mathrm{t}}$, which certainly will be *invariant* with respect to the group of translations of the lattice $\mathbb{Z}^d$ (cf. Subsection 3.3.7). In this situation we can substantially *refine the statement of Theorem* 2.14 by claiming the existence of $\mu \in \mathcal{G}^{\mathrm{st}}$ with the support property (2.97).

2.2.5 Possible generalizations

Here we briefly discuss certain important generalizations of the model (2.1) including multi-particle interactions and spin systems on graphs.

(i) N-particle interactions

The above method can be applied without principal changes to the interacting spin system described by a heuristic energy functional of the form

$$H(x) = \sum_{\ell} V_\ell(x_\ell) + \sum_{\{\ell_1,...,\ell_N\}} W_{\ell_1 \dots \ell_N}(x_{\ell_1}, ..., x_{\ell_N}). \tag{2.98}$$

The *N-particle interaction* potentials (taken here over all unordered finite sets $\{\ell_1, ..., \ell_N\}$ consisting of $N \geq 2$ distinct points) are given by *continuous symmetric* functions $W_{\ell_1 \dots \ell_N} : \mathbb{R}^{\nu N} \to \mathbb{R}$. Analogously to (2.21)–(2.23), one defines the local Hamiltonians $H_\Lambda(x|y)$ and the probability kernels $\pi_\Lambda(\mathrm{d}x|y)$ corresponding to the boundary conditions $y \in \Omega^{\mathrm{t}}$. Then, all previous statements for the Gibbs measures $\mu \in \mathcal{G}^{\mathrm{t}}$ (including their Definition 2.4 and Theorems 2.14, 2.15) hold true under the initial Assumption (**V**) (or its weaker version ($\mathbf{V}_1$)) combined with the following modifications of (**W**), (**J**):

Assumption ($\mathbf{W}_N$) *There exist constants* $R \geq 2$, $C_W \geq 0$ *and a symmetric matrix* $\mathbf{J} = (J_{\ell_1 \dots \ell_N} \geq 0)_{\mathbb{L}^N}$*, such that for all* $x_{\ell_1}, ..., x_{\ell_N} \in \mathbb{R}^\nu$

$$|W_{\ell_1 \dots \ell_N}(x_{\ell_1}, ..., x_{\ell_N})| \leq \frac{1}{2} J_{\ell_1 \dots \ell_N} \left(C_W + \sum\nolimits_{n=1}^{N} |x_{\ell_n}|^R \right). \tag{2.99}$$

Assumption ($\mathbf{J}_N$) *The matrix* $\mathbf{J}$ *is fastly decreasing, that is, for all* $p \geq 0$

$$||\mathbf{J}||_p := \sup_{\ell_1} \sum_{\{\ell_2,...,\ell_N\}} J_{\ell_1 \dots \ell_N} \left(1 + \max_{2 \leq n \leq N} |\ell_1 - \ell_n| \right)^p < \infty. \tag{2.100}$$

We suppose that $J_{\ell_1,...,\ell_N} = 0$ *if* $\ell_1 = \ell_n$ *for some* $n \leq N$.

To summarize, the related results for $N > 2$ differ only in the formulation of the exponential bound (2.51). The total strength of the interaction is now controlled by the "*effective*" matrix $\tilde{\mathbf{J}} = (\tilde{J}_{\ell_1 \ell_2})_{\mathbb{L} \times \mathbb{L}}$ with the entries $\tilde{J}_{\ell_1 \ell_2} := \sum_{\{\ell_3,...,\ell_N\}} J_{\ell_1 \dots \ell_N}$ and the finite norms $||\tilde{\mathbf{J}}||_p \leq ||\mathbf{J}||_p$.

(ii) General Hamiltonians

In a very broad setting, the interaction can be defined through a *family of potentials* $\{W_\Lambda\}$ indexed by all finite sets $\Lambda \Subset \mathbb{L}$, where each $W_\Lambda : \mathbb{R}^{\nu|\Lambda|} \to \mathbb{R}$ is a *continuos function* invariant under permutations of its coordinates. The local Hamiltonian in volume $\Lambda \Subset \mathbb{L}$ under the boundary condition $y \in \Omega^{\mathrm{t}}$ is then given by

$$H_\Lambda(x|y) := \sum_{\Delta \Subset \mathbb{L}:\ \Delta\cap\Lambda\neq\varnothing} W_\Delta(x_{\Delta\cap\Lambda}|y_{\Delta\cap\Lambda^c}), \quad x \in \Omega^{\mathrm{t}}. \tag{2.101}$$

Provided the one-particle potentials $V_\ell := W_{\{\ell\}}$ satisfy the former Assumption $(\mathbf{V}_1)$, we impose the following hypotheses on W_Λ with $|\Lambda| \geq 2$:

Assumption $(\mathbf{W}_\Lambda)$ *There exist constants* $R \geq 2$ and $C_W, J_\Lambda \geq 0$*, such that for each* $\Lambda \Subset \mathbb{L}$ *with* $|\Lambda| \geq 2$ *and for all* $x_\Lambda = (x_{\ell_1}, ..., x_{\ell_{|\Lambda|}}) \in \mathbb{R}^{\nu|\Lambda|}$

$$|W_\Lambda(x_{\ell_1}, ..., x_{\ell_{|\Lambda|}})| \leq \frac{1}{2} J_\Lambda \left(C_W + \sum\nolimits_{n=1}^{|\Lambda|} |x_{\ell_n}|^R \right). \tag{2.102}$$

Assumption $(\mathbf{J}_\Lambda)$ *The intensity* J_Λ *of the many-particle interaction is decreasing as the diameter of the sets* Λ *growths, that is for all* $p \geq 0$

$$||\mathbf{J}||_p := \sup_{\ell_1} \sum_{\Lambda\ni\ell_1:\ |\Lambda|\geq 2} J_\Lambda \left(1 + \max_{2\leq n\leq|\Lambda|} |\ell_1 - \ell_n|\right)^p < \infty. \tag{2.103}$$

Again, in the corresponding statements there occurs a new matrix $\tilde{\mathbf{J}} = (\tilde{J}_{\ell_1\ell_2})_{\mathbb{L}\times\mathbb{L}}$ with the entries $\tilde{J}_{\ell_1\ell_2} := \sum_{\Lambda\supseteq\{\ell_1,\ell_2\}:\ |\Lambda|\geq 2} J_\Lambda$ and with the norms $||\tilde{\mathbf{J}}||_p \leq ||\mathbf{J}||_p$. Going through the proof of Lemma 2.9, we observe that the only estimate on the interaction needed is as follows: for all $x, y \in \Omega^{\mathrm{t}}$

$$\sum_{\Lambda\ni\ell:\ |\Lambda|\geq 2} \left|W_\Lambda(x_\ell|y_{\Lambda\setminus\{\ell\}})\right| < \frac{1}{2}\left[||\mathbf{J}||_0 \left(A_1|x_\ell|^R + C_W\right) + \sum\nolimits_{\ell'(\neq\ell)} \tilde{J}_{\ell\ell'}|y_{\ell'}|^R\right], \tag{2.104}$$

with the parameter $\Delta_1 := A_1 - ||\mathbf{J}||_0/2 > ||\mathbf{J}||_0$. Then, by choosing in (2.51) any $\kappa \in (||\mathbf{J}||_0, \Delta_1)$, one immediately gets Dobrushin's bound $(\mathbf{D}_1)$ the same as in Corollary 2.11 (i).

(iii) Gibbs fields on graphs

The next (and very important in applications) step is to consider more general indexing sets $\mathbb{L}$ and thus pass from the regular lattice $\mathbb{Z}^d$ to an arbitrary *graph of bounded degree*. Here we sketch a situation, whereas a detailed study will be presented in Section 4.1.

Let us given the simple graph $\mathbb{G} := \mathbb{G}(\mathbb{V}, \mathbb{E})$ consisting of a countable set of *vertices* $v \in \mathbb{V}$ and a set of unordered *edges* $e = [v, v'] \in \mathbb{E}$. A standard choice for the distance $\rho(v, v')$ is the length of the shortest path γ connecting $v, v' \in \mathbb{V}$. For each vertex v

we define its degree $m(v) \leq \infty$ as the number of all nearest neighbors $v' \in \partial v$ with $\rho(v, v') = 1$. In the subsequent, we consider only the graphs having the *uniformly bounded degree* $m(\mathbb{G}) := \sup_{v \in \mathbb{V}} m(v) < \infty$. Furthermore, we impose the following regularity condition (substituting for Assumption $(\mathbf{L}_d)$ from Section 2.1):

Assumption $(\mathbf{G}_\delta)$ *There exists* $\delta_0 \geq 0$ *such that for all* $\delta > \delta_0$

$$\Xi_\delta := \sup_{o \in \mathbb{V}} \sum_{v} \exp\{-\delta\rho(\nu, o)\} < \infty. \tag{2.105}$$

For the lattice $\mathbb{G} := \mathbb{Z}^d$, (2.105) obviously holds with $\delta_0 = 0$. Setting $\Omega := [\mathbb{R}^\nu]^{\mathbb{G}}$, we define the subsets of (*exponentially*) *tempered configurations* (cf. (2.40))

$$\Omega^{(\mathrm{e})\mathrm{t}} := \bigcap_{o \in \mathbb{V},\ \delta > \delta_0} \Omega_{o,\delta}\ , \tag{2.106}$$

$$\Omega_{o,\delta} := \left\{ x \in \Omega \,\middle|\, ||x||_{o,\delta} := \left[\sum_v |x_v|^R \exp\{-\delta\rho(\nu, o)\} \right]^{1/R} < \infty \right\}.$$

Again, one has the *compact* embeddings $\Omega_{o,\delta} \hookrightarrow \Omega_{o,\delta'}$ whenever $\delta' > \delta$. On the graph $\mathbb{G}$, we now consider an interacting spin system with the formal Hamiltonian

$$H(x) := \sum_{v \in \mathbb{V}} V_v(x_v) + \frac{1}{2} \sum_{v \sim v'} W(x_v, x_{v'}), \tag{2.107}$$

where the potentials $W : \mathbb{R}^{2\nu} \to \mathbb{R}$ and $V_v : \mathbb{R}^\nu \to \mathbb{R}$ fulfill the former Assumptions $(\mathbf{W})$ and $(\mathbf{V})$ (or its modification $(\mathbf{V}_1)$). The matrix $\mathbf{J} := (J_{vv'})_{\mathbb{V} \times \mathbb{V}}$ in Assumption $(\mathbf{J})$ has the entries $J_{vv'} = J > 0$ if $v \sim v'$ and $J_{vv'} = 0$ otherwise. Fixed an inverse temperature $\beta > 0$, one defines the *local specification* $\Pi := \{\pi_\Lambda\}_{\Lambda \Subset \mathbb{V}}$: for all $\Lambda \Subset \mathbb{V}$ and $y \in \Omega$

$$\pi_\Lambda(B|y) := Z_\Lambda^{-1}(y) \int_{\Omega_\Lambda} \exp\{-\beta H_\Lambda(x_\Lambda|y)\} \mathbf{1}_B(x_\Lambda \times y_{\Lambda^c}) \mathrm{d}x_\Lambda, \quad B \in \mathcal{B}(\Omega), \tag{2.108}$$

where

$$H_\Lambda(x|y) := \sum_{v \in \Lambda} V_v(x_\ell) + \frac{1}{2} \sum_{v \in \Lambda,\ v' \in \partial v \cap \Lambda} W(x_v, x_{v'}) + \sum_{v \in \Lambda,\ v' \in \partial v \cap \Lambda^c} W(x_v, y_{v'}).$$

We confirm ourselves to the subset of *tempered Gibbs measures* $\mu \in \mathcal{G}^{(\mathrm{e})\mathrm{t}}$ supported by $\Omega^{(\mathrm{e})\mathrm{t}}$. Modifying the corresponding proofs for the system of weights $\exp\{-\delta\rho(\nu, o)\}$, $\delta > \delta_0$, one afterwords concludes that the set $\mathcal{G}^{(\mathrm{e})\mathrm{t}}$ is not empty (cf. Theorem 2.14) and all its element obey the à-priori bound (2.81) (cf. Theorem 2.15). Note that Assumption $(\mathbf{G}_\delta)$ is crucial for the validity of (2.81), while the existence of $\mu \in \mathcal{G}$ can be proved just *for any* graph with $m(\mathbb{G}) < \infty$.

In conclusion, we note that the situation drastically changes if $\mathbb{G}$ has *unbounded degree* $\sup_{v\in\mathbb{V}} m(v) = +\infty$. In particular, both Dobrushin's existence (see Condition $(\mathbf{D_1})$) and uniqueness (see Condition $(\mathbf{D_2})$) criteria do not apply directly, since the Dobrushin interdependence matrices in (2.45) and (2.173) are no longer strictly contractive in $l^\infty(\mathbb{G})$. So far there is no satisfactory theory of Gibbs distributions on such graphs, except some particular results available by comparison methods for the attractive harmonic interactions.

2.3 Uniqueness problem

In this section we present a number of conditions on the interaction which will suffice for $\mathcal{G}^{\mathrm{t}} = 1$ to be a singleton. We shall consider the cases of *high* ($\beta \ll 1$) and *law* ($\beta \gg 1$) *temperatures* in Subsections 2.3.2 and 2.3.3 respectively. A new issue, as compared with the previous uniqueness results (cf. e.g. [20, 71, 291]), is that we include the inter-particle interactions of possibly *superquadratic growth*. Furthermore, even for the mostly studied ferromagnetic quadratic interactions, this seems to be the first *elementary* treatment of the low temperature uniqueness in models with a unique ground state. Our approach is based on the *Dobrushin-Pechersky uniqueness criterion* to be formulated in Subsection 2.3.1. This is a modification of the fundamental Dobrushin criterion especially suited for non-compact spin spaces. A peculiarity in applying the Dobrushin-Pechersky theorem is that one first should control Dobrushin Compactness Condition $(\mathbf{D}_1)$ with some function h, which thereafter will participate in some new Contraction Condition $(\mathbf{DP}_2)$ has to be checked. This means that all proofs below will use strongly the à-priori moment bounds obtained in Subsection 2.2.2. To gain a complete insight into the subject, in Subsection 2.3.4 we shall revisit the original Dobrushin uniqueness criterion and examine to what extent it can be applied to the interactions obeying superquadratic growth or infinite range. Finally, in Subsection 2.3.5 we present a systematic account on the analytical properties, such as e.g. the decay of correlations for the Gibbs measures $\mu \in \mathcal{G}^{\mathrm{t}}$ and the spectral gaps for the associated Dirichlet operators $\mathbb{H}_\mu$, which typically occur in the uniqueness regime.

2.3.1 Dobrushin-Pechersky criterion

As already mentioned, we shall use a modification of the Dobrushin uniqueness theorem to the lattice systems with non-compact spin spaces, which was suggested by R. Dobrushin and E. Pechersky (see Theorem 1 in [92] and Theorem 4 in [230]). So far, such stronger version of Dobrushin's theorem has been stated and proven only for interactions of *finite range*, which gives rise to the following:

Assumption $(\mathbf{J_{fin}})$ *There exist* $r \geq 1$ *and* $J_\ell := J_{-\ell} \geq 0$, $|\ell| \leq r$, *such that* $J_{\ell\ell'} := J_{\ell'-\ell}$ *if* $|\ell - \ell'| \leq r$, *and* $J_{\ell\ell'} : = 0$ *otherwise.*

The *main advantage* of this approach is that one needs to check Dobrushin's condition of weak dependence not as usual for all boundary configurations, but only for such $y \in \Omega$ whose components y_ℓ, $\ell \in \mathbb{L}$, lye in a certain ball in $\mathbb{R}^\nu$. Moreover, the method straightforwardly extends to multi-particle interactions

$$W_\Lambda = W_{\Lambda+\ell} \quad \text{for all } \Lambda \Subset \mathbb{L} \text{ and } \ell \in \mathbb{L},$$
$$W_\Lambda \equiv 0 \quad \text{if } \operatorname{diam}\Lambda := \sup_{\ell,\ell' \in \Lambda} |\ell - \ell'| > r,$$

like those discussed in Subsection 2.2.5 (i). For this purpose, we introduce some quantities related to the *geometry* of the lattice $\mathbb{L} = \mathbb{Z}^d$. Let

$$\mathrm{a} := \mathrm{a}(r,d) = |\partial_r(\ell)|\,, \quad 2^d \le \mathrm{a} \le (2r+1)^d - 1, \tag{2.109}$$

denote the number of points in the r-vicinity of each $\ell \in \mathbb{L}$, and

$$\mathrm{b} := \mathrm{b}(r,d) = |\mathbb{L}/\mathbb{L}_0|\,, \quad 1 \le \mathrm{b} \le (r+1)^d, \tag{2.110}$$

be the number elements in the quotient group $\mathbb{L}/\mathbb{L}_0$ corresponding to a maximal subgroup $\mathbb{L}_0 \subset \mathbb{L}$ whose elements satisfy $|\ell - \ell'| > r$.

The Dobrushin–Pechersky uniqueness theorem requires the following two conditions to be fulfilled for the specification $\{\pi_\Lambda\}_{\Lambda \Subset \mathbb{L}}$. The first one is a *stronger version* of the *Dobrushin existence criterion* (cf. Condition $(\mathbf{D_1})$ in Subsection 2.3.4 (i)):

Compactness Condition $(\mathbf{DP}_1)$ *There exist a continuous compact function $h : \mathbb{R}^\nu \to \mathbb{R}_+$, a sequence $(I_\ell \ge 0)_{\ell \in \mathbb{L}}$, and a constant $\mathcal{C} \ge 0$, such that*

(i) The matrix $\mathbf{I} = (I_{\ell-\ell'})_{\mathbb{L}\times\mathbb{L}}$ is $l^\infty(\mathbb{L})$-contractive and, moreover,

$$||\mathbf{I}||_0 := \sum_{\ell \in \partial_r(0)} I_\ell \le \mathcal{I} < 1/\mathrm{a}^{\mathrm{b}} < 1. \tag{2.111}$$

(ii) For each $\ell \in \mathbb{L}$ and all configurations $y \in \Omega$,

$$\int_\Omega h(x_\ell)\pi_\ell(\mathrm{d}x|y) \le \mathcal{C} + \sum_{\ell' \in \partial_r(\ell)} I_{\ell-\ell'} h(y_{\ell'}). \tag{2.112}$$

In turn, the second condition is *a weaker version of* the well-known *Dobrushin uniqueness criterion* (cf. Condition $(\mathbf{D_2})$ in Subsection 2.3.4):

Contraction Condition $(\mathbf{DP}_2)$ *For a given $\mathcal{R} \ge 0$, there exists a sequence $(K_\ell \ge 0)_{\ell \in \mathbb{L}}$ such that*

(i) The matrix $\mathbf{K} = (K_{\ell-\ell'})_{\mathbb{L}\times\mathbb{L}}$ is $l^\infty(\mathbb{L})$-contractive, i.e.,

$$||\mathbf{K}||_0 := \sum_{\ell \in \partial_r(0)} K_\ell \le \mathcal{K} < 1. \tag{2.113}$$

(ii) For each $\ell \in \mathbb{L}$ and any pair of configurations $y, \tilde{y} \in \Omega$ satisfying

$$\max_{\ell' \in \partial_r(\ell)} \{h(y_{\ell'}),\ h(\tilde{y}_{\ell'})\} \le \mathcal{R},$$

the following estimate in the (half) total variation probability distance in $\mathbb{R}^\nu$ holds for the one-point projections

$$\begin{aligned}\mathbf{D}_{\mathrm{var}}\left(\mu_{\ell,y},\mu_{\ell,\tilde{y}}\right) &= \frac{1}{2}||\mu_{\ell,y}-\mu_{\ell,\tilde{y}}||_{\mathrm{TV}} := \sup_{B\in\mathcal{B}(\mathbb{R}^\nu)}\left[\mu_{\ell,y}-\mu_{\ell,\tilde{y}}\right](B) \\ &\le \sum_{\ell'\in\partial_r(\ell)} K_{\ell-\ell'}\delta(y_{\ell'}-\tilde{y}_{\ell'}),\end{aligned} \tag{2.114}$$

where the spin space $\mathbb{R}^\nu$ is equipped by the discrete metric

$$\delta(y_{\ell'}-\tilde{y}_{\ell'}) = \begin{cases} 0, & y_{\ell'}=\tilde{y}_{\ell'}; \\ 1, & y_{\ell'}\ne\tilde{y}_{\ell'}. \end{cases}$$

(iii) For each $\Lambda \Subset \mathbb{L}$, the mapping

$$(\varOmega,\ \mathcal{T}(\varOmega)) \ni y \to \mu_{\Lambda,y} \in (\mathcal{P}(\mathbb{R}^{\nu|\Lambda|}),\mathbf{D}_{\mathrm{var}})$$

is continuous, where $\mathcal{T}(\varOmega)$ denotes the usual product topology on $\varOmega$ (cf. Section 2.1).

Then, the Dobrushin–Pechersky theorem says that one always finds a constant $\mathcal{R}$ (which depends only on the parameters a, b, $\mathcal{I}$, $\mathcal{K}$, $\mathcal{C}$ and in principle can be written explicitly), such that to any local specification $\varPi = \{\pi_\Lambda\}_{\Lambda\Subset\mathbb{L}}$ obeying both Conditions $(\mathbf{DP}_1)$ and $(\mathbf{DP}_2)$ there corresponds *at most one* measure $\mu \in \mathcal{P}(\varOmega)$ solving the DLR equations and satisfying the *à-priori* bound

$$\sup_{\ell}\int_{\varOmega} h(x_\ell)\mu(\mathrm{d}x) < \infty. \tag{2.115}$$

The statement obviously generalizes to any (possibly infinite dimensional) *Polish space* X taken instead of the spin space $\mathbb{R}^\nu$. So far this criterion remained poorly recognized. Since the cited article [92], there have appeared a couple of subsequent works [45, 230] employing the above theorem in a different context of classical gases in $\mathbb{R}^\nu$. So, our paper seems to be the first one focused on its applications to the lattice spin systems. We mention that a Dobrushin like uniqueness criterion, assuming some local contraction condition for the probabilities $\mu_{\ell,y}(\mathrm{d}x_\ell)$, was also established in [32].

Remark 2.22 Our formulation of the Dobrushin-Pechersky theorem *slightly differs* from its original version in [92]. First, we add the *missing* condition (iii) in Assumption $(\mathbf{DP}_2)$, which is needed to justify the existence of proper *measurable couplings* for $\mu_{\ell,y}$ playing a crucial role in the proof (see Section 4.4). Furthermore, the continuity stated in (iii) means that the specification $\{\pi_\Lambda\}_{\Lambda\Subset\mathbb{L}}$ is regular and compact in the sense of Propositions 2.3 and 2.7. Indeed, the function h in Assumption $(\mathbf{DP}_1)$ *needs not to be compact*, in contrast to what was required in [92]. But if *h is compact,* the Dobrushin-Pechersky theorem implies the existence of the *exactly one* Gibbs measure satisfying (2.115). On the other hand, it is important to have a strong enough growth of h, so that the sets $\{y_\ell \mid h(y_\ell) \leq \mathcal{R}\}$ are bounded for each $\mathcal{R} > 0$. Actually, one may use *different functions* h to control respectively the *existence* and the *uniqueness* of $\mu \in \mathcal{G}^{\mathrm{t}}$. This remark will be relevant e.g. for quantum lattice systems, where the single spin spaces themselves are infinite dimensional and therefore a continuous growing function h may not be compact (see Subsection 3.2.5 below).

2.3.2 High-temperature uniqueness

In this subsection we prove that the set of tempered Gibbs measures corresponding to the lattice spin system (2.1) consists of *exactly one point*, provided the *temperature is large* enough ($\beta \ll 1$) or the strength of the *interaction is small* ($||\mathbf{J}||_0 \ll 1$). All other parameters of the interaction will be fixed. Although such results are rather expected (e.g. via cluster expansions, see [200] for an extended account), we are not sure that their *direct analytical proof* is known for superquadratic interactions. Below we present the corresponding Theorems 2.23–2.26, which will be verified by means of the Dobrushin-Pechersky criterion. Note that the case of $R = 2$ also could be treated by the fundamental Dobrushin uniqueness theorem (cf. Subsections 2.3.4 (i),(iii)), which usually produces better estimates on the critical parameters. However, Dobrushin's criterion is typically *not applicable* to the pair interactions $W_{\ell\ell'}$ growing fastly than quadratic (see the discussion in Subsection 2.3.4 (ii)).

(i) Description of results

Like as in Subsection 2.2.2, we allow here a more general situation of $P \geq R \geq 2$, which is described by Assumption $(\mathbf{V}_1)$ instead of $(\mathbf{V})$. If no further information on the interaction is available, our first result says that, keeping fixed $\beta > 0$, one always can achieve the uniqueness of $\mu \in \mathcal{G}^{\mathrm{t}}$ by taking a small enough $||\mathbf{J}||_0 \ll 1$.

Theorem 2.23 *Consider the spin system (2.1) on the lattice $\mathbb{L} = \mathbb{Z}^d$ with the interaction potentials $V_\ell, W_{\ell\ell'}$ satisfying Assumptions $(\mathbf{V}_1)$, $(\mathbf{J}_{\mathrm{fin}})$, and $(\mathbf{W})$. Then, for any $\beta > 0$ one finds a certain $\mathcal{J}(\beta) > 0$ such that, for all values of $||\mathbf{J}||_0 \leq \mathcal{J}(\beta)$, the corresponding set $\mathcal{G}^{\mathrm{t}}$ is singleton.*

We omit here the proof of this statement as the easier part of Theorems 2.25 and 2.26 below (or as a classical counterpart of Theorem 3.22 describing the more involved quantum case). It should be emphasized that the uniqueness holds simultaneously for the *whole class* of systems like (2.1), whose interactions potentials $V_\ell, W_{\ell\ell'}$ are controlled by the *same parameters* in Assumptions $(\mathbf{V}_1)$, $(\mathbf{J}_{\mathrm{fin}})$, and $(\mathbf{W})$. In a similar way one should understand all uniqueness results in the subsequent text.

To get more precise description of the uniqueness region, one has to refine the conditions on the interaction:

Assumption $(\mathbf{V_2})$ *Given $P \geq R$, there exist positive $A_1 \leq A_2$ and real $B_1 \leq B_2$, such that for all $\ell \in \mathbb{L}$ and $x_\ell \in \mathbb{R}^\nu$*

$$A_1|x_\ell|^P + B_1 \leq V_\ell(x_\ell) \leq A_2|x_\ell|^P + B_2. \tag{2.116}$$

If $P = R$, the value of $||\mathbf{J}||_0$ is small enough so that

$$\varDelta_1 := A_1 - \frac{1}{2}||\mathbf{J}||_0 > ||\mathbf{J}||_0. \tag{2.117}$$

Assumption $(\mathbf{W_1})$ *Additionally to* $(\mathbf{W})$ *the following holds: there exists a nonnegative C_1, such that for all $\ell, \ell' \in \mathbb{L}$ and $x_\ell, x_{\ell'} \in \mathbb{R}^\nu$*

$$|W_{\ell\ell'}(x_\ell, x_{\ell'}) - W_{\ell\ell'}(0, x_{\ell'})| \leq \frac{1}{2} J_{\ell-\ell'} |x_\ell| \cdot (C_1 + |x_\ell|^{R-1} + |x_{\ell'}|^{R-1}). \tag{2.118}$$

Remark 2.24 If $W_{\ell\ell'} \in C^1(\mathbb{R}^{2\nu})$, the following is sufficient for (2.118)

$$|\partial_{x_\ell} W_{\ell\ell'}(x_\ell, x_{\ell'})| \leq \frac{1}{2} J_{\ell-\ell'} (C_1 + |x_\ell|^{R-1} + |x_{\ell'}|^{R-1}). \tag{2.119}$$

The latter is surely true if $W_{\ell\ell'}(x_\ell, x_{\ell'}) := 2^{1-R} w_{\ell\ell'}(x_\ell - x_{\ell'})$, where $w_{\ell\ell'} \in C^1(\mathbb{R}^\nu)$ and $|w'_{\ell\ell'}(x_\ell)| \leq J_{\ell\ell'} \left(c_w + |x_\ell|^{R-1}\right)$ with some $c_w > 0$.

Respectively, we have the following *improvements* of Theorem 2.23, which allows us to control the uniqueness of $\mu \in \mathcal{G}^{\mathrm{t}}$ at the whole *temperature interval* $(0, \beta)$ and gives the *order parameter* (2.120).

Theorem 2.25 *Suppose that Assumptions* $(\mathbf{V}_2)$, $(\mathbf{J}_{\mathrm{fin}})$, *and* $(\mathbf{W})$ *hold. Then, for every $\beta_0 > 0$ one finds $\mathcal{J} := \mathcal{J}(\beta_0) > 0$, such that the set $\mathcal{G}^{\mathrm{t}}$ is singleton at all values of $\beta \leq \beta_0$ and $||\mathbf{J}||_0 \leq \mathcal{J}$.*

Theorem 2.26 *In the situation of Theorem 2.25, suppose additionally that Assumption* $(\mathbf{W}_1)$ *holds. Then, for every $\beta_0 > 0$ and $\mathcal{J}_0 < A_1/(\mathrm{a}^{\mathrm{b}} + 1/2)$, one finds a proper $\iota_0 := \iota_0(\beta_0, \mathcal{J}_0) > 0$ such that, at all values of $\beta \leq \beta_0$ and $||\mathbf{J}||_0 < \mathcal{J}_0$, the set $\mathcal{G}^{\mathrm{t}}$ is singleton if*

$$\beta^{1-R/P} ||\mathbf{J}||_0 =: \iota \leq \iota_0. \tag{2.120}$$

In the case of $P = R$ the statements of both theorems coincide. Their proofs will be done after some preparatory work in Subsections 2.3.2 (iii) and (iv).

(ii) Reduction to the case $\beta = 1$

To simplify things, we reduce the problem to the case of $\beta = 1$ by using *a space scaling* argument as described below. Let $\mu := \mu_\beta$ be a Gibbs distribution corresponding to the lattice system (2.1) at the temperature $\beta > 0$. This measure satisfies the DLR equations (2.34) with respect to the local specification $\{\pi_{\beta,\Lambda}\}_{\Lambda \Subset \mathbb{L}}$ given by (2.21). Setting

$$\alpha := (\beta ||\mathbf{J}||_0^\gamma)^{-1/R} \quad \text{with} \quad \gamma \in [0, 1], \tag{2.121}$$

let us define a *new probability* measure $\tilde{\mu} := \tilde{\mu}_\alpha$ on $(\varOmega, \mathcal{B}(\varOmega))$ by

$$\tilde{\mu}_\alpha(B) := \mu_\beta(\alpha B), \quad \alpha B := \left\{ x \in \varOmega \;\middle|\; \alpha^{-1} x \in \Delta \right\}, \quad B \in \mathcal{B}(\varOmega). \tag{2.122}$$

As is easy to see, $\tilde{\mu}_\alpha$ is a Gibbs distribution related to the local specification $\tilde{\Pi}_\alpha = \{\tilde{\pi}_{\alpha,\Lambda}\}_{\Lambda\Subset\mathbb{L}}$, where

$$\tilde{\pi}_{\alpha,\Lambda}(B|y) := \pi_{\beta,\Lambda}(\alpha B|\alpha y), \quad B \in \mathcal{B}(\Omega), \ y \in \Omega. \tag{2.123}$$

The kernels $\tilde{\pi}_{\alpha,\Lambda}(\mathrm{d}x|y)$ can be represented in the form (2.21)–(2.23) with $\beta = 1$ and the rescaled potentials

$$\tilde{V}_\ell(x_\ell) := \beta V_\ell(\alpha x_\ell), \qquad \tilde{W}_{\ell\ell'}(x_\ell, x_{\ell'}) := \beta W_{\ell\ell'}(\alpha x_\ell, \alpha x_{\ell'}). \tag{2.124}$$

They satisfy the same Assumptions $(\mathbf{W})$, $(\mathbf{V}_2)$, and $(\mathbf{J}_{\rm fin})$, that is

$$|\tilde{W}_{\ell\ell'}(x_\ell, x_{\ell'})| \leq \frac{1}{2}\tilde{J}_{\ell-\ell'}(\tilde{C}_W + |x_\ell|^R + |x_{\ell'}|^R), \tag{2.125}$$

$$\tilde{A}_1|x_\ell|^P + \tilde{B}_1 \leq \tilde{V}_\ell(x_\ell) \leq \tilde{A}_2|x_\ell|^P + \tilde{B}_2, \tag{2.126}$$

but with the constants

$$\begin{aligned} &\tilde{J}_{\ell-\ell'} := J_{\ell-\ell'}/||\mathbf{J}||_0^\gamma, \quad \tilde{C}_W = \beta||\mathbf{J}||_0^\gamma C_W, \\ &\tilde{A}_i := \beta^{1-P/R}||\mathbf{J}||_0^{-\gamma P/R}A_i, \quad \tilde{B}_i := \beta B_i, \quad i = 1, 2. \end{aligned} \tag{2.127}$$

So, we get the one-to-one correspondence between the Gibbs measures μ_β and $\tilde{\mu}_\alpha$. Moreover, the transform (2.123) preserves the class of tempered distributions $\mathcal{P}^{\rm t}(\Omega)$. So, the problems of existence and uniqueness of the Gibbs measures μ_β related to the initial system (2.1) at the inverse temperature $\beta > 0$ are thus reduced to the corresponding problems posed for the Gibbs measures $\tilde{\mu}_\alpha$ related to a similar system, but at $\beta = 1$ and with the potentials $\tilde{V}_\ell$, $\tilde{W}_{\ell\ell'}$. The measure μ_β is reconstructed back through the identity

$$\int_\Omega f(x)\mathrm{d}\mu_\beta(x) = \int_\Omega f(\alpha x)\mathrm{d}\tilde{\mu}_\alpha(x) \tag{2.128}$$

valid for all bounded measurable functions $f : \Omega \to \mathbb{R}$. Depending on the critical regime, below we shall use the described transform in two special cases of $\gamma = 0$ and $\gamma = 1$ (see the proofs of Theorems 2.25 and 2.26). From a technical point of view, the scaling (2.122) allows us to control the constants in the Dobrushin-Pechersky criterion for the modified specification $\tilde{\Pi}_\alpha$ *uniformly at all values* $\alpha \to +\infty/+0$, which happens to be impossible for the initial specification Π_β as respectively $\beta \to +0/+\infty$.

(iii) Uniqueness by small $||\mathbf{J}||_0$: proof of Theorem 2.25

Here we show that, for all values $0 < \beta \leq \beta_0$ and $||\mathbf{J}||_0 \leq \mathcal{J}(\beta_0)$, the modified specification (2.123) satisfies Conditions $(\mathbf{DP}_1)$ and $(\mathbf{DP}_2)$ of the Dobrushin-Pechersky criterion with the compact function $h(x_\ell) := |x_\ell|^R$.

Condition $(\mathbf{DP}_1)$: For this purpose we analyze the key estimate (2.51) in the case of one-point conditional distributions

$$\begin{aligned} \tilde{\mu}_{\ell,y}(\mathrm{d}x_\ell) &:= \mu_{\ell,\alpha y}(\alpha^{-1}\mathrm{d}x_\ell) \\ &= \tilde{Z}_\ell^{-1}(y)\int_{\mathbb{R}^\nu} \exp\left\{-\tilde{V}_\ell(x_\ell) - \sum_{\ell'(\neq\ell)} \tilde{W}_{\ell\ell'}(x_\ell, y_{\ell'})\right\}\mathrm{d}x_\ell. \end{aligned} \tag{2.129}$$

They depend on the parameter $\alpha := (\beta||\mathbf{J}||_0^\gamma)^{-1/R}$, where for further applications we consider all possible values of $\gamma \in [0,1]$. Let $||\mathbf{J}||_0$ varies in some bounded interval $[0, \mathcal{J}_0]$; in the case of $P = R$ one additionally requires (because of (2.117)) that $\mathcal{J}_0 < A_1/(\mathrm{a}^\mathrm{b} + 1/2)$. Next, we pick some $\kappa \in (\mathrm{a}^\mathrm{b}\mathcal{J}_0,\ A_1 - \mathcal{J}_0/2)$. At this point we refer to Lemma 2.49 below, which under Assumptions ($\mathbf{V}_2$) and ($\mathbf{W}$) concerns with the bound (2.51) in the *limit case* $\beta \to +0$. For the modified kernels, it can be rewritten as

$$\int\limits_{\mathbb{R}^\nu} \exp\left\{\kappa||\mathbf{J}||_0^{-\gamma}|x_\ell|^R\right\}\tilde{\mu}_{\ell,y}(\mathrm{d}x_\ell) \le \varGamma_0 \exp\left\{\sum_{\ell'\in\partial_r(\ell)} J_{\ell-\ell'}||\mathbf{J}||_0^{-\gamma}|y_{\ell'}|^R\right\}, \tag{2.130}$$

where the constant $\varGamma_0 := \varGamma_0(\beta_0, \mathcal{J}_0, \kappa) \ge 1$ is given explicitly by (2.258)–(2.260). This readily implies Condition ($\mathbf{DP}_1$) holding with the function $h(x_\ell) := |x_\ell|^R$, constant $\mathcal{C} := \kappa^{-1}\mathcal{J}_0^\gamma \log \varGamma_0$, and contractive matrix

$$I_{\ell-\ell'} := J_{\ell-\ell'}/\kappa, \quad ||\mathbf{I}||_0 \le \mathcal{I} := \mathcal{J}_0/\kappa < 1/\mathrm{a}^\mathrm{b}.$$

Thereafter, in the formulation of the Dobrushin-Pechersky theorem, we can fix some $\mathcal{K} < 1$ and the corresponding radius $\mathcal{R} := \mathcal{R}(\mathcal{C}, \mathcal{I}, \mathcal{K})$.

Condition ($\mathbf{DP}_2$): Below we use only Assumptions ($\mathbf{J}_{\mathrm{fin}}$) and ($\mathbf{W}$) together with the estimate (2.130) proved above. Given $\ell' \in \mathbb{L}$, let us consider a pair of boundary conditions $y, \tilde{y} \in \varOmega$ such that

$$y = \tilde{y} \ \text{ off } \ell', \quad ||y||_\infty^R,\ ||\tilde{y}||_\infty^R \le \mathcal{R}, \ \text{ where } \ ||y||_\infty := \sup_\ell |y_\ell|. \tag{2.131}$$

By (2.130), one has the uniform bound

$$\sup_{||y||_\infty^R \le \mathcal{R}} \int_{\mathbb{R}^\nu} \exp\left\{\kappa||\mathbf{J}||_0^{-\gamma}|x_\ell|^R\right\}\tilde{\mu}_{\ell,y}(\mathrm{d}x_\ell) \le \varGamma_0 \exp\left\{\mathcal{J}_0^{1-\gamma}\mathcal{R}\right\}. \tag{2.132}$$

For shorthand we denote

$$\Delta\tilde{W}_{\ell\ell'}(x_\ell) := \tilde{W}_{\ell\ell'}(x_\ell, y_{\ell'}) - \tilde{W}_{\ell\ell'}(x_\ell, \tilde{y}_{\ell'}),$$

which obeys by (2.125) and (2.127)

$$|\Delta\tilde{W}_{\ell\ell'}(x_\ell)| \le (J_{\ell-\ell'}/||\mathbf{J}||_0^\gamma) \cdot \left(\beta||\mathbf{J}||_0^\gamma C_W + \mathcal{R} + |x_\ell|^R\right). \tag{2.133}$$

Then, for any $\ell \ne \ell'$, the variation distance can be estimated as

$$\begin{aligned} ||\tilde{\mu}_{\ell,y} - \tilde{\mu}_{\ell,\tilde{y}}||_{\mathrm{TV}} &= \int_{\mathbb{R}^\nu} \left|1 - \tilde{Z}_\ell(y)\tilde{Z}_\ell^{-1}(\tilde{y})\exp\left\{\Delta\tilde{W}_{\ell\ell'}(x_\ell)\right\}\right|\tilde{\mu}_{\ell,y}(\mathrm{d}x_\ell) \\ &\le \mathcal{I}_{\ell\ell'}^{(1)} + \left|1 - \tilde{Z}_\ell(y)\tilde{Z}_\ell^{-1}(\tilde{y})\right| \cdot \mathcal{I}_{\ell\ell'}^{(2)}, \end{aligned} \tag{2.134}$$

where we set

$$\mathcal{I}_{\ell\ell'}^{(1)} := \int_{\mathbb{R}^\nu} \left|1 - \exp\left\{\Delta\tilde{W}_{\ell\ell'}(x_\ell)\right\}\right|\tilde{\mu}_{\ell,y}(\mathrm{d}x_\ell), \tag{2.135}$$

$$\mathcal{I}_{\ell\ell'}^{(2)} := \int_{\mathbb{R}^\nu} \exp|\tilde{\Delta}W_{\ell\ell'}(x_\ell)|\tilde{\mu}_{\ell,y}(\mathrm{d}x_\ell). \tag{2.136}$$

Introducing one more quantity

$$\mathcal{I}_{\ell\ell'} := \sup_{\substack{y,\tilde{y}\in\Omega \\ ||y||^R_\infty,\ ||\tilde{y}||^R_\infty\leq\mathcal{R}}} \int_{\mathbb{R}^\nu} \left|\Delta\tilde{W}_{\ell\ell'}(x_\ell)\right| \cdot \exp|\Delta\tilde{W}_{\ell\ell'}(x_\ell)|\tilde{\mu}_{\ell,y}(\mathrm{d}x_\ell), \tag{2.137}$$

and using elementary inequalities

$$|1-\exp(\pm w)| \leq w\exp w, \quad w \leq \exp w \leq e + w\exp w, \quad w\in\mathbb{R}_+, \tag{2.138}$$

one easily observes that

$$\left|1-\tilde{Z}_\ell(y)\tilde{Z}_\ell^{-1}(\tilde{y})\right| \leq \mathcal{I}^{(1)}_{\ell\ell'} \leq \mathcal{I}_{\ell\ell'}\ , \quad \mathcal{I}^{(2)}_{\ell\ell'} \leq e + \mathcal{I}_{\ell\ell'}\ . \tag{2.139}$$

Plugging (2.139) into (2.134), we arrive at

$$\mathbf{D}_{\mathrm{var}}\left(\tilde{\mu}_{\ell,y},\tilde{\mu}_{\ell,\tilde{y}}\right) \leq \frac{1}{2}\mathcal{I}_{\ell\ell'}\left(4+\mathcal{I}_{\ell\ell'}\right). \tag{2.140}$$

Herefrom we may restrict ourselves to the *particular case of* $\gamma = 0$. To estimate the right-hand side in (2.140), let us fix some positive $\epsilon < \kappa - \mathcal{J}_0$. Taking into account (2.132) and (2.133), we find that

$$\mathcal{I}_{\ell\ell'} \leq (J_{\ell-\ell'}/\epsilon)\Gamma_0\exp\left\{\mathcal{J}_0\mathcal{R} + (\mathcal{J}_0+\epsilon)\left(\beta_0 C_W + \mathcal{R}\right)\right\}. \tag{2.141}$$

Hence, for the pair of boundary conditions $y, \tilde{y}$ chosen as in (2.131)

$$\mathrm{D}_{\mathrm{var}}\left(\tilde{\mu}_{\ell,y}\ ,\ \tilde{\mu}_{\ell,\tilde{y}}\right) \leq K_{\ell-\ell'} := J_{\ell-\ell'}C_{2.142}(\beta_0,\mathcal{J}_0,\mathcal{R}), \tag{2.142}$$

where the constant $C_{2.142}(\beta_0,\mathcal{J}_0,\mathcal{R})$ can be written explicitly from (2.140) and (2.141). Thus, for each given $\mathcal{K} < 1$, by choosing small enough $||\mathbf{J}||_0 \leq \mathcal{J}(\beta_0) \leq \mathcal{J}_0$ one gets the required contractivity $||\mathbf{K}||_0 \leq \mathcal{K}$ of the matrix $(K_{\ell-\ell'})_{\mathbb{L}\times\mathbb{L}}$. Finally, by the triangle inequality, (2.142) extends to the Contraction Condition ($\mathbf{DP}_2$) valid for all $y, \tilde{y} \in \Omega$ obeying (2.131). ■

(iv) Uniqueness by small β: proof of Theorem 2.26

It is now convenient to look at the modified specification (2.123) corresponding to the *particular choice of* $\gamma = 1$. For all values of $\beta \leq \beta_0$ and $||\mathbf{J}||_0 \leq \mathcal{J}_0 < A_1/(\mathrm{a}^{\mathrm{b}}+1/2)$, the validity of Condition ($\mathbf{DP}_1$) with the compact function $h(x_\ell) := |x_\ell|^R$ and $\mathcal{I} := \mathcal{J}_0/\kappa < 1/\mathrm{a}^{\mathrm{b}}$ has been already checked in the proof of Theorem 2.25. Fixed some $\mathcal{K} < 1$ and $\mathcal{R} := \mathcal{R}(\mathcal{C},\mathcal{I},\mathcal{K}) > 0$, it remains to show that the matrix $(K_{\ell-\ell'})_{\mathbb{L}\times\mathbb{L}}$ in Condition ($\mathbf{DP}_2$) can be made contractive by small values of $\iota := \beta^{1-R/P}||\mathbf{J}||_0$.

Let us conventionally rewrite each probability measure (2.129) as

$$\tilde{\mu}_{\ell,y}(\mathrm{d}x_\ell) = \bar{Z}_\ell^{-1}(y)\exp\left\{-\bar{H}_\ell(x_\ell|y)\right\}\mathrm{d}x_\ell, \tag{2.143}$$

where

$$\bar{H}_\ell(x_\ell|y) := \tilde{V}_\ell(x_\ell) + \sum\nolimits_{\ell' \in \partial_r(\ell)} \bar{W}_{\ell\ell'}(x_\ell, y_{\ell'}),$$
$$\bar{W}_{\ell\ell'}(x_\ell, y_{\ell'}) := \tilde{W}_{\ell\ell'}(x_\ell, y_{\ell'}) - \tilde{W}_{\ell\ell'}(0, y_{\ell'}).$$

By the above construction $\bar{W}_{\ell\ell'}(0, y_{\ell'}) = 0$, whereas by Assumption $(\mathbf{W}_1)$

$$\sup_{|y_{\ell'}|^R \leq \mathcal{R}} |\bar{W}_{\ell\ell'}(x_\ell, y_{\ell'})| \leq \frac{1}{2} (J_{\ell-\ell'}/||\mathbf{J}||_0) \cdot \left(|x_\ell|^R + \mathcal{L}_1 |x_\ell|\right), \tag{2.144}$$
$$\mathcal{L}_1 := (\beta_0 ||\mathbf{J}||_0)^{1-1/R} C_1 + \mathcal{R}^{1-1/R}.$$

According to (2.140), the proof is reduced to getting a proper bound for

$$\mathcal{I}_{\ell\ell'} := \sup_{\substack{y, \tilde{y} \in \Omega \\ ||y||_\infty^R,\ ||\tilde{y}||_\infty^R \leq \mathcal{R}}} \int_{\mathbb{R}^\nu} \left|\Delta \bar{W}_{\ell\ell'}(x_\ell)\right| \cdot \exp |\Delta \bar{W}_{\ell\ell'}(x_\ell)| \tilde{\mu}_{\ell,y}(\mathrm{d}x_\ell), \tag{2.145}$$

where we set $\Delta \bar{W}_{\ell\ell'}(x_\ell) := \bar{W}_{\ell\ell'}(x_\ell, y_{\ell'}) - \bar{W}_{\ell\ell'}(x_\ell, \tilde{y}_{\ell'})$. Having regard of (2.125)–(2.127) and (2.143)–(2.145), we find that

$$\mathcal{I}_{\ell\ell'} \leq (\tilde{X}/\tilde{Y}) \cdot (J_{\ell-\ell'}/||\mathbf{J}||_0) \exp\left\{\beta\left(B_2 - B_1\right)\right\}, \tag{2.146}$$

with

$$\tilde{X} := \int_{\mathbb{R}^\nu} \left(\iota |x_\ell|^R + \iota^{1/R} \mathcal{L}_1 |x_\ell|\right) \tag{2.147}$$
$$\times \exp\left\{-A_1 |x_\ell|^P + \frac{3}{2}\left[\iota |x_\ell|^R + \iota^{1/R} \mathcal{L}_1 |x_\ell|\right]\right\} \mathrm{d}x_\ell,$$
$$\tilde{Y} := \int_{\mathbb{R}^\nu} \exp\left\{-A_2 |x_\ell|^P - \frac{1}{2}\left[\iota |x_\ell|^R + \iota^{1/R} \mathcal{L}_1 |x_\ell|\right]\right\} \mathrm{d}x_\ell. \tag{2.148}$$

In the above integrals we have already made the change of variables $x_\ell \mapsto \iota^{1/R} x_\ell$, which by (2.146) yields us that

$$\sup_{\ell \in \mathbb{L},\ \ell' \in \partial_r(\ell)} \mathcal{I}_{\ell\ell'} = \mathcal{O}(\iota^{1/R}),\ \ \iota \to 0.$$

Turning back to (2.140) and (2.142) we conclude that, for each $\ell \in \mathbb{L}$ and $y, \tilde{y} \in \Omega$ as in (2.131),

$$\mathbf{D}_{\mathrm{var}}\left(\tilde{\mu}_{\ell,y}(\mathrm{d}x_\ell),\ \tilde{\mu}_{\ell,\tilde{y}}(\mathrm{d}x_\ell)\right) \leq K \sum_{\ell' \in \partial_r(\ell)} (J_{\ell-\ell'}/||\mathbf{J}||_0) \tag{2.149}$$

with a certain $K := K(\beta_0, \mathcal{J}_0, \mathcal{R}) = \mathcal{O}(\iota^{1/R})$, $\iota \to 0$. Setting $K_{\ell-\ell'} := K$ for $\ell' \in \partial_r(\ell)$ and $K_{\ell-\ell'} := 0$ otherwise, we thus get the matrix $(K_{\ell-\ell'})_{\mathbb{L}\times\mathbb{L}}$ satisfying Condition $(\mathbf{DP}_2)$. ■

Corollary 2.27 *Suppose that $P > R$ and Assumption $(\mathbf{W}_1)$ holds with $C_W = C_1 = 0$, then the result of Theorem 2.26 is true without imposing the global bound $||\mathbf{J}||_0 \leq \mathcal{J}_0$. If in addition $(\mathbf{V}_2)$ holds with $B_1 = B_2$, we can also drop the restriction $\beta \leq \beta_0$. So, the uniqueness of $\mu \in \mathcal{G}^t$ can be achieved alone by small values of the order parameter (2.120).*

Proof. Let us turn back to the estimate (2.130) in the case of $\gamma = 1$. By plugging $\kappa := \mathrm{k}||\mathbf{J}||_0$ with any $\mathrm{k} > \mathrm{a}^{\mathrm{b}}$ into (2.257)–(2.260), one observes that the constant Γ_0 participating in Lemma 2.49 (and hence $\mathcal{C}$ and $\mathcal{R}$ respectively in Conditions $(\mathbf{DP}_1)$ and $(\mathbf{DP}_2)$) depends only on ι and $\beta\,(B_2 - B_2)$. In a similar way analyzing (2.146)–(2.148), we conclude that the coefficients $K_{\ell-\ell'}$ depend on ι, $\mathcal{R}$, and $\beta\,(B_2 - B_2)$. The final answer now follows by setting $B_1 = B_2$. ■

(v) Comments on Theorems 2.23, 2.25, and 2.26

(i) As may be seen from the proofs, we pass to the modified specification (2.123) since Condition $(\mathbf{DP}_1)$ in Theorem 2.25 (respectively $(\mathbf{DP}_2)$ in Theorem 2.30) does not hold for the initial kernels $\mu_{\ell,y}(\mathrm{d}x_\ell)$ uniformly for $\beta \to +0$ (respectively $\beta \to \infty$). Furthermore, a proper space scaling allows us to determine *the order parameter* (2.120) in Theorem 2.26 (and the corresponding one (2.162) in Theorem 2.30).

(ii) The proof of Theorem 2.23 repeats, with certain reductions, the corresponding steps in the proof of Theorem 2.25 and hence is omitted here. Since β is fixed, a passage to the rescaled measures $\tilde{\mu}_{\ell,y}(\mathrm{d}x_\ell)$ is not needed.

(iii) As the only example, in their paper R. Dobrushin and E. Pechersky considered a system of scalar spins $x_\ell \in \mathbb{R}$ with the heuristic Hamiltonian

$$H(x) := \frac{J}{2} \sum_{\ell,\ell'\,:\,|\ell-\ell'|=1} (x_\ell - x_{\ell'})^R + \sum_{\ell} x_\ell^P, \tag{2.150}$$

where P, R are even integers such that $P > R$ (cf. Theorem 7 in [92]). This is the simplest model of ferromagnetic type with the *convex* potentials of superquadratic growth. Hence, one expects here nothing else than $|\mathcal{G}^{\mathrm{t}}| = 1$. After the change of variables $x_\ell \to \beta^{-1/R} x_\ell$, the problem is reduced to the study of Gibbs measures related to the same Hamiltonian (2.150), but at the temperature $\tilde{\beta} = 1$ and with the coupling $\tilde{J} = J\beta^{1-R/P}$. This example clearly shows that the phase diagram is governed by the order parameter (2.120), which in a general situation is confirmed by Theorem 2.26.

(iv) We would like to indicate one more method for estimating the Dobrushin coefficients $K_{\ell-\ell'}$ in Condition $(\mathbf{DP}_2)$, which however requires stronger regularity of the potentials $W_{\ell\ell'} \in C^1(\mathbb{R}^{2\nu})$:

Assumption $(\mathbf{W}_2)$ *Let additionally to* $(\mathbf{W})$ *the following hold: there exists $C_2 > 0$ such that for all $\ell, \ell' \in \mathbb{L}$ and $x_\ell, x_{\ell'} \in \mathbb{R}^\nu$*

$$|\partial_{x_{\ell'}} W_{\ell\ell'}(x_\ell, x_{\ell'}) - \partial_{x_{\ell'}} W_{\ell\ell'}(0, x_{\ell'})| \leq \frac{1}{2} J_{\ell-\ell'} |x_\ell| \cdot (C_2 + |x_\ell|^{R-2} + |x_{\ell'}|^{R-2}). \tag{2.151}$$

Note that this condition is surely fulfilled if $W_{\ell\ell'} \in C^2(\mathbb{R}^{2\nu})$ and

$$\left|\partial^2_{x_\ell x_{\ell'}} W_{\ell\ell'}(x_\ell, x_{\ell'})\right|_{\mathcal{L}(\mathbb{R}^\nu)} \leq \frac{1}{2} J_{\ell-\ell'}(C_2 + |x_\ell|^{R-2} + |x_{\ell'}|^{R-2}). \tag{2.152}$$

For $\gamma = 1$, (2.151) implies the following bound on the rescaled potentials $\tilde{W}_{\ell\ell'}$

$$\sup_{||y||^R_\infty \leq \mathcal{R}} |\partial_{x_{\ell'}} \tilde{W}_{\ell\ell'}(x_\ell, y_{\ell'}) - \partial_{x_{\ell'}} \tilde{W}_{\ell\ell'}(0, y_{\ell'})|$$
$$\leq \frac{1}{2} \left(J_{\ell-\ell'}/||\mathbf{J}||_0\right) \cdot \left(|x_\ell|^{R-1} + \mathcal{L}_2|x_\ell|\right), \tag{2.153}$$

with a constant $\mathcal{L}_2 := (\beta_0||\mathbf{J}||_0)^{1-2/R} C_2 + \mathcal{R}^{1-2/R}$.

Alternative proof of Theorem 2.26. Assuming that $(\mathbf{V}_2)$, $(\mathbf{J}_{\text{fin}})$, and $(\mathbf{W}_2)$ hold, we need to check Condition $(\mathbf{DP}_2)$. For any given $f \in L^\infty(\mathbb{R}^\nu)$, the mapping

$$\mathbb{R}^\nu \ni y_{\ell'} \longmapsto \int_{\mathbb{R}^\nu} f(x_\ell) \tilde{\mu}_{\ell,y}(\mathrm{d}x_\ell) \tag{2.154}$$

is continuously differentiable. Having regard to (2.129), we calculate its partial derivatives along directions $y^{(i)}_{\ell'} \in \mathbb{R}$, $1 \leq i \leq \nu$,

$$\partial_{y^{(i)}_{\ell'}} \int_{\mathbb{R}^\nu} f(x_\ell) \tilde{\mu}_{\ell,y}(\mathrm{d}x_\ell) = -\mathbf{Cov}_{\tilde{\mu}_{\ell,y}(\mathrm{d}x_\ell)} \left\{ f(x_\ell); \partial_{y^{(i)}_{\ell'}} \tilde{W}_{\ell\ell'}(x_\ell, y_{\ell'}) \right\}. \tag{2.155}$$

For all $y, \tilde{y}$ as in (2.131), one gets by the mean-value theorem

$$||\tilde{\mu}_{\ell,y} - \tilde{\mu}_{\ell,\tilde{y}}||_{\mathrm{TV}} = \sup_{||f||_\infty \leq 1} \int_{\mathbb{R}^\nu} f(x_\ell) \left[\tilde{\mu}_{\ell,y}(\mathrm{d}x_\ell) - \tilde{\mu}_{\ell,\tilde{y}}(\mathrm{d}x_\ell)\right]$$
$$\leq \sup_{||f||_\infty \leq 1,\ ||y||^R_\infty \leq \mathcal{R}} \left| \mathbf{Cov}_{\tilde{\mu}_{\ell,y}(\mathrm{d}x_\ell)} \left\{ f(x_\ell); \left(\partial_{y_{\ell'}} \tilde{W}_{\ell\ell'}(x_\ell, y_{\ell'}), y_{\ell'} - \tilde{y}_{\ell'} \right) \right\} \right|$$
$$\leq \sup_{||y||^R_\infty \leq \mathcal{R}} 2 \left\{ \int_{\mathbb{R}^\nu} \left(\partial_{y_{\ell'}} \tilde{W}_{\ell\ell'}(x_\ell, y_{\ell'}) - \partial_{y_{\ell'}} \tilde{W}_{\ell\ell'}(0, y_{\ell'}), y_{\ell'} - \tilde{y}_{\ell'} \right)^2 \tilde{\mu}_{\ell,y}(\mathrm{d}x_\ell) \right\}^{1/2}$$
$$\leq 2\mathcal{R} \left(J_{\ell-\ell'}/||\mathbf{J}||_0\right) \sup_{||y||^R_\infty \leq \mathcal{R}} \left\{ \int_{\mathbb{R}^\nu} \left[|x_\ell|^{R-1} + \mathcal{L}_2|x_\ell|\right]^2 \tilde{\mu}_{\ell,y}(\mathrm{d}x_\ell) \right\}^{1/2}.$$

The integral in the last line tends to zero as $\iota := \beta^{1-R/P}||\mathbf{J}||_0$ gets small, which can be verified by the change of variables $x_\ell \mapsto \iota^{1/R} x_\ell$. Indeed, by (2.125)–(2.127) we have for $Q \geq 2$

$$\int_{\mathbb{R}^\nu} |x_\ell|^Q \tilde{\mu}_{\ell,y}(\mathrm{d}x_\ell) \leq \iota^{Q/R} \exp\left\{\beta_0(||\mathbf{J}||_0 C_W + B_2 - B_1) + \mathcal{R}\right\}$$
$$\times \frac{\int_{\mathbb{R}^\nu} |x_\ell|^Q \exp\left\{-A_1|x_\ell|^P + \frac{1}{2}\iota|x_\ell|^R\right\} \mathrm{d}x_\ell}{\int_{\mathbb{R}^\nu} \exp\left\{-A_2|x_\ell|^P - \frac{1}{2}\iota|x_\ell|^R\right\} \mathrm{d}x_\ell} =: \mathcal{I}_Q = \mathcal{O}(\iota^{Q/R}), \quad \iota \to 0,$$

uniformly for all $\ell \in \mathbb{L}$, $||y||^R_\infty \leq \mathcal{R}$. The proof is completed by setting

$$K_{\ell-\ell'} := \mathcal{R} \left(J_{\ell-\ell'}/||\mathbf{J}||_0\right) \left\{ (\mathcal{I}_{2R-2})^{1/2} + \mathcal{L}_2 (\mathcal{I}_2)^{1/2} \right\} = \mathcal{O}(\iota^{1/R}), \quad \iota \to 0.$$

■

2.3.3 Low-temperature uniqueness

In this subsection we consider the Hamiltonians (2.1) with the *unique ground state*, which is assumed to be *stable* in a certain sense. Based on the Dobrushin-Pechersky criterion, we provide an elementary proof of the uniqueness result for $\mu \in \mathcal{G}^{\mathrm{t}}$, which holds if the strength of the interaction is *small* ($||\mathbf{J}||_0 \ll 1$) or the inverse temperature is *large* ($\beta \gg 1$). The corresponding Theorems 2.28–2.30 might be viewed as a complementary to the Pirogov-Sinai theory, in so far as they cover the case of *non-translation invariant* interactions of *superquadratic growth.* On the other hand, there is a principal distinction from the high-temperature situation dealt with in Subsection 2.3.4. Namely, no reasonable type of interactions (including even the ferromagnetic ones $J|x_\ell - x_{\ell'}|^2 \geq 0$) can be treated by the original Dobrushin uniqueness theorem as $\beta \to \infty$. Technically this is caused by the fact that there are missing contraction estimates for $\mu_{\ell,y}(\mathrm{d}x_\ell)$ which could be valid *uniformly* for all boundary conditions $y \in \Omega$ (see the discussion in Subsection 2.3.5).

(i) Hamiltonians with the unique ground state

The following guarantees that the configuration $x \equiv 0$ will be the *unique ground state* for the system (2.1):

Assumption $(\mathbf{W}_3)$ *The pair potentials vanish at origin, i.e., $W_{\ell\ell'}(0,0) = 0$. Furthermore, they satisfy Assumption $(\mathbf{W})$ with $C_W = 0$, that means for all $\ell, \ell' \in \mathbb{L}$ and $x_\ell, x_{\ell'} \in \mathbb{R}^\nu$*

$$|W_{\ell\ell'}(x_\ell, x_{\ell'})| \leq \frac{1}{2} J_{\ell-\ell'}(|x_\ell|^R + |x_{\ell'}|^R). \tag{2.156}$$

Assumption $(\mathbf{V}_3)$ *The one-particle potentials possess the unique global minimum $V_\ell(0) = 0$, so that $V(x_\ell) > 0$ if $x_\ell \neq 0$. Furthermore, there exist*

$$P \geq R \geq 2, \quad A_3 \geq A_4 > \frac{3}{2}||\mathbf{J}||_0, \quad a_3 \geq a_4 > 0,$$

such that for all $\ell \in \mathbb{L}$ and $x_\ell \in \mathbb{R}^\nu$

$$A_4|x_\ell|^R + a_4|x_\ell|^2 \leq V_\ell(x_\ell) \leq A_3|x_\ell|^P + a_3|x_\ell|^2. \tag{2.157}$$

According to the above conditions, the local energies $H_\Lambda(x_\Lambda)$ attain the *global minimum* at $x \equiv 0$ and their behavior in the neighborhood of zero is essentially determined by the quadratic terms in the left- and right-hand sides of (2.157). We stress that $H_\Lambda(x_\Lambda)$ are allowed to be globally *non-convex functions* as well as to have *other critical points* and *local extrema* away from zero. Our first result here controls the uniqueness on a temperature interval $\beta \in [\beta^0, \infty)$ by small values of $||\mathbf{J}||_0$.

Theorem 2.28 *Let $\mathbb{L} := \mathbb{Z}^d$ and suppose that Assumptions $(\mathbf{V}_3)$, $(\mathbf{J}_{\mathrm{fin}})$, and $(\mathbf{W}_3)$ hold. Then, for every $\beta^0 > 0$ one finds $\mathcal{J} := \mathcal{J}(\beta^0) > 0$ such that the set $\mathcal{G}^{\mathrm{t}}$ is singleton at all values $\beta \geq \beta^0$ and $||\mathbf{J}||_0 \leq \mathcal{J}$.*

If the one-particle potentials are identical, i.e., $V_\ell \equiv V$ for all $\ell \in \mathbb{L}$, one easily can apply the *Laplace integral method* (cf. Section II in [106]). The following conditions on the phase $V \in C(\mathbb{R}^\nu)$ are typical in that case:

Assumption $(\mathbf{V}_4)$ *There exist $A_4 > 0$ and $R \geq 2$ such that for all $x_\ell \in \mathbb{R}^\nu$*

$$V(x_\ell) \geq A_4 |x_\ell|^R. \tag{2.158}$$

Furthermore, V has a unique, non-degenerate global minimum $V(0) = 0$. This means that $V \in C^3(\mathcal{U})$ in a zero's neighborhood $\mathcal{U} \subset \mathbb{R}^\nu$, its gradient $V'(0) = 0$, and the corresponding matrix of second derivatives (Hessian) is positive definite

$$V''(0) := \left(\partial^2_{x^i_\ell x^j_\ell} V(0)\right)_{i,j=1}^{\nu} \geq (a_4/2) \cdot \mathbf{Id}_\nu > 0. \tag{2.159}$$

Theorem 2.29 *The uniqueness result of Theorem 2.28 is true under Assumptions $(\mathbf{V}_4), (\mathbf{J}_{\text{fin}})$, and $(\mathbf{W}_3)$.*

Both these theorems will be proved in Subsection 2.3.3 (ii). We now examine whether $|\mathcal{G}^{\text{t}}| = 1$ can be achieved, fixed all other parameters, alone by the *law temperature β^{-1}*. To this end, we impose a stronger version of Assumptions $(\mathbf{W}_1)$, $(\mathbf{W}_3)$:

Assumption $(\mathbf{W}_4)$ *For all $\ell, \ell' \in \mathbb{L}$ and $x_\ell, x_{\ell'} \in \mathbb{R}^\nu$*

$$|W_{\ell\ell'}(x_\ell, x_{\ell'})| \leq \frac{1}{2} J_{\ell-\ell'} \left(|x_\ell|^R + |x_{\ell'}|^R\right), \tag{2.160}$$

$$|W_{\ell\ell'}(x_\ell, x_{\ell'}) - W_{\ell\ell'}(0, x_{\ell'})| \leq \frac{1}{2} J_{\ell-\ell'} |x_\ell| \cdot \left(|x_\ell|^{R-1} + |x_{\ell'}|^{R-1}\right). \tag{2.161}$$

In this situation we have the following complement to the previous statements.

Theorem 2.30 *Suppose that Assumptions $(\mathbf{V}_3)$, $(\mathbf{J}_{\text{fin}})$ and $(\mathbf{W}_4)$ hold with $P \geq R > 2$.*
Then, for each $\beta^0 > 0$ and $\mathcal{J}_0 < A_4/\left(\mathrm{a}^{\mathrm{b}} + 1/2\right)$ one finds a proper $\varsigma_0 := \varsigma_0(\beta^0, \mathcal{J}_0) > 0$ such that the corresponding set $\mathcal{G}^{\text{t}}$ is singleton at all values of $\beta \geq \beta^0$ and $||\mathbf{J}||_0 \leq \mathcal{J}_0$ related by

$$\beta^{1-R/2} ||\mathbf{J}||_0 =: \varsigma \leq \varsigma_0. \tag{2.162}$$

Remark 2.31 A periodic configuration $x \in \Omega$ (in our case, $x \equiv 0$) is said to be a *ground state* of $H(x)$ if it minimizes all local Hamiltonians $H_\Lambda(x)$, $\Lambda \Subset \mathbb{L}$, defined by (2.24) (cf. [259]). In contrast to systems with finite spin spaces, the uniqueness of the ground state itself does not yet entail the low-temperature uniqueness of the Gibbs measures in our model (see related examples constructed in [229]). To gain the uniqueness of $\mu \in \mathcal{G}^{\text{t}}$, one should impose additional *stability properties* of the ground state, like that in (2.159) claiming that the global minimum of V has *the positive mass.* If the potential V has another local minima away from zero, there is a possibility of phase transitions (in dimensions $d \geq 3$) at intermediate temperatures $\beta_* \leq \beta \leq \beta^*$ (cf. [93]) and typically there is uniqueness of $\mu \in \mathcal{G}^{\mathrm{t}}$ for small $\beta \to +0$ (cf. Theorem 2.28). The most powerful and universal method for studying the low temperature behavior in spin systems with multiple phases is the *Pirogov-Sinai theory*; see e.g. [200, 259, 292] for its in-depth presentation and [182, 205, 263] for the concrete applications to the uniqueness problem. To some extent our approach is an *elementary counterpart* to this theory, which nevertheless allows us to cover unbounded spin spaces, superquadratic interactions, and non-translation invariant Gibbs states.

(ii) Uniqueness by small $||\mathbf{J}||_0$: proof of Theorems 2.28 and 2.29

Again, we shall verify Conditions $(\mathbf{DP}_1)$ and $(\mathbf{DP}_2)$ of the Dobrushin–Pechersky criterion for the modified specification (2.123) instead of the initial one (2.17). We look at the corresponding family of one-point conditional distributions (2.129) depending on the parameter $\gamma \in [0, 1]$.

Condition $(\mathbf{DP}_1)$: The proof will be based on a refinement of the exponential bound (2.52) for $\beta \to \infty$, which makes the context of Lemma 2.52 below. It says that under Assumptions $(\mathbf{V}_3)$, $(\mathbf{W}_3)$ there exists $\Gamma^0 := \Gamma^0(\beta^0, \mathcal{J}_0) \geq 1$ (which is explicitly given by (2.267)–(2.269), such that

$$\int_\Omega \exp\left\{\beta\kappa|x_\ell|^R\right\}\mu_{\ell,y}(\mathrm{d}x) \leq \Gamma^0 \exp\left\{\beta \sum_{\ell' \in \partial_r(\ell)} J_{\ell-\ell'}|y_{\ell'}|^R\right\} \tag{2.163}$$

simultaneously for all $\beta \geq \beta^0$, $||\mathbf{J}||_0 \leq \mathcal{J}_0 < 2A_4/3$, and $\kappa \leq A_4 - \mathcal{J}_0/2$. If Assumption $(\mathbf{V}_4)$ holds instead of $(\mathbf{V}_3)$, the validity of (2.163) is stated in Remark 2.54. By (2.129) the above bound is equivalent to the following one

$$\int_\Omega \exp\left\{\kappa||\mathbf{J}||_0^{-\gamma}|x_\ell|^R\right\}\tilde{\mu}_{\ell,y}(\mathrm{d}x_\ell) \leq \Gamma^0 \exp\left\{\sum_{\ell' \in \partial_r(\ell)} J_{\ell-\ell'}||\mathbf{J}||_0^{-\gamma}|y_{\ell'}|^R\right\},$$

which immediately implies

$$\int_\Omega |x_\ell|^R \tilde{\mu}_{\ell,y}(\mathrm{d}x_\ell) \leq \kappa^{-1}\left[||\mathbf{J}||_0^\gamma \log \Gamma^0 + \sum_{\ell' \in \partial_r(\ell)} J_{\ell-\ell'}|y_{\ell'}|^R\right]. \tag{2.164}$$

Picking here any $\kappa \in (\mathrm{a}^{\mathrm{b}}\mathcal{J}_0, A_4 - \mathcal{J}_0/2]$, we get Condition $(\mathbf{DP}_1)$ with the constants $\mathcal{I} := \mathcal{J}_0/\kappa < 1/\mathrm{a}^{\mathrm{b}}$ and $\mathcal{C} := \kappa^{-1}||\mathbf{J}||_0^\gamma \log \Gamma^0$. It remains to set $\gamma = 0$, fix some $\mathcal{K} < 1$, and find the corresponding value of $\mathcal{R} := \mathcal{R}(\mathcal{C}, \mathcal{I}, \mathcal{K})$.

Condition $(\mathbf{DP}_2)$ is checked in perfect analogy with the proof of Theorem 2.25. Substituting $\Gamma_0 := \Gamma^0$ and $C_W = 0$ in (2.141), we get the following bound with any positive $\epsilon < \kappa - \mathcal{J}_0$

$$\mathcal{I}_{\ell\ell'} \leq (J_{\ell-\ell'}/\epsilon) \cdot \Gamma^0 \exp\left\{(2\mathcal{J}_0 + \epsilon)\mathcal{R}\right\}. \tag{2.165}$$

Hence, for all $\beta \geq \beta^0$ and $||\mathbf{J}||_0 < \mathcal{J}_0$,

$$\frac{1}{2}||\tilde{\mu}_{\ell,y} - \tilde{\mu}_{\ell,\tilde{y}}||_{\mathrm{TV}} \leq K_{\ell-\ell'} := J_{\ell-\ell'}\mathcal{C}_{2.166}, \tag{2.166}$$

where the constant $\mathcal{C}_{2.166} := \mathcal{C}_{2.166}(\beta^0, \mathcal{J}_0, \mathcal{R})$ can be written explicitly from (2.140) and (2.165). By choosing small enough $||\mathbf{J}||_0 < \mathcal{J}(\beta^0) < \mathcal{J}_0$, we make the norm of matrix $(K_{\ell-\ell'})_{\mathbb{L}\times\mathbb{L}}$ smaller than a given $\mathcal{K} < 1$ and thus prove Condition $(\mathbf{DP}_2)$. ■

(iii) Uniqueness by large β: proof of Theorem 2.30

Condition $(\mathbf{DP}_1)$ has been already examined by proving Theorem 2.29. Now it suffices to put everywhere $\gamma = 1$. As is clear from (2.164) and (2.267)–(2.269), the constant Γ^0 (and hence $\mathcal{I}$, $\mathcal{C}$, and $\mathcal{R}$) may be taken the same for all $\beta \geq \beta^0$ and $||\mathbf{J}||_0 > 0$ satisfying the constraint $\beta^{1-R/2}||\mathbf{J}||_0 =: \varsigma \leq \varsigma_0$. To check Condition $(\mathbf{PD_2})$ we proceed analogously to the proof of Theorem 2.26. From here let us fix some $\mathcal{R} \geq 1$. Keeping the same notation and repeating the estimates (2.145)–(2.148) with $C_1 = 0$, we find that

$$\mathcal{I}_{\ell\ell'} \leq \frac{\varsigma^{1/R}\int\limits_{\mathbb{R}^\nu}\left(\varsigma^{1-1/R}|x_\ell|^R + \mathcal{R}|x_\ell|\right)\exp\left\{-a_4|x_\ell|^2 + \frac{3}{2}\mathcal{R}\varsigma^{1/R}|x_\ell|\right\}\mathrm{d}x_\ell}{\int\limits_{\mathbb{R}^\nu}\exp\left\{-\left[a_3|x_\ell|^2 + \beta^{1-P/2}A_3|x_\ell|^P + \frac{1}{2}\varsigma|x_\ell|^R + \frac{1}{2}\mathcal{R}\varsigma^{1/R}|x_\ell|\right]\right\}\mathrm{d}x_\ell}.$$

In doing so we have used Assumption $(\mathbf{V}_3)$ and made the change of variables $x_\ell \mapsto \varsigma^{1/R}x_\ell$. This tells us that $\sup_{\ell\in\mathbb{L},\ \ell'\in\partial_r(\ell)}\mathcal{I}_{\ell\ell'} = \mathcal{O}(\varsigma^{1/R})$ as $\varsigma \to 0$. The latter implies by (2.140) that, for each $\ell \in \mathbb{L}$ and $y, \tilde{y} \in \Omega$ obeying (2.131),

$$\frac{1}{2}||\tilde{\mu}_{\ell,y} - \tilde{\mu}_{\ell,\tilde{y}}||_{\mathrm{TV}} \leq K(\beta^0, \mathcal{J}_0) = \mathcal{O}(\varsigma^{1/R}) \text{ as } \varsigma \to 0. \tag{2.167}$$

Furthermore, this estimate is uniform with respect to all $\beta \geq \beta^0$ and $||\mathbf{J}||_0 < \mathcal{J}_0$ in the domain (2.162). Hence $(\mathbf{DP}_2)$ is obeyed, which completes the proof. ■

2.3.4 Dobrushin's uniqueness criterion

Here we clarify to what extent the *Dobrushin uniqueness theorem* could be applicable to the model (2.1). Three particular situations, which were not covered so far in the literature, will be considered:

- *superquadratic* interactions;
- quadratic interactions with *infinite* range;
- further improvements for *scalar* spins.

To this end, we follow the idea from the earlier author's papers [19]–[21], which suggested an *analytical* way to estimate the coefficients of Dobrushin's matrix via the functional (e.g., Poincaré or log-Sobolev) inequalities for the one-point conditional Gibbs distributions. Those papers were the first that could treat anharmonic systems (both in the classical and in the quantum cases) with the pair interactions $W_{\ell\ell'}(x_\ell, x_{\ell'})$

of *at most quadratic* growth and *finite* range, and their results are optimal in such general framework (unless no additional information on the structure of the interaction is available).

We conclude with a short historical comment. For the interacting systems of spins taking values in a compact Riemannian manifold, the similar idea to get a bound on Dobrushin's coefficients through the spectral gap for the associated Dirichlet operators goes back to [89, 285]. For the scalar ferromagnetic systems with the harmonic pair potentials $J_{\ell\ell'}(x_\ell - x_{\ell'})^2 \geq 0$, the Dobrushin matrix has been also estimated in [43, 71, 247, 289], however by using specific methods which are based on the correlations inequalities.

(i) Weak dependence and Dobrushin's contraction condition

We first recall the original statement due to R. Dobrushin (cf. Theorem 4 in [91]), but in the form adapted to our concrete setting.

Let $\mathbb{L}$ be any countable indexing set. Let the single spin space, in our case $X := \mathbb{R}^\nu$, be equipped with some metric ρ which makes it a *Polish space*. We suppose that the embedding $(\mathbb{R}^\nu, |\cdot|) \hookrightarrow (\mathbb{R}^\nu, \rho)$ is continuous, so that (by the general Kuratowski theorem, cf. [226], page 21, Theorem 3.9) the both Borel algebras induced on $\mathbb{R}^\nu$ respectively by $|\cdot|$ and ρ have to coincide. Analogously, for each $\Lambda \Subset \mathbb{L}$ we define $\Omega_\Lambda := \mathbb{R}^{\nu|\Lambda|}$ as a Polish space with the metric $\rho_\Lambda(x_\Lambda, \tilde{x}_\Lambda) := \sum_{\ell\in\Lambda} \rho(x_\ell, \tilde{x}_\ell)$. Let us given a local specification $\Pi := \{\pi_\Lambda\}$ consisting of the *probability* kernels $\pi_\Lambda(\mathrm{d}x|y) \in \mathcal{P}(\Omega)$, $\Lambda \Subset \mathbb{L}$, $y \in \Omega$, such that

$$\int_\Omega \rho(x_\ell, 0)\pi_\ell(\mathrm{d}x|y) < \infty. \tag{2.168}$$

In view of (2.21), such set-up could be satisfied only by the interactions of *finite range* (i.e., when $W_{\ell\ell'} \equiv 0$ as $|\ell - \ell'| > r$). The Dobrushin criterion presumes a *weak dependence* of the one-point distributions $\mu_{\ell,y}(\mathrm{d}x_\ell) := \mathbb{P}_\ell^{-1} \circ \pi_\ell(\mathrm{d}x|y)$ on the values of boundary conditions $y \in \Omega$ on sites $\ell' \neq \ell$. To this end, we introduce the (L^1-) *Wasserstein probability distance* related to the metric ρ (see e.g. [91, 101, 238, 280])

$$\mathbf{W}_\rho\left(\mu_{\ell,y}, \mu_{\ell,\tilde{y}}\right) := \sup_{f\in \mathrm{Lip}_1(\mathbb{R}^\nu,\rho)} \left| \int_{\mathbb{R}^\nu} f(x_\ell)\left[\mu_{\ell,y}(\mathrm{d}x_\ell) - \mu_{\ell,\tilde{y}}(\mathrm{d}x_\ell)\right]\right|, \tag{2.169}$$

where

$$\mathrm{Lip}_1(\mathbb{R}^\nu, \rho) := \left\{ f : \mathbb{R}^\nu \to \mathbb{R} \;\middle|\; [f]_\rho := \sup_{s\neq\tilde{s}} \frac{|f(s) - f(\tilde{s})|}{\rho(s, \tilde{s})} \leq 1 \right\} \tag{2.170}$$

is the unit ball in the space of Lipschitz continuous functions on $\mathbb{R}^\nu$. Then, in order that there is at most one *"tempered"* Gibbs measure $\mu \in \mathcal{G}$ with the *uniformly bounded moments*

$$\sup_\ell \int_\Omega \rho(x_\ell, 0)\mu(\mathrm{d}x_\ell) < \infty, \tag{2.171}$$

the fulfillment of the following is sufficient:

Contraction Condition $(\mathbf{D}_2)$ *There exist nonnegative constants* $D_{\ell\ell'}$, $\ell \neq \ell'$, *such that*

(i) The matrix $\mathbf{D} = (D_{\ell\ell'})_{\mathbb{L}\times\mathbb{L}}$ *is* $l^\infty(\mathbb{L})$*-contractive, i.e.,*

$$||\mathbf{D}||_0 := ||\mathbf{D}||_{\mathcal{L}(l^\infty(\mathbb{L}))} = \sup_{\ell} \sum_{\ell'(\neq\ell)} D_{\ell\ell'} < 1. \tag{2.172}$$

(ii) For each $\ell \in \mathbb{L}$ *and any pair of configurations* $y, \tilde{y} \in \Omega$

$$\mathbf{W}_\rho \left(\mu_{\ell,y}(\mathrm{d}x_\ell), \mu_{\ell,\tilde{y}}(\mathrm{d}x_\ell)\right) \leq \sum_{\ell'(\neq\ell)} D_{\ell\ell'}\rho(y_{\ell'}, \tilde{y}_{\ell'}). \tag{2.173}$$

What is the same, one has to check the l^∞-contractivity (2.172) of the *Dobrushin interdependence matrix* $\mathbf{D} = (D_{\ell\ell'})_{\mathbb{L}\times\mathbb{L}}$ with the entries

$$D_{\ell\ell'} := \sup_{\substack{y,\tilde{y}\in\Omega \\ y=\tilde{y} \text{ off } \ell'}} \left\{ \frac{\mathbf{W}_\rho\left(\mu_{\ell,y}\,,\, \mu_{\ell,\tilde{y}}\right)}{\rho(y_{\ell'}, \tilde{y}_{\ell'})} \right\}, \quad \ell \neq \ell'. \tag{2.174}$$

Remark 2.32 We stress the following issues:

(i) In the original proof of R. Dobrushin ([91]; see also [92, 94]) there remained an *open question* about measurability of the optimal couplings if the spin spaces are not longer discrete. In Subsection 4.4.1 we shall explain how to bridge this gap. In the later versions of Dobrushin's criterion (see [32, 108, 122, 176, 178]) this problem was partially overcome by using a dual scheme based on the Kantorovich-Rubinstein relation (4.194), (4.195) for Wasserstein distances. The same measurability question emerges in the proof of the Dobrushin-Pechersky criterion, see Remark 2.22.

(ii) The known proofs of Dobrushin's criterion exploit only the assertion that the *spectral radius* $r_{\mathrm{sp}}(\mathbf{D}) := \lim_{n\to\infty} ||\mathbf{D}^n||_0^{1/n}$ of the operator $\mathbf{D} := (D_{\ell\ell'})_{\mathbb{L}\times\mathbb{L}} \in l^\infty(\mathbb{L})$ is strictly smaller than 1. Since by a general fact $r_{\mathrm{sp}}(\mathbf{D}) \leq ||\mathbf{D}||_0$, the l^∞-contractivity condition (2.172) is even *stronger* than one needs. Nevertheless, (2.172) is more convenient for applications, in so far as it can be easily verified in terms of the Dobrushin coefficients $D_{\ell\ell'}$.

(iii) If the metric ρ introduced on $\mathbb{R}^\nu$ is *discrete* (i.e., $\rho(s, \tilde{s}) := \mathbf{1}_{s\neq s}$), then in Condition $(\mathbf{D_2})$ there appears the *variation distance* (2.114). This leads to the global version, with $\mathcal{R} = \infty$, of Condition $(\mathbf{PD_2})$ in the Dobrushin-Pechersky theorem (cf. Subsection 2.3.1), which however cannot hold for unbounded interactions.

(iv) A general way how to modify this criterion for interactions of possibly *infinite range* will be suggested by Theorem 4.46. In that case the kernels $\pi_\ell(\mathrm{d}x|y)$ are defined as elements from $\mathcal{P}(\Omega)$ only for some subset of *tempered configurations* $y \in \Omega^{\mathrm{t}}$, so that the above statement formally does not apply. In Subsection 4.4.2 below we also will be interested in the rate of convergence of $\pi_\Lambda(\mathrm{d}x|y)$ to $\mu \in \mathcal{G}^{\mathrm{t}}$ as $\Lambda \nearrow \mathbb{L}$, which is much stronger than the uniqueness result alone. Further applications of the weak dependence condition (2.172)–(2.174) to the mixing properties of the Gibbs measures and the ergodicity of the corresponding stochastic dynamics will be pointed out in Subsections 2.3.5 and 4.5.1.

(ii) Superquadratic interactions

It is well-known that the standard choice of $\rho(x_\ell, x_{\ell'}) = |x_\ell - x_{\ell'}|$ allows us to consider pair interactions of the form $w_{\ell\ell'}(x_\ell - x_{\ell'})$, where $w_{\ell\ell'} : \mathbb{R}^\nu \to \mathbb{R}$ are convex functions with $\sup_{\mathbb{R}^\nu} |w''_{\ell\ell'}| < \infty$ (cf. e.g. [20, 21]). It is naturally to ask whether it would be possible to cover the case of $R \geq 2$ by constructing a proper *non-Euclidean metric* ρ depending on the growth of the interaction. For simplicity, we here consider the *scalar spins* $x_\ell \in \mathbb{R}$ only. Merely speaking, the conditions imposed below will permit for $W_{\ell\ell'}(x_\ell, x_{\ell'})$ to have at the infinity not more than polynomial growth of the order $R \leq 1 + P/2$, whereas $V_\ell(x_\ell)$ are growing not slowly as $|x_\ell|^P$ with $P \geq 2$.

Assumption $(\mathbf{V}_5)$ *Suppose* $(\mathbf{V}_1)$ *holds, and let each of* V_ℓ *can be written as*

$$V_\ell = U_\ell + Q_\ell, \tag{2.175}$$

where, respectively, $U_\ell \in C^2(\mathbb{R})$ *is strictly convex and* $Q_\ell \in C_{\mathrm{b}}(\mathbb{R})$ *is globally bounded. Furthermore, this decomposition is uniform in the following sense: there exist* $a_U, A_U, \delta_Q > 0$ *such that for all* $\ell \in \mathbb{L}$ *and* $x_\ell \in \mathbb{R}$

$$U''_\ell(x_\ell) \geq a_U + A_U |x_\ell|^{P-2} \tag{2.176}$$

and

$$\mathbf{Osc} Q_\ell := \sup_{x_\ell} Q_\ell(x_\ell) - \inf_{x_\ell} Q_\ell(x_\ell) \leq \delta_Q. \tag{2.177}$$

Assumption $(\mathbf{W}_5)$ *Suppose that* $W_{\ell\ell'} \in C^2(\mathbb{R} \times \mathbb{R})$*, and let additionally to* $(\mathbf{W})$ *the following hold: there exist* $a_W \in \mathbb{R}$ *and* $b_W, B_W > 0$ *such that for all* $\ell, \ell' \in \mathbb{L}$ *and* $x_\ell, x_{\ell'} \in \mathbb{R}$

$$\begin{aligned} \partial^2_{x_\ell} W_{\ell\ell'}(x_\ell, x_{\ell'}) &\geq J_{\ell\ell'} a_W, \\ |\partial^2_{x_\ell x_{\ell'}} W_{\ell\ell'}(x_\ell, x_{\ell'})| &\leq J_{\ell\ell'} \left(b_W + B_W |x_\ell|^{R-2} + B_W |x_{\ell'}|^{R-2} \right), \end{aligned} \tag{2.178}$$

where $\alpha := a_U + a_W ||\mathbf{J}||_0 > 0$ *and* $2 \leq R \leq 1 + P/2$.

It is clear that the above conditions (2.176) and (2.177) are *mutually competitive*. Namely, the bounded perturbations Q_ℓ may produce multiple wells of the potential energy responsible for phase transitions, while the strictly convex terms U_ℓ ensure the uniqueness. The next result states the uniqueness of $\mu \in \mathcal{G}^{\mathrm{t}}$ if the pair interaction is not too strong.

Theorem 2.33 *Let* $\nu = 1$*, and consider the spin system (2.1) with the interaction of finite range satisfying Assumptions* $(\mathbf{V}_5)$*,* $(\mathbf{J}_1)$*, and* $(\mathbf{W}_5)$*. Then, for every* $\beta_0 > 0$ *one finds a proper* $\mathcal{J} := \mathcal{J}(\beta_0)$ *such that the set* $\mathcal{G}^{\mathrm{t}}$ *is singleton at all values* $\beta \leq \beta_0$ *and* $||\mathbf{J}||_0 < \mathcal{J}$*.*

Proof. Let us introduce a new metric on $\mathbb{R}$ by

$$\rho(s,\tilde{s}) := \left|\int_s^{\tilde{s}} \varphi(t)\mathrm{d}t\right| \quad \text{where} \;\; \varphi(t) := (\alpha + A_U t^{P-2})^{1/2}. \tag{2.179}$$

By Theorems 2.14 and 2.15, the set $\mathcal{G}^{\mathrm{t}}$ is nonempty and all its elements obey the moment estimate (2.171). Fix any function $f \in C^1(\mathbb{R}) \cap \mathrm{Lip}_1(\mathbb{R},\rho)$, which thus satisfies $|f'(s)| \le \varphi(s)$, $s \in \mathbb{R}$. Let us given some distinct $\ell, \ell' \in \mathbb{L}$ and $y, \tilde{y} \in \Omega$ coinciding off ℓ'. Recall that by (2.155)

$$\partial_{y_{\ell'}} \int_{\mathbb{R}} f(x_\ell)\mu_{\ell,y}(\mathrm{d}x_\ell) = -\beta \mathbf{Cov}_{\mu_{\ell,y}(\mathrm{d}x_\ell)} \left\{ f(x_\ell); \partial_{y_{\ell'}} W_{\ell\ell'}(x_\ell, y_{\ell'}) \right\}. \tag{2.180}$$

Applying the mean-value theorem (cf. Theorem 1, §2, Section I in [70]), we find from (2.179), (2.180) that

$$\begin{aligned} &\left| \int_{\mathbb{R}^\nu} f(x_\ell) \left[\mu_{\ell,y}(\mathrm{d}x_\ell) - \mu_{\ell,\tilde{y}}(\mathrm{d}x_\ell)\right] \right| \cdot \rho^{-1}(y_{\ell'}, \tilde{y}_{\ell'}) \qquad (2.181)\\ \le\;& \beta \sup_{y\in\Omega} \left\{ \left| \mathbf{Cov}_{\mu_{\ell,y}} \left(f(x_\ell); \partial_{y_{\ell'}} W_{\ell\ell'}(x_\ell, y_{\ell'}) \right) \right| \cdot \varphi^{-1}(y_{\ell'}) \right\} \\ \le\;& \beta \sup_{y\in\Omega} \left\{ (\mathbf{Var}_{\mu_{\ell,y}} f)^{1/2} (\mathbf{Var}_{\mu_{\ell,y}} \partial_{y_{\ell'}} W_{\ell\ell'}(x_\ell, y_{\ell'}))^{1/2} \varphi^{-1}(y_{\ell'}) \right\}. \end{aligned}$$

Let us first assume that $Q_\ell = 0$, which implies by (2.176)–(2.179) that

$$\partial^2_{x_\ell} H_\ell(x_\ell|y) = U''_\ell(x_\ell) + \sum\nolimits_{\ell'(\neq\ell)} \partial^2_{x_\ell} W_{\ell\ell'}(x_\ell, y_{\ell'}) \ge \varphi^2(x_\ell) > 0, \tag{2.182}$$

and hence $\mu_{\ell,y}$ is the *log-concave* measure on $\mathbb{R}$. This enables us to use the *Brascamp-Lieb inequality* (cf. Theorem 4.1 in [65]) in the form

$$\mathbf{Var}_{\mu_{\ell,y}} f \le \frac{1}{\beta} \int_{\mathbb{R}} |f'(x_\ell)|^2 \left(\partial^2_{x_\ell} H_\ell(x_\ell|y)\right)^{-1} \mu_{\ell,y}(\mathrm{d}x_\ell) \le \frac{1}{\beta}[f]^2_\rho. \tag{2.183}$$

In particular, by (2.178), (2.179)

$$\begin{aligned} &\mathbf{Var}_{\mu_{\ell,y}(\mathrm{d}x_\ell)} \partial_{y_{\ell'}} W_{\ell\ell'}(x_\ell, y_{\ell'}) \qquad (2.184)\\ &\le \frac{1}{\beta}\varphi^2(y_{\ell'}) \sup_{x_\ell, y_{\ell'} \in\mathbb{R}} \left\{ \frac{\partial^2_{x_\ell y_{\ell'}} W_{\ell\ell'}(x_\ell, y_{\ell'})}{\varphi(x_\ell)\varphi(y_{\ell'})} \right\}^2 \le \frac{1}{\beta} C_{2.184} J_{\ell\ell'} \varphi^2(y_{\ell'}), \end{aligned}$$

with $C_{2.184} := 4(B_W + c_W)(A_U^{-1} + \alpha^{-1})$. Adding a bounded potential Q_ℓ with the total oscillation $\delta_Q < \infty$, leads by the well-known *perturbation argument* (cf. Lemma 1.2 in [186]) to the extra factor $\exp(2\beta\delta_Q)$ in the right-hand side of (2.183) and (2.184). Hence, in the situation described by Assumptions $(\mathbf{V}_5)$ and $(\mathbf{W}_5)$, we have that

$$\rho^{-1}(y_{\ell'}, \tilde{y}_{\ell'}) \cdot \left| \int_{\mathbb{R}^\nu} f(x_\ell) \left[\mu_{\ell,y}(\mathrm{d}x_\ell) - \mu_{\ell,\tilde{y}}(\mathrm{d}x_\ell)\right] \right| \le C_{2.184} J_{\ell\ell'} e^{2\beta\delta_Q}. \tag{2.185}$$

By the standard approximation by convolutions (for details see the proof of Theorem 4.61) this estimate extends to all $f \in \mathrm{Lip}_1(\mathbb{R}, \rho)$. Thus, the Dobrushin coefficients are bounded by the right-hand side in (2.185), i.e.,

$$D_{\ell\ell'} \le C_{2.186} J_{\ell\ell'}, \quad C_{2.186} := 4e^{2\beta\delta_Q}(B_W + c_W)(A_U^{-1} + \alpha^{-1}). \tag{2.186}$$

By choosing sufficiently small $||\mathbf{J}||_0 < \mathcal{J}(\beta_0)$, this implies Contraction Condition $(\mathbf{D_2})$ at all values $\beta \le \beta_0$. ■

Remind that for $\mathbb{L} := \mathbb{Z}^d$ and $R \le P$ the uniqueness was already treated by means of the Dobrushin-Pechersky criterion, see Theorems 2.23, 2.25, and 2.26. In contrary, the proof of Theorem 2.33 demonstrates that the Dobrushin criterion is applicable only under the *essential restriction* $R < 1 + P/2$.

(iii) Quadratic interactions with infinite range

In this case Dobrushin's criterion is not applicable in the original form, therefore we shall use its modification given by Theorem 4.46 below.

(a) Vector spins, $\nu \ge 1$: Consider the system of ν-dimensional spins interacting via the pair potentials $W_{\ell\ell'}(x_\ell, x_{\ell'}) := w_{\ell\ell'}(x_\ell - x_{\ell'})$ of at most quadratic growth. Suppose that the functions $w_{\ell\ell'} \in C^2(\mathbb{R}^\nu)$ fulfill the operator estimate on their second derivatives with certain $a_W, b_W \in \mathbb{R}$

$$J_{\ell\ell'} a_W \cdot \mathbf{Id}_\nu \le w''_{\ell\ell'}(x_\ell) \le J_{\ell\ell'} b_W \cdot \mathbf{Id}_\nu, \tag{2.187}$$

which corresponds to the choice of $P \ge R = 2$ and $A_U = B_W = 0$ in Assumptions $(\mathbf{V}_5)$ and $(\mathbf{W}_5)$. The one-particle potentials $V_\ell = U_\ell + Q_\ell$ are *uniformly convex at infinity*, that means

$$\begin{gathered} U''_\ell(x_\ell) \ge a_U \mathbf{Id}_\nu > 0, \quad \mathbf{Osc}\ Q_\ell \le \delta_Q < \infty, \\ \alpha := a_U + a_W ||\mathbf{J}||_0 > 0. \end{gathered} \tag{2.188}$$

The matrix $\mathbf{J}$ has possibly *infinite range* $r \le \infty$ and satisfies Assumption $(\mathbf{J})$.

Theorem 2.34 *The set $\mathcal{G}^{\mathrm{t}}$ is singleton if the following relation holds:*

$$a_U ||\mathbf{J}||_0^{-1} + a_W > b_W e^{2\beta\delta_Q}. \tag{2.189}$$

Proof. For $p > d$ and $\varepsilon \in [0, 1]$, we define (cf. 2.71)

$$||\mathbf{J}||_{p,\varepsilon} := \sup_\ell \sum_{\ell'} J_{\ell\ell'}(1 + \varepsilon|\ell - \ell'|)^p \le ||\mathbf{J}||_p < \infty.$$

As was already shown in the proof of Lemma 2.12 (cf. (2.69)), for any $\iota > 0$ one finds a small enough $\varepsilon > 0$ such that

$$||\mathbf{J}||_{p,\varepsilon} - ||\mathbf{J}||_0 < \iota. \tag{2.190}$$

The metric ρ on the spin space $\mathbb{R}^\nu$ is just the Euclidean one. The coefficients $D_{\ell\ell'}$ are defined by (2.174), whereby the supremum is taken over all $y, \tilde{y} \in \Omega^{\mathrm{t}}$. To estimate them we now use the *Poincaré inequality*

$$\mathbf{Var}_{\mu_{\ell,y}(\mathrm{d}x_\ell)} f \leq \frac{1}{C_{\mathrm{SG}}} \int_{\mathbb{R}^\nu} |f'(x_\ell)|^2 \mu_{\ell,y}(\mathrm{d}x_\ell), \quad \forall f \in C^1_{\mathrm{b}}(\mathbb{R}^\nu), \tag{2.191}$$

which is valid uniformly for all $\mu_{\ell,y}$ with the (*spectral gap*) constant

$$C_{\mathrm{SG}} \geq \beta e^{-2\beta\delta_Q} (a_U + a_W ||\mathbf{J}||_0) > 0 \tag{2.192}$$

(cf. Corollary 1.4 in [186]). Repeating the previous proof with $\varphi \equiv 1$, we arrive at the following bounds

$$D_{\ell\ell'} \leq \frac{1}{C_{\mathrm{SG}}} \beta J_{\ell\ell'} b_W, \quad ||\mathbf{D}||_{p,\varepsilon} \leq e^{2\beta\delta_Q} \frac{b_W ||\mathbf{J}||_{p,\varepsilon}}{(a_U + a_W ||\mathbf{J}||_0)}. \tag{2.193}$$

Thus, by (2.189) and (2.190) we achieve that $||\mathbf{D}||_{p,\varepsilon} < 1$ and hence by Theorem 4.46 get the uniqueness in the class of all $\mu \in \mathcal{G}^{\mathrm{t}}$ satisfying

$$\sup_\ell (1 + |\ell|)^{-p} \mathbf{E}_\mu(|x_\ell|) < \infty, \quad \forall p > d.$$

Since by Theorem 2.15 this class coincides with $\mathcal{G}^{\mathrm{t}}$, we thus conclude that $|\mathcal{G}^{\mathrm{t}}| = 1$. ■

Remark 2.35 In the simplest way the uniqueness criterion can be written down in the case of $\mathbb{L} := \mathbb{Z}^d$ and the *nearest-neighbor interaction*

$$W_{\ell\ell'}(x_\ell, x_{\ell'}) := J|x_\ell - x_{\ell'}|^2/2 \quad \text{as} \quad |\ell - \ell'| = 1.$$

Now $a_W = b_W = 1$ and $||\mathbf{J}||_0 = 2dJ$, so that

$$D_{\ell\ell'} \leq e^{2\beta\delta_Q} \frac{J}{(a_U + 2dJ)} \quad \text{as} \quad |\ell - \ell'| = 1, \quad ||\mathbf{D}||_0 \leq e^{2\beta\delta_Q} \frac{2dJ}{(a_U + 2dJ)}, \tag{2.194}$$

and the sufficient condition (2.189) reads as

$$e^{2\beta\delta_Q} < 1 + \frac{a_U}{2dJ}. \tag{2.195}$$

All this justifies that the uniqueness of $\mu \in \mathcal{G}^{\mathrm{t}}$ can be obtained by choosing sufficiently small one of the following parameters: the inverse temperature β, the intensity of the pair interaction J, or the total oscillation δ_Q of the perturbations Q_ℓ. For the discussion of the quantum case and some illustrative examples see Subsection 3.2.5.

(b) Scalar case, $\nu = 1$: There is a possibility to extend the previous framework by using the results about spectral gap estimates for probability measures on the real line recently obtained in [121]. Such technique is based on *Hardy*-type analytical criteria for the Poincaré inequality (2.191) and is restricted to the *scalar spins* $x_\ell \in \mathbb{R}$. Namely, let each of the one-particle potentials possess a decomposition

$$V_\ell = U_\ell + Q_\ell + \varPhi_\ell, \tag{2.196}$$

where U_ℓ and Q_ℓ are the same as before and $\varPhi_\ell \in C(\mathbb{R})$ fulfills for $\beta \leq \beta^* < \infty$

$$\gamma(\beta) := \sup_\ell \int_{\mathbb{R}} \left(e^{\beta|\varPhi_\ell(x_\ell)|} - 1 \right) \mathrm{d}x_\ell < \infty. \tag{2.197}$$

Then, by Theorems 3.4 and 4.1 in [121], each of the one-point measures $\mu_{\ell,y}(\mathrm{d}x_\ell)$ satisfies the Poincaré inequality (2.191) with the uniform constant

$$C_{\mathrm{SG}} \geq \frac{1}{4} e^{-2\beta\delta_Q} \left[1 + 4(\alpha\beta)^{-1} + \gamma(\beta) \right]^{-2}, \tag{2.198}$$

where $\alpha > 0$ was defined in (2.188). For the Dobrushin coefficients this implies the following modification of the bound (2.193)

$$D_{\ell\ell'} \leq 4 J_{\ell\ell'} b_W \beta e^{2\beta\delta_Q} \left[1 + 4(\alpha\beta)^{-1} + \gamma(\beta) \right]^{2}. \tag{2.199}$$

Hence, for β varying in the compact interval $[\beta_*, \beta^*] \subset \mathbb{R}_+$, the uniqueness can be achieved by taking small enough $||\mathbf{J}||_0 < \mathcal{J}(\beta_*, \beta^*)$. The estimate (2.198) is, however, too rough to study the asymptotics of C_{SG} as $\beta \to +0$. Furthermore, it does not yet give the basic result (2.192) as $\varPhi_\ell \equiv 0$. So, the results obtained here do not seem to be final. Typical examples of such perturbations Φ_ℓ have been constructed in the quoted paper. A new issue as compared with Theorem 2.33 is that the resulting V_ℓ might be outside the standard class of convex at infinity potentials.

2.3.5 Dirichlet operators, spectral gaps, and decay of correlations

It is a well known and remarkable fact (cf. [95, 96], [202]–[204], [269, 270]) that in the *compact spin* setting the following properties are *equivalent* for any Gibbs specification $\varPi = \{\pi_\Lambda\}_{\Lambda \Subset \mathbb{L}}$: (i) the Dobrushin-Shlosman *"constructive criterion"* generalizing $(\mathbf{D_2})$; (ii) the *exponential decay of correlations* for all $\pi_\Lambda(\mathrm{d}x|y)$, uniformly in the volume and boundary condition; (iii) the *exponential relaxation* of the corresponding Glauber dynamics, expressed by means of the *log-Sobolev* and *Poincaré inequalities* for $\pi_\Lambda(\mathrm{d}x|y)$. In the literature the above list of properties is usually referred to as the *complete analyticity.* In statistical mechanics, such properties indicate the absence of phase transitions and bring together the notions of *thermal* and *dynamical equilibrium.* In the series of papers of N. Yoshida [289]– [291] this equivalence was extended to classical *ferromagnetic systems* with unbounded scalar spins. In doing so, proper correlation inequalities were heavily used, which however does not work for more general interactions. So far, the relations between (i)–(iii) for anharmonic lattice systems are not fully established. Here we briefly discuss possible approaches to the mentioned problems in context of the model (2.1). In more detail this topic will be continued in Sections 4.3–4.5.

(i) Dobrushin contraction technique in the infinite volume

In the framework of his uniqueness criteria, R. Dobrushin has elaborated a special comparison method for Gibbsian fields (cf. Theorems 3 and 4 in [91]). Among standard applications of this method (see e.g. [94, 95, 108, 134, 176]) is a result about the *exponential decay* of truncated correlations $\mathbf{Cov}_\mu(x_\ell; x_{\ell'})$, as the distance $|\ell - \ell'|$ gets large, for the (*unique*) Gibbs measure $\mu \in \mathcal{G}^{\mathrm{t}}$.

Let us turn to the situation of $\nu = 1$ and $R \geq 2$ dealt with in Subsection 2.3.4 (ii). Set

$$\mathcal{R} := \sup_\ell \int_\Omega \rho^2(x_\ell, 0)\mu(\mathrm{d}x_\ell), \tag{2.200}$$

which is finite by (2.81) and (2.179). Since $||\mathbf{D}||_0 < 1$ and $D_{\ell\ell'} = 0$ as $|\ell - \ell'| > r$, one finds a small enough $\sigma > 0$ such that

$$||\mathbf{D}||_\sigma := \sup_\ell \sum_{\ell'} D_{\ell\ell'} \exp\{-\sigma|\ell - \ell'|\} < 1. \tag{2.201}$$

For a given domain $\Lambda \Subset \mathbb{L}$, let us consider cylinder functions $f,\ g \in \mathrm{Lip}(\mathbb{R}^\Lambda, \rho)$ which have finite Lipschitz seminorms (generalizing (2.170))

$$[f]_{\Lambda,\rho} := \sum_{\ell\in\Lambda} \left[\sup_{x=\tilde{x} \text{ off } \ell} \frac{|f(x) - f(\tilde{x})|}{\rho(x_\ell, \tilde{x}_\ell)} \right]. \tag{2.202}$$

Then, Theorem 3.7 and Corollary 1.7 in [108] say that (2.200)–(2.202) imply the *exponential mixing property*

$$|\mathbf{Cov}_\mu(f; g \circ t_{\ell'})| \leq C_{2.203} \exp(-\sigma|\ell'|)\, [f]_{\Lambda,\rho}[g]_{\Lambda,\rho}, \tag{2.203}$$

where $C_{2.203} := \mathcal{R}(1 - ||\mathbf{D}||_\sigma)^{-1}$ and $t_{\ell'}$ is the shift along direction ℓ' (i.e., $(t_{\ell'}x)_\ell := x_{\ell+\ell'}$, $\ell \in \mathbb{L}$). In particular, for $\beta \leq \beta_0$ and $||\mathbf{J}||_0 \leq \mathcal{J}(\beta_0)$, the spin-spin correlations are exponentially decaying as

$$|\mathbf{Cov}_\mu(x_\ell; x_{\ell'})| \leq C_{2.204} \exp\{-\sigma|\ell - \ell'|\} \tag{2.204}$$

with $C_{2.204} := C_{2.203}\,(a_U + a_W||\mathbf{J}||_0)^{-1}$. Under the assumptions of Lemma 2.49, one further observes that $C_{2.203}(\beta) = \mathcal{O}\left(\beta^{-P/R}\right)$ as $\beta \to +0$, which gives the same β-asymptotics for the covariances in (2.203) and (2.204). In the case of $\nu \geq 1$ and $R = 2$ considered in Subsection 2.3.4 (iii), we have respectively the asymptotic $\mathcal{O}(\beta^{-1})$ as $\beta \to +0$.

Remark 2.36 The estimate (2.203) trivially includes the *main result* of the paper [28], which was concerned with the *convex superquadratic* potentials $V_\ell(x_\ell) := (1 + x_\ell^2)^{2\mathrm{q}+1}$ and $W_{\ell\ell'}(x_\ell, x_{\ell'}) := J_{\ell-\ell'}(x_\ell - x_{\ell'})^{2\mathrm{q}+2}$ with $\mathrm{q} \geq 1$.

(ii) Dirichlet forms and operators

Here we point out some intrinsic connections with the *theory of Dirichlet operators* being an important part of the modern stochastic analysis. This material constitutes a background for the analytical approach to the Gibbs states $\mu \in \mathcal{G}^{\mathrm{t}}$ which will be developed in Chapter 4.

Let $\mu \in \mathcal{G}^{\mathrm{t}}$ be any (possibly *non-unique*) Gibbs measure corresponding to the Hamiltonian (2.1). Let the potentials V_ℓ and $W_{\ell\ell'}$ be regular enough, so that the mappings

$$\Omega_p \ni x \mapsto b_\ell(x) := -\beta \left[V_\ell'(x_\ell) + \sum_{\ell'(\neq \ell)} \partial_{x_\ell} W_{\ell\ell'}(x_\ell, x_{\ell'}) \right] \in \mathbb{R}^\nu \tag{2.205}$$

are well defined and continuous for each $p > d$. The vector field $b := (b_\ell)_{\ell \in \mathbb{L}} : \Omega_p \to \Omega$ is called the *logarithmic derivative* of the measure μ and its components $b_\ell^i : \Omega_p \to \mathbb{R}$ respectively the *partial logarithmic derivatives* along the basic directions $e_\ell^i := \{\delta_{\ell\ell'}\delta_{ii'} \mid \ell' \in \mathbb{L}, 1 \le i' \le \nu\}$ in Ω. Furthermore, we assume that $|b_\ell| \in L^2(\mu)$ for each ℓ, which could be concluded from the à-priori bound (2.81) by knowing the growth of V_ℓ and $W_{\ell\ell'}$. In particular, this is the case if V_ℓ and $W_{\ell\ell'}$ satisfy Assumptions $(\mathbf{V}_8)$ and $(\mathbf{W}_8)$ from Subsection 4.5.3 below. Denote by $\mathcal{F}C_{\mathrm{b}}^\infty(\Omega)$ the set of all *smooth cylinder functions* $f : \Omega \to \mathbb{R}$ which can be represented as $f(x) = f_\Lambda(x_\Lambda)$ with some $\Lambda \Subset \mathbb{L}$ and $f_\Lambda \in C_{\mathrm{b}}^\infty(\mathbb{R}^{\nu|\Lambda|})$. It is well-known that $\mathcal{F}C_{\mathrm{b}}^\infty(\Omega)$ is dense in all Banach spaces $L^q(\mu)$, $1 \le q \le \infty$. Consider a differential expression

$$\mathbb{H}_\mu f(x) = -\sum_\ell \left[\Delta_\ell f + (b_\ell, \partial_{x_\ell} f)\right], \quad f \in \mathcal{F}C_{\mathrm{b}}^\infty(\Omega), \tag{2.206}$$

which satisfies the *integration by parts formula* (resulting from Proposition 2.37)

$$(\mathbb{H}_\mu f, g)_{L^2(\mu)} = \mathcal{E}_\mu(f, g) := \int_\Omega \sum_\ell (\partial_{x_\ell} f, \partial_{x_\ell} g) \mathrm{d}\mu, \quad f, g \in \mathcal{F}C_{\mathrm{b}}^\infty(\Omega). \tag{2.207}$$

We used in (2.206) the standard notation $\Delta_\ell f(x) := \sum_{i=1}^\nu \partial^2_{x_\ell^i} f(x)$. The symmetric bilinear form (2.207) is closable in $L^2(\mu)$; its closure $(\mathcal{E}_\mu, \mathcal{D}(\mathcal{E}_\mu))$ is a canonical *Dirichlet form* in the sense of [4, 26, 199]. The corresponding Friedrichs extension $(\mathbb{H}_\mu, \mathcal{D}(H_\mu))$, which is a nonnegative self-adjoint operator in the complexification of $L^2(\mu)$, is called the *Dirichlet operator* associated with the Gibbs measure $\mu \in \mathcal{G}^{\mathrm{t}}$.

In this respect we mention the *equivalent description* of the Gibbs measures $\mu \in \mathcal{G}^{\mathrm{t}}$ in terms of their logarithmic derivatives via integration by parts formulas. Let $C_{\mathrm{b}}^1(\Omega_p)$ denote the set of all functions $f : \Omega_p \to \mathbb{R}$ which are bounded and continuous together with their partial derivatives $\partial_{x_\ell} f : \Omega_p \to \mathbb{R}^\nu$, $\ell \in \mathbb{L}$. As usual, $C_0^1(\Omega_p)$ will be its subset consisting of all functions with bounded support (i.e., $f \in C_{\mathrm{b}}^1(\Omega_p)$ such that $f(x) = 0$ if $|x|_p > r(f)$).

Proposition 2.37 *(cf. [17, 18, 22, 23]). Denote by $\mathcal{M}^{\mathrm{t}}$ the set of all probability measures on Ω such that $\mu(\Omega_p) = 1$ with some $p = p(\mu) > d$ and such that the integration by parts formula*

$$\int_{\Omega} \frac{\partial}{\partial x_\ell^i} f(x) \mathrm{d}\mu(x) = -\int_{\Omega} f(x_\ell) b_\ell^i(x) \mathrm{d}\mu(x), \quad \forall \ell \in \mathbb{L}, \ 1 \le i \le \nu, \tag{2.208}$$

holds for all functions $f \in C_0^1(\Omega_p)$. Then $\mathcal{M}^{\mathrm{t}} = \mathcal{G}^{\mathrm{t}}$.

The integration by parts characterization (2.208) can be used as an *alternative* way to establish the existence (cf. Theorem 2.14) and à-priori estimates (cf. Theorem 2.15) for the Gibbs measures $\mu \in \mathcal{G}^{\mathrm{t}}$, see the joint papers [22, 23]. More about the peculiarities of this approach and its generalization to the quantum case can be found in Subsection 4.5.3.

By Defnition 2.4 the set of tempered Gibbs measures $\mathcal{G}^{\mathrm{t}}$ is always convex. The subset of all its *extreme* points (which cannot be written as combinations $\theta\mu_1 + (1-\theta)\mu_2$ with $\theta \in (0,1)$ and $\mu_1 \neq \mu_2$) will be denoted by $\mathrm{ex}(\mathcal{G}^{\mathrm{t}})$. By Theorem 7.7 in [122], a measure $\mu \in \mathrm{ex}(\mathcal{G}^{\mathrm{t}})$ iff it is trivial on the tail σ-algebra $\bigcap_{\Lambda \Subset \mathbb{L}} \mathcal{B}(\Omega_{\Lambda^c})$. In statistical mechanics only such measures, also called *pure phases*, could describe possible macrostates of physical models. For a further role of the extreme measures $\mu \in \mathrm{ex}(\mathcal{G}^{\mathrm{t}})$ within the DLR approach see e.g. Theorem 3.41 and Proposition 3.52 below. In stochastic analysis respectively there is the following *description of the subset* $\mathrm{ex}(\mathcal{G}^{\mathrm{t}})$:

Proposition 2.38 *(cf. [17, 18]). The following assertions are equivalent:*

(i) The measure μ is an extreme point in $\mathcal{G}^{\mathrm{t}}$;

(ii) The Dirichlet form $\mathcal{E}_\mu$ is irreducable, that is any $f \in \mathcal{D}(\mathcal{E}_\mu)$ satisfying $\mathcal{E}_\mu(f,f) = 0$ is a constant μ-almost everywhere;

(iii) The corresponding sub-Markov semigroup (or the equilibrium dynamics) $\mathbb{T}_t := \exp(-t\mathbb{H}_\mu)$, $t \ge 0$, is ergodic in $L^2(\mu)$, that is

$$\lim_{t\to\infty} ||\mathbb{T}_t f - \mathbf{E}_\mu f||_{L^2(\mu)} = 0, \quad \text{for each } f \in L^2(\mu). \tag{2.209}$$

The above statement extends the famous result of R. Holley and D. Stroock for the Ising model proved in [144] and further motivates the study of spectral properties of the operators $\mathbb{H}_\mu$. In particular, this raises an important question about the *strong uniqueness* of the dynamics $\mathbb{T}_t$, $t \ge 0$, or equivalently, the *essential self-adjointness* of the operator $\mathbb{H}_\mu \restriction \mathcal{F}C_{\mathrm{b}}^\infty(\Omega)$. For lattice spin systems, the essential self-adjointness of the infinite dimensional Dirichlet operators on natural domains like $\mathcal{F}C_{\mathrm{b}}^\infty(\Omega)$ was shown e.g. in [15, 16, 20, 166, 167, 191, 210]. However, all the techniques developed so far are principally limited to the pair potentials $W_{\ell\ell'}$ having *at most quadratic growth* ($R = 2$) and to the one-particle potentials V_ℓ obeying certain *coercivity* and *semi-monotonicity* properties. The corresponding Theorem 4.61 to be proved below imposes the *most general assumptions* of such type and can be straightforwardly extended to N-particle interactions.

In a similar way, for $\Lambda \Subset \mathbb{L}$ and $y \in \Omega^{\mathrm{t}}$, we define *the Dirichlet operators* $\mathbb{H}_{\Lambda,y}$ associated in $L^2(\Omega_\Lambda, \mu_{\Lambda,y})$ with the local Gibbs measures $\mu_{\Lambda,y}(\mathrm{d}x_\Lambda) := \pi_\Lambda(\mathrm{d}x_\Lambda|y) \circ \mathbb{P}_\Lambda^{-1}$. These are elliptic differential operators of second order

$$\mathbb{H}_{\Lambda,y} f := -\sum_{\ell \in \Lambda} \left[\Delta_\ell f + (b_\ell^{\Lambda,y}, \partial_{x_\ell} f) \right], \quad f \in \mathcal{F}C_{\mathrm{b}}^\infty(\Omega_\Lambda), \tag{2.210}$$

where the corresponding logarithmic derivatives $b^{\Lambda,y} = (b_\ell^{\Lambda,y})_{\ell \in \Lambda} : \Omega_\Lambda \to \Omega_\Lambda$ are given by

$$\begin{aligned} b_\ell^{\Lambda,y}(x) &:= -\beta \cdot \partial_{x_\ell} H_\Lambda(x|y) \qquad (2.211) \\ &= -\beta \left[V_\ell'(x_\ell) + \sum_{\ell' \in \Lambda} \partial_{x_\ell} W_{\ell\ell'}(x_\ell, x_{\ell'}) + \sum_{\ell' \in \Lambda^c} \partial_{x_\ell} W_{\ell\ell'}(x_\ell, y_{\ell'}) \right]. \end{aligned}$$

In finite dimensions the essential self-adjointness of $\mathbb{H}_{\Lambda,y}$ on $C_{\mathrm{b}}^\infty(\mathbb{R}^{\nu|\Lambda|})$ takes place under a much weaker sufficient condition $|b^{\Lambda,y}|_{\mathbb{R}^{\nu|\Lambda|}} \in L^4(\Omega_\Lambda, \mu_{\Lambda,y})$, see Theorem 1 in [194] and Theorem 4.62 in Subsection 4.5.4.

The next step would be to prove the *Poincaré inequality* for $\mathbb{H}_\mu$, which says there exists a spectral gap constant $C_{\mathrm{SG}} > 0$ such that

$$\mathcal{E}_\mu(f,f) := \int_\Omega \sum_{\ell \in \mathbb{L}} |\partial_{x_\ell} f|^2 \mathrm{d}\mu \geq C_{\mathrm{SG}} ||f||^2_{L^2(\mu)}, \quad \forall f \in \mathcal{D}(\mathcal{E}_\mu), \quad \mathbf{E}_\mu f = 1. \tag{2.212}$$

In other words, $0 \in \mathbb{R}$ is an isolated, simple eigenvalue and $\mathbb{H}_\mu \geq C_{\mathrm{SG}} \mathbf{1}$ on the orthogonal complement to the constants in $L^2(\mu)$. By the spectral theorem, this is equivalent to the *exponential ergodicity* of the corresponding semigroup

$$||\mathbb{T}_t f - \mathbf{E}_\mu f||_{L^2(\mu)} \leq e^{-tC_{\mathrm{SG}}} ||f - \mathbf{E}_\mu f||_{L^2(\mu)}, \quad \forall t \geq 0, \; f \in L^2(\mu). \tag{2.213}$$

If (2.212) or (2.213) holds *for all* $\mu \in \mathcal{G}^{\mathrm{t}}$, then Proposition 2.38 immediately will yield the uniqueness result $|\mathcal{G}^{\mathrm{t}}| = 1$. So far, a direct spectral analysis of the infinite volume Dirichlet operators $\mathbb{H}_\mu$ was possible only in models with the strictly convex Hamiltonians $H(x)$; the associated $\mu \in \mathcal{G}^{\mathrm{t}}$ belong then to a subclass of *log-concave measures*, cf. [4, 16]. The other way mostly followed in the literature (see [53, 186], [289]–[291]) is to look for the spectral gap estimates which are valid *uniformly* for all finite volume operators $\mathbb{H}_{\Lambda,y}$ with the constant

$$C_{\mathrm{USG}} := \inf \left\{ C_{\mathrm{SG}}(\Lambda, y) \mid \Lambda \Subset \mathbb{L}, \, y \in \Omega^{\mathrm{t}} \right\} > 0. \tag{2.214}$$

In many cases (2.214) leads to the uniqueness of $\mu \in \mathcal{G}^{\mathrm{t}}$ and hence, via a thermodynamic limit $\Lambda \nearrow \mathbb{L}$, to the global spectral gap (2.212). We start to discuss this approach in the next item and then shall focus on its applications in Sections 4.3, 4.5.

(iii) Uniform spectral gap estimates in finite volumes

Here we show that under Dobrushin's weak dependence condition $(\mathbf{D_2})$, the one-point variance (Brascamp-Lieb or Poincaré) estimates (2.183) and (2.191) guarantee by themselves a similar property for the whole family of conditional distributions $\mu_{\Lambda,y}(\mathrm{d}x_\Lambda)$, $\Lambda \Subset \mathbb{L}$, $y \in \Omega$. This will be done in a very short way by using a new technique based on the *Efron-Stein-Wu inequality* for variances.

Let us start with the superquadratic interactions dealt with in Subsection 2.3.4 (ii). For simplicity, we have assumed there the dimension $\nu = 1$ and the the range of interaction $r < \infty$. For functions $f \in C^1(\mathbb{R}^\Lambda) \cap \mathrm{Lip}(\mathbb{R}^\Lambda, \rho)$, let us introduce the *weighted Sobolev seminorms*

$$||\nabla f||_{\Lambda,y} := \left[\int_{\mathbb{R}^\Lambda} \left(\sum_{\ell\in\Lambda} |\partial_{x_\ell} f(x_\Lambda)|^2 \varphi^{-2}(x_\ell) \right) \mu_{\Lambda,y}(\mathrm{d}x_\Lambda) \right]^{1/2}, \tag{2.215}$$

where ρ and φ are connected by (2.179). Obviously $||\nabla f||_{\Lambda,y} \le [f]_{\Lambda,\rho}$, where the latter was defined in (2.202). Then, (2.183) may be rewritten as the Poincaré-type inequality

$$\mathbf{Var}_{\mu_{\ell,y}(\mathrm{d}\tilde{x}_\ell)} f(\tilde{x}_\ell \times x_{\Lambda\setminus\{\ell\}}) \le \beta^{-1} e^{2\beta\delta_Q} ||\partial_{\tilde{x}_\ell} f(\tilde{x}_\ell \times x_{\Lambda\setminus\{\ell\}})||^2_{\ell,y}. \tag{2.216}$$

A principal issue is to use the classical *Efron-Stein inequality* for variances, which recently was generalized by L. Wu (cf. Theorem 2.1 in [288]) to a local Gibbs specification obeying Dobrushin Contraction Condition $(\mathbf{D_2})$. We shall apply the mentioned inequality to each measure $\mu_{\Lambda,y}(\mathrm{d}x_\Lambda)$ and its family of one-point conditional distributions $\mu_\ell(\mathrm{d}x_\ell | x_\Lambda \times y_{\Lambda^c})$, $\ell \in \Lambda$. This yields us (for more details see Proposition 4.49 in Subsection 4.5.1) that

$$\begin{aligned} &(1 - ||\mathbf{D}_\Lambda||_0)\mathbf{Var}_{\mu_{\Lambda,y}(\mathrm{d}x_\Lambda)}(f(x_\Lambda)) \\ &\le \mathbf{E}_{\mu_{\Lambda,y}(\mathrm{d}x_\Lambda)} \left(\sum_{\ell\in\Lambda} \mathbf{Var}_{\mu_\ell(\mathrm{d}\tilde{x}_\ell | x_\Lambda \times y_{\Lambda^c})}(f(\tilde{x}_\ell \times x_{\Lambda\setminus\{\ell\}})) \right), \end{aligned} \tag{2.217}$$

where $||\mathbf{D}_\Lambda||_0 \le ||\mathbf{D}||_0 < 1$ is the $l^\infty(\Lambda)$-norm of the Dobrushin matrix $\mathbf{D} = (D_{\ell\ell'})_{\Lambda\times\Lambda}$. Substituting (2.186), (2.216) into the right-hand side of (2.217) and using the consistency property (2.26), we immediately get that

$$\mathbf{Var}_{\mu_{\Lambda,y}}(f) \le \frac{1}{C_{2.218}} ||f||^2_{\Lambda,y}, \tag{2.218}$$

with one and the same constant $C_{2.218} := [1 - C_{2.186}||\mathbf{J}||_0]\, \beta e^{-2\beta\delta_Q}$.

For the inter-particle interaction of at most quadratic growth with $\nu \ge 1$ and $r \le +\infty$, see Subsection 2.3.4 (iii), this technique leads to the following result (concerned with high temperatures or weak couplings):

Theorem 2.39 *In the context of Theorem 2.34, for all $\beta \le \beta_0$ and*

$$||\mathbf{J}||_0 < \mathcal{J}(\beta_0) := a_U \left[b_W e^{2\beta\delta_Q} - a_W \right]^{-1}, \tag{2.219}$$

the family of local Gibbs distributions $\mu_{\Lambda,y}$, $\Lambda \Subset \mathbb{L}$, $y \in \Omega^{\mathrm{t}}$, *satisfies the Poincaré inequality*

$$\mathbf{Var}_{\mu_{\Lambda,y}}(f) \le \frac{1}{C_{\mathrm{USG}}} \int_{\mathbb{R}^{\Lambda}} |\nabla f(x_\Lambda)|^2 \mu_{\Lambda,y}(\mathrm{d}x_\Lambda), \quad f \in C^1_{\mathrm{b}}(\mathbb{R}^{\nu|\Lambda|}), \tag{2.220}$$

with the uniform constant

$$C_{\mathrm{USG}} \ge C_{2.221} := \beta \left[(a_U + a_W ||\mathbf{J}||_0)\, e^{-2\beta\delta_Q} - b_W ||\mathbf{J}||_0 \right]. \tag{2.221}$$

If the corresponding Dirichlet operator $\mathbb{H}_{\Lambda,y}$ *(cf. (2.210)) is essentially self-adjoint on* $C^\infty_{\mathrm{b}}(\mathbb{R}^{\nu|\Lambda|})$, *this implies the spectral gap estimate* $\mathbb{H}_{\Lambda,y} \ge C_{\mathrm{USG}}\mathbf{1}$ *for its restriction on the subspace* $L^2(\mu_{\Lambda,y}) \ominus \{\mathrm{const}\}$.

Remark 2.40 With a more technical effort, one may next prove that the local Gibbs distributions $\mu_{\Lambda,y}(\mathrm{d}x_\Lambda)$ possess the *exponential decay* of correlations similar to (2.203). We claim that for all $\beta \le \beta_0$, $||\mathbf{J}||_0 < \mathcal{J}(\beta_0)$, and $\sigma > 0$ being chosen from (2.201)

$$|\mathbf{Cov}_{\mu_{\Lambda,y}}(f; g \circ \tau_{\ell'})| \le C_{2.222} \exp(-\sigma|\ell'|)\, ||f||_{\Lambda,y} ||g||_{\Lambda,y}, \tag{2.222}$$

with some universal (i.e., independent of Λ and y) constant $C_{2.222}$ which behaves like $\mathcal{O}(\beta^{-1})$ *as* $\beta \to +0$. To this end, we can use two different arguments: either (i) the *iteration procedure* based on the $\mathbf{\Gamma}_2$-*criterion* and conditional integration (2.26) (cf. Proposition 6.2 in [186] for $R = 2$ and $\varphi \equiv$ *const*); or (ii) the *analytical representation* of covariances via the *Witten-Laplacian* (cf. Theorem 3.2 in [53] for $R \ge 2$). Obviously, (2.218) and (2.222) imply the same properties for the unique Gibbs measure $\mu \in \mathcal{G}^{\mathrm{t}}$ as a limit point of $\{\pi_\Lambda(\mathrm{d}x_\ell|y)\}_{\Lambda\in\mathcal{L}}$ as $\mathcal{L} \nearrow \mathbb{L}$. As compared with the similar result (2.203), the improvement achieved in (2.222) is that the Lipschitz norm $[f]_{\Lambda,\rho}$ is replaced by the Sobolev one $||f||_{\Lambda,y}$.

(iv) Low temperature case

In the low temperature limit $\beta \to \infty$, the decay of correlations in systems of scalar spins was studied in [29]–[31] and [236] *uniformly in finite volumes* and, respectively, in [36, 210, 211] directly in the *infinite volume*. The methods used fall into two groups: (i) *cluster expansions* and (ii) *Witten-Laplacian techniques*; however the latter were applied so far only to the Hamiltonians with the *unique ground state*. As is commonly recognized, a specific feature of the low temperature case is that the most of finite volume results do not hold uniformly with respect to *boundary conditions*. This leads to principal difficulties in applying Dobrushin's contraction technique. So, it might be possible that the infinite volume Dirichlet operator $\mathbb{H}_\mu$ possesses a spectral gap $C_{\mathrm{SG}} > 0$, while at the same time $C_{\mathrm{USG}} := \inf_{\Lambda,y} C_{\mathrm{SG}}(\Lambda, y) = 0$. Since the *Witten-Laplacians* on zero-forms are unitary equivalent to the Dirichlet operators, they also can be used to study the *spectral properties* of $\mathbb{H}_\mu$ associated with the Gibbs measures $\mu \in \mathcal{G}^{\mathrm{t}}$, cf. Theorem 3.3 in [210]. To compare the related results let us analyze the lattice $P(\varphi)$-model, which fits the hypotheses of all mentioned papers as well as the sufficient uniqueness conditions from Subsection 2.3.3.

Example 2.41 Let the one-particle potentials be given by a polynomial

$$V_\ell(x_\ell) := V(x_\ell) := \sum\nolimits_{\mathbf{s}=2}^{P} b^{(\mathbf{s})} x_\ell^{\mathbf{s}} \geq 0, \quad x_\ell \in \mathbb{R}, \tag{2.223}$$

of even degree $P \geq 2$ and with the positive coefficients $b^{(P)}$ and $b^{(2)} := V''(0)/2$. We assume that $V(0) = 0$ is *the only global minimum,* i.e., $V_\ell(x_\ell) > 0$ whenever $x_\ell \neq 0$. Respectively, let

$$W_{\ell\ell'}(x_\ell, x_{\ell'}) := w(x_\ell - x_{\ell'}) \ \text{ if } \ |\ell - \ell'| \leq r < \infty,$$

where the function $w \in C^3(\mathbb{R})$ vanishes at the origin and obeys

$$|w^{(k)}(x_\ell)| \leq J \left(c_w + |x_\ell|^{R-k} \right), \quad k = 0, 1, 2, \tag{2.224}$$

with certain $J > 0$ *and* $R \leq 1 + P/2$. Then Theorem 1.1 in [30] tells us that, fixed the boundary condition $y = 0$, for any $\varepsilon \in (0, 1)$ there exist positive $\beta^0 := \beta^0(\varepsilon)$ and $\mathcal{J}_0 := \mathcal{J}_0(\varepsilon)$ such that

$$|\mathbf{Cov}_{\mu_{\Lambda,0}}(x_\ell; x_{\ell'})| \leq (\beta V''(0))^{-1} (1 + \varepsilon^{-1}) \exp\{-\varepsilon|\ell - \ell'|\}, \tag{2.225}$$

uniformly for all $\beta \geq \beta^0$, $J \leq \mathcal{J}_0$, and $\ell, \ell' \in \Lambda \Subset \mathbb{L}$. As the uniqueness result of Theorems 2.28–2.30 applies to this model, from (2.225) we have the similar covariance estimate

$$|\mathbf{Cov}_\mu(x_\ell; x_{\ell'})| \leq (\beta V''(0))^{-1} (1 + \varepsilon^{-1}) \exp\{-\varepsilon|\ell - \ell'|\} \tag{2.226}$$

for the *unique* Gibbs measure $\mu \in \mathcal{G}^{\mathrm{t}}$ being the limit point of $\pi_\Lambda(\mathrm{d}x|0)$ as $\Lambda \nearrow \mathbb{L}$. Furthermore, starting from the corresponding finite volume result of Theorem 1.1 in [31], in the thermodynamic limit one obtains asymptotic formulas for $\mathbf{Cov}_\mu(x_\ell; x_{\ell'})$. The same argument is expected to work for the spectral gap estimates as $\Lambda \nearrow \mathbb{L}$. This provides us with an *alternative proof* of the infinite volume results obtained for the *pure* Gibbs states $\mu \in \mathrm{ext}(\mathcal{G}^{\mathrm{t}})$ as $\beta \to \infty$ in [210, 211]. Actually the most of statements there are reduced to the *unique Gibbs measure,* since by Theorems 2.28–2.30 we already know that $|\mathcal{G}^{\mathrm{t}}| = |\mathrm{ext}(\mathcal{G}^{\mathrm{t}})| = 1$.

2.4 Further properties of the Gibbs kernels

Here we look in more detail at the exponential moments

$$\int_\Omega \exp\{\lambda|x_\ell|^R\}\pi_\Lambda(\mathrm{d}x|y), \quad \int_\Omega \exp\{\lambda|x_\ell - < x_\ell >_{\Lambda,y}|^2\}\pi_\Lambda(\mathrm{d}x|y).$$

In particular, we shall analyze their dependence on the *boundary conditions* $y \in \Omega^{\mathrm{t}}$ and the *asymptotic behavior* as $\lambda \to \infty$ and $\beta \to +0/\infty$. The results of Subsection 2.4.3 have been already used for proving the uniqueness Theorems 2.25, 2.26 and 2.28–2.30.

2.4.1 Exponential bound of Bellissard and Høegh-Krohn

In this subsection we give a substantial improvement of the exponential bound (2.46). For the *ferromagnetic* systems of *scalar* spins, this bound first was discovered by J. Bellissard and R. Høegh-Krohn in 1982 (see Proposition III.1 and Theorem III.2 in [42]) and since then is frequently cited in the literature. However, over the years it remained open a question about it possible extensions to more general interactions.

To this end we shall apply elementary arguments based on the *integration by parts* for the measures $\mu_{\ell,y}(\mathrm{d}x)$, cf. Proposition 2.37. Under the natural *coercivity* hypothesis (2.229) on the one-particle potentials, such technique will provide us, as $R \geq 2$ and $\lambda \to \infty$, even with the more accurate asymptotics $\mathcal{O}(\lambda^{R/(R-1)})$ than the one with $\mathcal{O}(\lambda^2)$ in (2.46). In contrast to the original paper [42] dealing exceptionally with the scalar ferromagnets, our method covers also *multi-dimensional* spins as well as *non-translation invariant* and *many-particle* interactions. Its further advantage is that all the constants in the estimate (2.46), and hence the ones in the resulting Dobrushin's Condition $(\mathbf{D_1})$ with $h(x_\ell) := |x_\ell|$, are calculated *explicitly*, which in principle is impossible by the asymptotical methods used for similar aims in [42, 237, 259].

Additionally to the basic Assumptions $(\mathbf{W})$, $(\mathbf{J})$, and $(\mathbf{V_1})$, we suppose that the interaction potentials are given by continuously differentiable functions $V_\ell \in C^1(\mathbb{R}^\nu)$, $W_{\ell\ell'} \in C^1(\mathbb{R}^{2\nu})$ satisfying the following:

Assumption $(\mathbf{W}_6)$ *There exists* $C_6 > 0$, *such that for all* $\ell, \ell' \in \mathbb{L}$, $x_\ell, x_{\ell'} \in \mathbb{R}^\nu$

$$\left|\frac{\partial}{\partial x_\ell} W_{\ell\ell'}(x_\ell, x_{\ell'})\right| \leq \frac{1}{2} J_{\ell\ell'}(C_6 + |x_\ell|^{R-1} + |x_{\ell'}|^{R-1}). \tag{2.227}$$

Assumption $(\mathbf{V}_6)$ *The functions* $V_\ell \in C^1_{\mathrm{exp}}(\mathbb{R}^\nu)$ *have at most exponential growth, which means*

$$|V_\ell(x_\ell)| + |V'_\ell(x_\ell)|_{\mathbb{R}^\nu} \leq C_\ell \exp\{C_\ell |x_\ell|\} \tag{2.228}$$

with certain $C_\ell > 0$. *Moreover, there exist* $A_6 > ||\mathbf{J}||_0$ *and* $B_6 \in \mathbb{R}$ *such that for all* $\ell \in \mathbb{L}$ *and* $x_\ell \in \mathbb{R}^\nu$

$$(V'_\ell(x_\ell), x_\ell) \geq A_6 |x_\ell|^R + B_6. \tag{2.229}$$

Remark 2.42 The *coercivity* (or *one-sided growth*) estimate (2.229) is typically fulfilled by the polynomials of even degree with a positive leading coefficient like that in (2.10). Clearly, (2.229) ensures by itself that the potentials V_ℓ grow not slowly as $C|x_\ell|^R$ and hence

$$\int_\Omega \exp\{\lambda |x_\ell|\} \left[(1 + |\partial x_\ell H_\ell(x_\ell|y)|\right] \pi_\Lambda(\mathrm{d}x|y) < \infty, \quad \forall \lambda > 0. \tag{2.230}$$

Furthermore, similarly to Proposition 2.3, one can check that the map $(x, y) \to \partial_{x_\ell} H_\ell(x_\ell|y)$ is uniformly continuous on bounded sets in $\Omega_p \times \Omega_p$, $p > d$.

These preparations enables us to get the following refinement of the exponential bound (2.46):

Theorem 2.43 *Let Assumptions* ($\mathbf{W}_6$), ($\mathbf{J}$), *and* ($\mathbf{V}_6$) *be fulfilled, and set*

$$\varDelta_6 := A_6 - ||\mathbf{J}||_0 > 0. \tag{2.231}$$

Then, for every $\beta > 0$, *there exist corresponding* $\mathcal{L}$, $\mathcal{M} > 0$ *such that for all* $\ell \in \mathbb{L}$, $y \in \varOmega^{\mathrm{t}}$, *and* $\lambda > 0$

$$\begin{aligned} &\int_{\varOmega} \exp\{\lambda|x_\ell|\}\pi_\ell(\mathrm{d}x|y) \\ &\leq \exp\left\{\lambda\mathcal{L} + \lambda^{R/(R-1)}\mathcal{M} + 2\lambda\varDelta_6^{-1/R}\sum_{\ell'(\neq\ell)}(J_{\ell\ell'})^{1/R}|y_{\ell'}|\right\}. \end{aligned} \tag{2.232}$$

Proof. Let us apply the *integration by parts formula*, cf. (2.208),

$$\int_{\mathbb{R}^\nu} \frac{\partial}{\partial x_\ell^i} f(x_\ell)\mu_{\ell,y}(\mathrm{d}x_\ell) = \beta\int_{\mathbb{R}^\nu} f(x_\ell)\frac{\partial}{\partial x_\ell^i}H_\ell(x_\ell|y)\mu_{\ell,y}(\mathrm{d}x_\ell), \tag{2.233}$$

to the test functions

$$f_i(x_\ell) := x_\ell^i \exp\{\lambda|x_\ell|\}, \qquad x_\ell = (x_\ell^i)_{i=1}^\nu \in \mathbb{R}^\nu.$$

After summing over $1 \leq i \leq \nu$, one gets the identity

$$\begin{aligned} &\int_{\mathbb{R}^\nu} (\nu + \lambda|x_\ell|)\exp\{\lambda|x_\ell|\}\mu_{\ell,y}(\mathrm{d}x_\ell) \\ &= \beta\int_{\mathbb{R}^\nu}\left(x_\ell, V_\ell'(x_\ell) + \sum_{\ell'(\neq\ell)}\partial_{x_\ell}W_{\ell\ell'}(x_\ell, y_{\ell'})\right)\exp\{\lambda|x_\ell|\}\mu_{\ell,y}(\mathrm{d}x_\ell). \end{aligned} \tag{2.234}$$

Herefrom, by Young's inequality

$$\mathrm{a}^{R-1}\mathrm{b} \leq \left(1 - \frac{1}{R}\right)\mathrm{a}^R + \frac{1}{R}\mathrm{b}^R, \quad \mathrm{a}, \mathrm{b} \in \mathbb{R}_+, \tag{2.235}$$

and Assumptions ($\mathbf{V}_6$), ($\mathbf{W}_6$), we find that

$$\begin{aligned} &\int_{\mathbb{R}^\nu} \exp\{\lambda|x_\ell|\}\mu_{\ell,y}(\mathrm{d}x_\ell) \\ \leq\ &\frac{\beta}{2}\int_{\mathbb{R}^\nu}\left(\sum_{\ell'(\neq\ell)}J_{\ell\ell'}|y_{\ell'}|^R - \varDelta_6|x_\ell|^R + 2\varPsi\right)\exp\{\lambda|x_\ell|\}\mu_{\ell,y}(\mathrm{d}x_\ell). \end{aligned} \tag{2.236}$$

For shorthand, we here set

$$\varPsi := (1/\varDelta_6)^{1/(R-1)}\left(\lambda\beta^{-1} + C_6||\mathbf{J}||_0/2\right)^{R/(R-1)} + (1+\nu)\beta^{-1} + |B_6|. \tag{2.237}$$

Using in the right-hand side of (2.236) the easy-to-check inequality

$$\max_{\mathrm{a}\in\mathbb{R}_+} \left(\mathrm{b}^R - \mathrm{a}^R\right) e^{\lambda \mathrm{a}} \le \mathrm{b}^R e^{\lambda \mathrm{b}} \le \left(\frac{R}{\lambda e}\right)^R e^{2\lambda \mathrm{b}}, \quad \lambda, \mathrm{b} \in \mathbb{R}_+, \tag{2.238}$$

we further get that

$$\int_{\mathbb{R}^\nu} \exp\{\lambda |x_\ell|\} \mu_{\ell,y}(\mathrm{d}x_\ell) \tag{2.239}$$
$$\le \frac{1}{2}\beta\varDelta_6 \left(\frac{R}{\lambda e}\right)^R \exp\left\{2\lambda\varDelta_6^{-1/R}\left[2\varPsi + \sum_{\ell'(\neq\ell)} J_{\ell\ell'}|y_{\ell'}|^R\right]^{1/R}\right\}.$$

This readily implies the required estimate (2.232) for $\lambda \ge 1$, whereby the constants can be written explicitly as

$$\mathcal{L} := \log\left\{\frac{1}{2}\beta\varDelta_6\left(\frac{R}{e}\right)^R\right\} + 4\left[\left(\frac{(1+\nu)\beta^{-1} + |B_6|}{\varDelta_6}\right)^{1/R} + \left(\frac{C_6||\mathbf{J}||_0}{\varDelta_6}\right)^{1/(R-1)}\right],$$
$$\mathcal{M} := 4(\beta\varDelta_6)^{-1/(R-1)}. \tag{2.240}$$

The case of $\lambda \le 1$ is then trivial, since by (2.232) and Jensen's inequality

$$\log \int_{\mathbb{R}^\nu} \exp\{\lambda|x_\ell|\}\mu_{\ell,y}(\mathrm{d}x_\ell) \le \lambda\left[\mathcal{L} + \mathcal{M} + 2\varDelta_6^{-1/R}\sum_{\ell'(\neq\ell)}(J_{\ell\ell'})^{1/R}|y_{\ell'}|\right].$$

■

Corollary 2.44 *(i) Additionally to* $(\mathbf{W}_6)$, $(\mathbf{J})$, *and* $(\mathbf{V}_6)$, *suppose that the following relation holds*

$$\frac{1}{2}\varDelta_6^{1/R} > ||\mathbf{J}^{1/R}||_0 := \sup_\ell \sum_{\ell(\neq\ell')}(J_{\ell\ell'})^{1/R} \ge ||\mathbf{J}||_0^{1/R}. \tag{2.241}$$

Then, the kernels $\pi_\ell(\mathrm{d}x|y)$ *obey Dobrushin's bound* $(\mathbf{D}_1)$ *with the compact function* $h(x_\ell) := |x_\ell|$, *constant* $\mathcal{C} := \mathcal{L} + \lambda^{1/(R-1)}\mathcal{M}$, *and* l^∞*-contractive matrix* $(I_{\ell\ell'})_{\mathbb{L}\times\mathbb{L}}$ *with the entries* $I_{\ell\ell'} := 2(J_{\ell\ell'}/\varDelta_6)^{1/R}$.

Remark 2.45 (i) Repeating (2.234)–(2.237) for $\lambda = 0$, we get that

$$\int_\varOmega |x_\ell|^R \pi_\ell(\mathrm{d}x|y) \le \mathcal{C} + \sum_{\ell'(\neq\ell)} I_{\ell\ell'}|y_{\ell'}|^R, \tag{2.242}$$

$$\text{with } \mathcal{C} := \varDelta_6^{-1}\left(\nu\beta^{-1} + |B_6| + \frac{1}{2}C_6^{R/(R-1)}||\mathbf{J}||_0\right), \quad I_{\ell\ell'} := \frac{1}{2}\varDelta_6^{-1}J_{\ell\ell'}. \tag{2.243}$$

Integrating by parts the test functions $f_i(x_\ell) := x_\ell^i|x_\ell|^Q$, one finds that the similar to (2.242) estimates hold for the family of compact functions $h_Q(x_\ell) := |x_\ell|^{R+Q}$, $Q \ge 0$.

Furthermore, (2.243) shows that $\mathcal{C} := \mathcal{C}(\beta)$ behaves like $\mathcal{O}(1+\beta^{-1})$ as $\beta \to +0/\infty$. This observation could be used (alternatively to Corollaries 2.50 and 2.53) for proving the uniqueness results from Subsections 2.3.2 and 2.3.3.

(ii) The statement of Theorem 2.43 can be made more precise for *attractive harmonic interactions* $W_{\ell\ell'}(x_\ell, x_{\ell'}) := J_{\ell-\ell'}|x_\ell - x_{\ell'}|^2/2$. Suppose that the potentials V_ℓ fulfill Assumption $(\mathbf{V}_6)$ with $R = 2$ and any (arbitrarily small) $A_6 > 0$. Then we get the following refinement of the Dobrushin bound (2.242)

$$\int_{\Omega} |x_\ell|^2 \pi_\ell(\mathrm{d}x|y) \le \frac{1}{4A_6 + ||\mathbf{J}||_0} \left(\nu\beta^{-1} + |B_6| + \sum_{\ell'(\neq \ell)} J_{\ell-\ell'}|y_{\ell'}|^2 \right). \tag{2.244}$$

The corresponding matrix $(I_{\ell\ell'})_{\mathbb{L}\times\mathbb{L}}$ is always *contractive* due to the estimate $||\mathbf{I}||_0 \le \left(1 + 4A_6||\mathbf{J}||_0^{-1}\right)^{-1} < 1$.

2.4.2 Covariance estimates for $\pi_\Lambda(\mathrm{d}x|y)$

In various applications (such as e.g. the uniqueness problem or the validity of the Poincáre and log-Sobolev inequalities for $\mu \in \mathcal{G}^{\mathrm{t}}$) it is important to have à-priori information about correlations functions calculated with respect to $\pi_\Lambda(\mathrm{d}x|y)$. Especially one looks for the bounds which are *uniform* in volumes $\Lambda \Subset \mathbb{L}$ and boundary conditions $y \in \Omega^{\mathrm{t}}$. All results available by now concern, however, the *scalar* spins $x_\ell \in \mathbb{R}$ interacting via the *attractive harmonic* potentials like $J_{\ell-\ell'}|x_\ell - x_{\ell'}|^2 \ge 0$ (cf. e.g. Section 6 in [290]). In order to cover the case of *multi-component* spins and *general* pair (or N-particle) potentials, we here propose *a new approach* which is conceptually close to the Dobrushin iterative technique discussed in Subsection 2.2.2.

Our considerations will be restricted to the lattice system (2.1) with the pair interaction of at most quadratic growth. Additionally to the previous Assumptions $(\mathbf{W})$, $(\mathbf{J})$, and $(\mathbf{V}_1)$ holding with $R = 2$, we suppose that $V_\ell \in C^1(\mathbb{R}^\nu)$ and $W_{\ell\ell'} \in C^1(\mathbb{R}^{2\nu})$ satisfy the following:

Assumption $(\mathbf{W}_7)$ *For all* $\ell, \ell' \in \mathbb{R}$ and $x_\ell, \tilde{x}_\ell \in \mathbb{R}^\nu$

$$\sup_{x_{\ell'} \in \mathbb{R}^\nu} \left| \frac{\partial}{\partial x_\ell} W_{\ell\ell'}(x_\ell, x_{\ell'}) - \frac{\partial}{\partial \tilde{x}_\ell} W_{\ell\ell'}(\tilde{x}_\ell, x_{\ell'}) \right| \le \frac{1}{2} J_{\ell\ell'}|x_\ell - \tilde{x}_\ell|. \tag{2.245}$$

Assumption $(\mathbf{V}_7)$ *There exist* $A_7 > ||\mathbf{J}||_0$ *and* $B_7 \in \mathbb{R}$, *such that for all* $\ell \in \mathbb{L}$ *and* $x_\ell \in \mathbb{R}^\nu$

$$(V_\ell'(x_\ell) - V_\ell'(\tilde{x}_\ell), x_\ell - \tilde{x}_\ell)_{\mathbb{R}^\nu} \ge A_7|x_\ell - \tilde{x}_\ell|^2 + B_7. \tag{2.246}$$

Note that the above conditions guarantee the existence of $\mu \in \mathcal{G}^{\mathrm{t}}$ (cf. Subsection 2.2.3), but tell nothing about their uniqueness. Similarly to Remark 2.42, one observes that *semi-monotonicity* property (2.246) certainly fulfills for polynomials like (2.10). Furthermore, Assumption $(\mathbf{V}_7)$ implies $(\mathbf{V}_6)$ with $A_6 = A_7$ and respectively $(\mathbf{V}_1)$ with $A_1 = A_7/2$ and $R = 2$. Again, our strategy is to start with the *one-point estimates* for $\pi_\ell(\mathrm{d}x|y)$ and then iterate them by the consistency property (2.26) in domains $\Lambda \Subset \mathbb{L}$.

Lemma 2.46 *Suppose that the above assumptions hold, and set*

$$\Delta_7 := A_7 - ||\mathbf{J}||_0 > 0. \tag{2.247}$$

Then, for every $\beta > 0$ *and* $\kappa < \Delta_7$*, there exist corresponding* $\mathcal{E}$, $\mathcal{F} > 0$ *such that for all* $\ell \in \mathbb{L}$ *and* $y, \tilde{y} \in \Omega^{\mathrm{t}}$

$$\begin{aligned}&\int_\Omega \int_\Omega \exp\left\{\frac{1}{4}\beta\kappa|x_\ell - \tilde{x}_\ell|^2\right\} \pi_\ell(\mathrm{d}x|y)\pi_\ell(\mathrm{d}\tilde{x}|\tilde{y}) \\ &\leq \exp\left\{\mathcal{E} + \beta\mathcal{F}\sum\nolimits_{\ell'(\neq\ell)} J_{\ell\ell'}|y_{\ell'} - \tilde{y}_{\ell'}|^2\right\}. \end{aligned} \tag{2.248}$$

Proof. To some extent we proceed similarly to the proof of Theorem 2.43, but now we use the integration by parts with respect to the coupled measure $\mu_{\ell,y}(\mathrm{d}x_\ell) \times \mu_{\ell,\tilde{y}}(\mathrm{d}\tilde{x}_\ell)$ on $\mathbb{R}^{2\nu}$. Namely, let us apply the integration by parts formula

$$\begin{aligned}&\int_{\mathbb{R}^{2\nu}} \left(\frac{\partial}{\partial x_\ell^i} g(x_\ell, \tilde{x}_\ell) - \frac{\partial}{\partial \tilde{x}_\ell^i} g(x_\ell, \tilde{x}_\ell)\right) \mu_{\ell,y}(\mathrm{d}x_\ell)\mu_{\ell,\tilde{y}}(\mathrm{d}\tilde{x}_\ell) \\ &= \beta \int_{\mathbb{R}^{2\nu}} g(x_\ell, \tilde{x}_\ell)\left(\frac{\partial}{\partial x_\ell^i} H_\ell(x_\ell|y) - \frac{\partial}{\partial \tilde{x}_\ell^i} H_\ell(\tilde{x}_\ell|\tilde{y})\right) \mu_{\ell,y}(\mathrm{d}x_\ell)\mu_{\ell,\tilde{y}}(\mathrm{d}\tilde{x}_\ell). \end{aligned} \tag{2.249}$$

to the test functions

$$g_i(x_\ell, \tilde{x}_\ell) := (x_\ell^i - \tilde{x}_\ell^i)\exp\left\{\frac{1}{4}\beta\kappa|x_\ell - \tilde{x}_\ell|^2\right\}, \quad x_\ell = (x_\ell^i)_{i=1}^\nu, \ \tilde{x}_\ell = (\tilde{x}_\ell^i)_{i=1}^\nu \in \mathbb{R}^\nu,$$

and their derivatives

$$\partial_{x_\ell^i} g(x_\ell, \tilde{x}_\ell) = -\partial_{\tilde{x}_\ell^i} g(x_\ell, \tilde{x}_\ell) = \left(1 + \frac{1}{2}\beta\kappa|x_\ell^i - \tilde{x}_\ell^i|^2\right)\exp\left\{\frac{1}{4}\beta\kappa|x_\ell - \tilde{x}_\ell|^2\right\}. \tag{2.250}$$

Doing so is correct since all $g_i, \partial_{x_\ell^i} g \in L^1\left(\mu_{\ell,y} \times \mu_{\ell,\tilde{y}}\right)$ for $\kappa < \Delta_7$. After summing in (2.250) over $1 \leq i \leq \nu$ and taking into account $(\mathbf{W}_7)$, $(\mathbf{J})$, and $(\mathbf{V}_7)$, we arrive at

$$\begin{aligned}&\int_{\mathbb{R}^{2\nu}} \exp\left\{\frac{1}{4}\beta\kappa|x_\ell - \tilde{x}_\ell|^2\right\} \mu_{\ell,y}(\mathrm{d}x_\ell)\mu_{\ell,\tilde{y}}(\mathrm{d}\tilde{x}_\ell) \\ &\leq \beta \int_{\mathbb{R}^{2\nu}} \Phi(|x_\ell - \tilde{x}_\ell|^2)\exp\left\{\frac{1}{4}\beta\kappa|x_\ell - \tilde{x}_\ell|^2\right\} \mu_{\ell,y}(\mathrm{d}x_\ell)\mu_{\ell,\tilde{y}}(\mathrm{d}\tilde{x}_\ell). \end{aligned} \tag{2.251}$$

In the integrand we have

$$\Phi(s) := -(\Delta_7 - \kappa)s + \frac{1}{8}\sum_{\ell'(\neq\ell)} J_{\ell\ell'}|y_{\ell'} - \tilde{y}_{\ell'}|^2 + \beta^{-1}(1 + 2\nu + \beta|B_7|), \tag{2.252}$$

which is a linear function of $s := |x_\ell - \tilde{x}_\ell|^2$. Estimating $\sup_{s\geq 0}\left\{\Phi(s)\exp(\frac{1}{4}\beta\kappa s)\right\}$ with the help of (2.238), we obtain the desired bound (2.248) with the constants

$$\mathcal{E} := \log(4\varsigma) + \frac{1}{2\varsigma}(1 + 2\nu + \beta|B_7|) - 1, \quad \mathcal{F} := \frac{1}{16\varsigma}, \quad \varsigma := \kappa^{-1}\Delta_7 - 1. \tag{2.253}$$

■

Employing conditional integration in volumes $\Lambda \Subset \mathbb{L}$, from (2.248) we readily get the *uniform variance* bound for the kernels $\pi_\Lambda(\mathrm{d}x|y)$.

Theorem 2.47 *For each* $\beta > 0$ *and* $\kappa < \varDelta_7$*, there exists a corresponding* $C_{2.254} := C_{2.254}(\beta, \kappa) > 0$ *such that for all* $\ell \in \mathbb{L}$*,* $\Lambda \Subset \mathbb{L}$*, and* $y \in \Omega^{\mathrm{t}}$

$$\int_{\Omega} \exp\left\{\frac{1}{4}\beta\kappa|x_\ell - < x_\ell >_{\Lambda,y}|^2\right\} \pi_\Lambda(\mathrm{d}x|y) \le C_{2.254}, \tag{2.254}$$

where we define the mean values

$$< x_\ell >_{\Lambda,y} := \int_{\Omega} x_\ell \pi_\Lambda(\mathrm{d}x|y) \in \mathbb{R}^\nu.$$

Proof. The line of reasoning is close to that used in the proof of Lemma 2.12. Applying Jensen's inequality, we first observe that

$$\begin{aligned} &\log\left\{\int_{\Omega} \exp\left(\frac{1}{4}\beta\kappa|x_\ell - < x_\ell >_{\Lambda,y}|^2\right) \pi_\Lambda(\mathrm{d}x|y)\right\} \\ \le &\log\left\{\int_{\Omega}\int_{\Omega} \exp\left(\frac{1}{4}\beta\kappa|x_\ell - \tilde{x}_\ell|^2\right) \pi_\Lambda(\mathrm{d}x|y)\pi_\Lambda(\mathrm{d}\tilde{x}|y)\right\} =: \tilde{n}_\ell(\Lambda|y). \end{aligned} \tag{2.255}$$

In order to estimate the last line in (2.255), let us integrate the both sides of (2.248) with respect to $\pi_\Lambda(\mathrm{d}x|y) \times \pi_\Lambda(\mathrm{d}\tilde{x}|y)$. By the consistency property (2.26) this yields us that

$$\tilde{n}_\ell(\Lambda|y) \le \mathcal{E} + \log\left\{\int_{\Omega}\int_{\Omega} \exp\left(\beta\mathcal{F}\sum_{\ell'\in\Lambda} J_{\ell\ell'}|x_\ell - \tilde{x}_\ell|^2\right) \pi_\Lambda(\mathrm{d}x|y)\pi_\Lambda(\mathrm{d}\tilde{x}|y)\right\}. \tag{2.256}$$

Now let us fix any $\kappa \in (\varDelta_7 - ||\mathbf{J}||_0/4,\ \varDelta_7)$. Then by (2.71)–(2.69) and (2.253) it holds for each $p > d$ and a small enough $\varepsilon := \varepsilon(p) \in (0, 1)$

$$4\kappa^{-1}\mathcal{F}||\mathbf{J}||_0 \le 4\kappa^{-1}\mathcal{F}||\mathbf{J}||_{p,\varepsilon} < 1.$$

Applying Hölder's inequality (2.73) and summing over $\ell \in \Lambda$, we conclude from (2.74), (2.75), and (2.256) that for all $\ell_0 \in \mathbb{L}$

$$\tilde{n}_{\ell_0}(\Lambda|y) \le \sum_{\ell\in\Lambda} \tilde{n}_\ell(\Lambda|y) \cdot (1 + \varepsilon|\ell_0 - \ell|)^{-p} \le \mathcal{E}\frac{\boldsymbol{\Xi}_{p,\varepsilon}}{1 - 4\kappa^{-1}\mathcal{F}||\mathbf{J}||_{p,\varepsilon}} =: \mathcal{E}_p.$$

The final answer then follows with $C_{2.254} := \inf_{p>d} \exp \mathcal{E}_p$. ■

Remark 2.48 The covariance estimate (2.254) holds *independently* of whether or not the family of $\mu_{\Lambda,y}(\mathrm{d}x_\Lambda)$ satisfies the Poincaré or log-Sobolev inequalities uniformly in $\Lambda \Subset \mathbb{L}$ and $y \in \Omega^{\mathrm{t}}$ (cf. Subsections 2.3.5 and 4.3.2). Typically ones proceeds in the inverse direction and makes use of those inequalities to derive the à-priori bounds on correlations. On the other hand, it is well known (cf. Theorem 3.3 in [1]) that the exponential integrability (2.254) is a *necessary* condition for the validity of the log-Sobolev inequality for the measures $\mu_{\Lambda,y}(\mathrm{d}x_\Lambda)$.

2.4.3 Asymptotic analysis

To complete the arguments used in proving the uniqueness criteria in Subsections 2.3.2 and 2.3.3, here we analyze the behavior of the constants in Dobrushin's Compactness Condition ($\mathbf{D_1}$) as $\beta \to +0/+\infty$. Furthermore, the results obtained below can be used to describe the concentration properties of the measures $\pi_\Lambda(\mathrm{d}x|y)$. We start with the uniform exponential bound in the high temperature regime.

Lemma 2.49 *Suppose that $2 \leq R \leq R$, and let Assumptions ($\mathbf{V}_2$), ($\mathbf{J}$), and ($\mathbf{W}$) from Subsection 2.3.2 (i) be fulfilled for all $||\mathbf{J}||_0 \leq \mathcal{J}_0 < 2A_1/3$. Then, for any $\beta_0 > 0$ and $\kappa < A_1 - \mathcal{J}_0/2$, there exists a proper $\Gamma_0 := \Gamma_0(\beta_0, \mathcal{J}_0, \kappa) \geq 1$ such that*

$$\int_\Omega \exp\left\{\beta\kappa|x_\ell|^R\right\}\pi_\ell(\mathrm{d}x|y) \leq \Gamma_0 \exp\left\{\beta \sum\nolimits_{\ell'(\neq \ell)} J_{\ell\ell'}|y_{\ell'}|^R\right\}, \tag{2.257}$$

simultaneously for all $\ell \in \mathbb{L}$, $y \in \Omega^{\mathrm{t}}$, $||\mathbf{J}||_0 \leq \mathcal{J}_0$, and $\beta \leq \beta_0$.

Proof. Recall that, for each $||\mathbf{J}||_0 \leq \mathcal{J}_0$ and $\beta \leq \beta_0$, Lemma 2.9 gives us the required bound (2.257) with the constant

$$\Gamma_0(\beta, ||\mathbf{J}||_0, \kappa) := \{X_1/Y_1\}\exp\left\{\beta\left(C_W||\mathbf{J}||_0 - B_1 + B_2\right)\right\}, \tag{2.258}$$

where we set

$$X_1 := \int_{\mathbb{R}^\nu} \exp\left\{-A_1|x_\ell|^P + (\kappa + ||\mathbf{J}||_0/2)\,\beta^{1-R/P}|x_\ell|^R\right\}\mathrm{d}x_\ell, \tag{2.259}$$

$$Y_1 := \int_{\mathbb{R}^\nu} \exp\left\{-A_2|x_\ell|^P - (||\mathbf{J}||_0/2)\beta^{1-R/P}|x_\ell|^R\right\}\mathrm{d}x_\ell. \tag{2.260}$$

For convenience, in the above integrals we have already made the change of variables $x_\ell \to \beta^{-1/P}x_\ell$. Since both X_1 and Y_1 are monotone functions of the parameters $||\mathbf{J}||_0$ and β, we may put Γ_0 equal to the right-hand side in (2.258) at the endpoints $\mathcal{J}_0$ and β_0. ■

As a sequel, we get the following asymptotics for the coefficients in the Dobrushin compactness criterion at *small* β (cf. Remark 2.10 (ii)).

Corollary 2.50 *There exists a positive $C_{2.261} := C_{2.261}(\beta_0, \mathcal{J}_0, \kappa)$ such that the Dobrushin-type condition*

$$\int_\Omega |x_\ell|^R \pi_\ell(\mathrm{d}x|y) \leq \beta^{-1}C_{2.261} + \kappa^{-1}\sum_{\ell'} J_{\ell\ell'}|y_{\ell'}|^R \tag{2.261}$$

holds for all $\ell \in \mathbb{L}$, $y \in \Omega^{\mathrm{t}}$, $||\mathbf{J}||_0 \leq \mathcal{J}_0$, and $\beta \leq \beta_0$.

Proof. The result follows by Jensen's inequality applied to (2.257), whereby we set $\mathrm{C} := (\kappa\mathcal{J}_0)^{-1}\log\Gamma_0$. ■

This enables us to refine the statements of Lemma 2.12 and Theorem 2.15.

Corollary 2.51 *In the situation of Lemma 2.49 the following holds:*

(i) For any given $p > d$ there exist positive $\Gamma_p := \Gamma_p(\beta_0, \mathcal{J}_0, ||\mathbf{J}||_p, \kappa)$ and $\mathcal{J}_p := \mathcal{J}_p(\beta_0, \mathcal{J}_0, ||\mathbf{J}||_p, \kappa)$ such that

$$\int_\Omega \exp\left\{\beta\kappa|x_\ell|^R\right\} \pi_\Lambda(\mathrm{d}x|y) \leq \Gamma_p \exp\left\{\beta\mathcal{J}_p\|y_{\Lambda^c}\|_p^R\right\}, \tag{2.262}$$

for all $\ell \in \Lambda \Subset \mathbb{L}$, $y \in \Omega_p$, $||\mathbf{J}||_0 \leq \mathcal{J}_0$, and $\beta \leq \beta_0$. In particular,

$$\limsup_{\Lambda \nearrow \mathbb{L}} \int_\Omega \exp\left\{\beta\kappa|x_\ell|^R\right\} \pi_\Lambda(\mathrm{d}x|y) \leq \Gamma_p. \tag{2.263}$$

(ii) The corresponding Gibbs measures $\mu \in \mathcal{G}^{\mathrm{t}}$ obey

$$\limsup_{\beta \leq \beta_0} \left\{\sup_\ell \int_\Omega \exp\left\{\beta\kappa|x_\ell|^R\right\} \mu(\mathrm{d}x)\right\} \leq C_{2.264}, \tag{2.264}$$

with one and the same $C_{2.264} := C_{2.264}(\beta_0, \mathcal{J}_0, ||\mathbf{J}_p||, \kappa) \geq 1$.

Proof. (i) Going through the proof of Lemma 2.12 and plugging (2.257) into (2.72), we arrive at (2.262) holding with the constants

$$\log\Gamma_p := \frac{\Xi_{p,\varepsilon}}{1 - \Delta_1^{-1}||\mathbf{J}||_{p,\varepsilon}} \log\Gamma_0, \qquad \mathcal{J}_p := \frac{\varepsilon^{-p}||\mathbf{J}||_{p,\varepsilon}}{1 - \Delta_1^{-1}||\mathbf{J}||_{p,\varepsilon}}. \tag{2.265}$$

According to (2.71)–(2.70), above we fixed a sufficiently small $\varepsilon > 0$ such that

$$||\mathbf{J}||_0 \leq ||\mathbf{J}||_{p,\varepsilon} < \Delta_1 := A_1 - ||\mathbf{J}||_0/2.$$

(ii) The result follows from (2.262) combined with (2.76) and (2.83). ■

As $\beta \to \infty$, such analysis happens to be highly *nontrivial*, except the special case when the Hamiltonian (2.1) admits a *unique* ground state.

Lemma 2.52 *Let Assumptions* ($\mathbf{V}_3$), ($\mathbf{J}$), *and* ($\mathbf{W}_3$) *from Subsection 2.3.3 (i) be fulfilled for all $||\mathbf{J}||_0 \leq \mathcal{J}_0 < 2A_4/3$. Then, for any $\beta^0 > 0$ there exists a proper $\Gamma^0 := \Gamma^0(\beta^0, \mathcal{J}_0) \geq 1$ such that*

$$\int_\Omega \exp\left\{\beta\kappa|x_\ell|^R\right\} \pi_\ell(\mathrm{d}x|y) \leq \Gamma^0 \exp\left\{\beta \sum\nolimits_{\ell'(\neq\ell)} J_{\ell\ell'}|y_{\ell'}|^R\right\}, \tag{2.266}$$

simultaneously for all $\ell \in \mathbb{L}$, $y \in \Omega^{\mathrm{t}}$, $\kappa \leq A_4 - \mathcal{J}_0/2$, $||\mathbf{J}||_0 \leq \mathcal{J}_0$, and $\beta \geq \beta^0$.

Proof. Again, we adapt to the present situation the estimates (2.53)–(2.57) used in proving Lemma 2.9. Performing the scaling $x_\ell \to \beta^{-1/2}x_\ell$, we obtain the required bound (2.266) holding, for each $||\mathbf{J}||_0 \leq \mathcal{J}_0$ and $\beta \geq \beta^0$, with the constant

$$\Gamma^0(\beta, ||\mathbf{J}||_0) := X_2/Y_2, \tag{2.267}$$

Here we set

$$X_2 := \int_{\mathbb{R}^\nu} \exp\left\{ -\left[a_4|x_\ell|^2 + \left(A_4 - \kappa - \frac{||\mathbf{J}||_0}{2} \right) \beta^{1-R/2}|x_\ell|^R \right] \right\} \mathrm{d}x_\ell, \qquad (2.268)$$

$$Y_2 := \int_{\mathbb{R}^\nu} \exp\left\{ -\left[a_3|x_\ell|^2 + A_3\beta^{1-P/2}|x_\ell|^P + \frac{||\mathbf{J}||_0}{2}\beta^{1-R/2}|x_\ell|^R \right] \right\} \mathrm{d}x_\ell. \qquad (2.269)$$

By a monotonicity argument, we may put $\Gamma^0(\beta^0, \mathcal{J}_0)$ equal to the right-hand side in (2.267) at the endpoints $\mathcal{J}_0$ and β^0. ∎

An important application of the above lemma concerns the validity of Dobrushin's compactness criteria at *large* β.

Corollary 2.53 *For each fixed (though arbitrarily small)* $\mathrm{c} > 0$ *there exists a proper* $\beta^0 := \beta^0(\mathrm{c}, \mathcal{J}_0) > 0$ *such that*

$$\int_\Omega |x_\ell|^R \pi_\ell(\mathrm{d}x|y) \leq \mathrm{c} + (A_4 - \mathcal{J}_0/2)^{-1} \sum_{\ell'} J_{\ell\ell'}|y_{\ell'}|^R, \qquad (2.270)$$

for all $\ell \in \mathbb{L}$, $y \in \Omega^{\mathrm{t}}$, $||\mathbf{J}||_0 \leq \mathcal{J}_0$, *and* $\beta \geq \beta^0$.

Proof. (i) The statement follows by Jensen's inequality applied to (2.266), where we set $\kappa := A_4 - \mathcal{J}_0/2$. The constant $\mathrm{c} := \log \Gamma^0/\beta\kappa$ obviously tends to zero as $\beta \to \infty$. ∎

Remark 2.54 (i) From the proof Lemma 2.52 it is clear the following: Fixed all other parameters except the temperature, the bound (2.266) holds true with a certain constant $\Gamma^0(\tau_0)$, which can be taken the same *for all* $(\beta, ||\mathbf{J}||_0) \in \mathbb{R}_+ \times \mathbb{R}_+$ fulfilling the constraint

$$\tau := A_3\beta^{1-P/2} + (||\mathbf{J}||_0/2)\beta^{1-R/2} \leq \tau_0 < \infty. \qquad (2.271)$$

In particular, for every $P \geq R > 2$ *and* $\Gamma^0 > (a_3/a_4)^{\nu/2}$, one finds a corresponding $\tau_0 := \tau_0(\Gamma^0)$ such that (2.266) holds in the phase domain (2.271).

(ii) Much better estimates can be obtained by the asymptotic *Laplace method* for multiple integrals. To illustrate the idea, we assume that all one-particle potentials are identical, i.e., $V \equiv V_\ell \in C(\mathbb{R}^\nu)$; otherwise one has to treat the upper and lower bounds for V_ℓ. Furthermore, let V have a *unique, non-degenerate* global minimum $V(0) = 0$, as it was described by Assumption $(\mathbf{V}_4)$ in Subsection 2.3.3. To be more concrete, let us suppose that $R > 2$. Then a straightforward application of the Laplace method (cf. Sect. II, §4, Theorem 4.1 in [106]) gives us the identical asymptotics

$$X,\ Y \sim (2\pi/\beta)^{\nu/2} \frac{1 + \mathcal{O}(\beta^{-1})}{|\det V''(0)|^{1/2}}, \quad \text{as } \beta \to \infty,$$

for both integrals in (2.54), (2.55). Thus, Lemma 2.52 holds with $\lim_{\beta\to\infty} \Gamma^0(\beta, \mathcal{J}_0) = 1$.

Proceeding analogously to the proof of Corollary 2.50, we can improve the statements of Lemma 2.12 and Theorem 2.15.

Corollary 2.55 *In the situation of Lemma 2.52 the following holds:*

(i) For any given $p > d$ there exist positive $\Gamma_p := \Gamma_p(\beta^0, \mathcal{J}_0, ||\mathbf{J}||_p)$ and $\mathcal{J}_p := \mathcal{J}_p(\beta^0, \mathcal{J}_0, ||\mathbf{J}||_p)$ such that

$$\int_\Omega \exp\left\{\beta\kappa|x_\ell|^R\right\}\pi_\Lambda(\mathrm{d}x|y) \leq \Gamma_p \exp\left\{\beta\mathcal{J}_p\|y_{\Lambda^c}\|_p^R\right\}, \tag{2.272}$$

for all $\kappa \leq \Delta_4 := A_4 - ||\mathbf{J}||_0/2$, $\ell \in \Lambda \Subset \mathbb{L}$, $y \in \Omega_p$, $||\mathbf{J}||_0 \leq \mathcal{J}_0$, and $\beta \geq \beta^0$. In particular,

$$\limsup_{\Lambda \nearrow \mathbb{L}} \int_\Omega \exp\left\{\beta\kappa|x_\ell|^R\right\}\pi_\Lambda(\mathrm{d}x|y) \leq \Gamma_p. \tag{2.273}$$

(ii) The corresponding Gibbs measures $\mu \in \mathcal{G}^{\mathrm{t}}$ obey

$$\limsup_{\beta \geq \beta^0} \left\{\sup_\ell \int_\Omega \exp\left\{\beta\kappa|x_\ell|^R\right\}\mu(\mathrm{d}x)\right\} \leq C_{2.274}, \tag{2.274}$$

with one and the same constant $C_{2.274} := C_{2.274}(\beta^0, \mathcal{J}_0, ||\mathbf{J}||_p) \geq 1$.

A useful sequel of the estimate (2.272) is the following *localization result* valid in the *low temperature* regime. It says that any perturbation of the local energy $H_\Lambda(x_\Lambda|y)$ in the single spin variables x_ℓ, $\ell \in \mathbb{L}$, taken away from its global minimum at $x_\Lambda = 0$, can change the partition function $Z_\Lambda(y)$ only by an *exponentially small* error.

Corollary 2.56 *Under the assumptions of Corollary 2.55, the following estimate holds with $\Delta_4 := A_4 - ||\mathbf{J}||_0/2$ for all $\xi > 0$*

$$\begin{aligned}\int_{\{x\in\Omega:\ |x_\ell|\geq\xi\}} \pi_\Lambda(\mathrm{d}x|y) &= \frac{1}{Z_\Lambda(y)}\int_{\Omega_\Lambda} \mathbf{1}_{\{x_\Lambda:\ |x_\ell|\geq\xi\}} \exp\left\{-\beta H_\Lambda(x_\Lambda|y)\right\}\mathrm{d}x_\Lambda \\ &\leq \Gamma_p \exp\left\{-\beta\left[\Delta_4\xi^R - \mathcal{J}_p\|y_{\Lambda^c}\|_p^R\right]\right\}\end{aligned} \tag{2.275}$$

with the parameter $\Delta_4 := A_4 - ||\mathbf{J}||_0/2 \geq 0$. Herefrom, in particular,

$$\sup_{\ell\in\Lambda\Subset\mathbb{L}} \int_{\{x\in\Omega\ |\ |x_\ell|\geq\xi\}} \pi_\Lambda(\mathrm{d}x|0) \quad\leq\quad \Gamma_p \exp\left\{-\beta\Delta_4\xi^R\right\}, \tag{2.276}$$

$$\limsup_{\Lambda\nearrow\mathbb{L}} \int_{\{x\in\Omega\ |\ |x_\ell|\geq\xi\}} \pi_\Lambda(\mathrm{d}x|y) \quad\leq\quad \Gamma_p \exp\left\{-\beta\Delta_4\xi^R\right\}. \tag{2.277}$$

Proof. The claim follows by Chebyshev's inequality applied to (2.272). ■

Remark 2.57 A similar to (2.277) localization result, but for the Hamiltonians $H_\Lambda(x_\Lambda)$ with empty boundary condition, cf. (2.24), was proven in [30]. Namely, Theorem 1.6 there says that there exists some universal $\beta^0 > 0$ such that for all $\xi \in (0,1)$, $\beta \geq \beta^0$, and $||\mathbf{J}||_0 \leq \mathcal{J}_0 := \xi^4/\beta^0$,

$$\sup_{\ell\in\Lambda\Subset\mathbb{L}} \int_{\{x\in\Omega\ |\ |x_\ell|\geq\xi\}} \pi_\Lambda(\mathrm{d}x) \leq \beta^0 \exp\left\{-\xi^2\beta/\beta^0\right\}. \tag{2.278}$$

The proof is rather involved and presumes a number of analytical conditions on V_ℓ and $W_{\ell\ell'}$. For technical reasons, the growth of pair interaction is not allowed there to be too fast: in our assumptions this corresponds to the restriction $R \leq 1 + P/2$ used in Subsection 2.3.4 (ii).

Chapter 3

Systems of Interacting Quantum Oscillators

3.1 Euclidean Gibbs measures

This Chapter is concerned with models of quantum anharmonic lattice systems, see (1.4). Our aim will be to give a complete description of the thermodynamic properties of such systems by using the *Euclidean* (i.e., *path integral*) approach.

Usually, *Gibbs states* of quantum models are defined as positive normalized functionals on algebras of observables, satisfying the *Kubo-Martin-Schwinger* (*KMS*) condition, see [66], which reflects the consistency between the dynamic and thermodynamic properties of the system proper to the thermodynamic equilibrium. For a subsystem located in a finite $\Lambda \subset \mathbb{L}$ and thus described by the local Hamiltonian H_Λ, the *KMS* condition is formulated by means of the unitary operators $\exp(\imath t H_\Lambda)$, $t \in \mathbb{R}$. To describe the dynamics of the whole model one has to take the infinite volume limit of $\exp(\imath t H_\Lambda)$, which certainly exists for finite rank H_Λ, e.g. for spin models. However for the quantum lattice models like (1.4), such limits do not make sense and therefore the *KMS* condition for the whole system cannot be formulated. This produces a fundamental problem and actually there is no canonical way to define Gibbs states, and hence to give a complete description of the thermodynamic properties of such models. Thus, we shall follow an alternative way, which allows to bridge this gap with the help of path integrals.

In [3], an approach employing the fact that the local Hamiltonians H_Λ generate stochastic processes has been initiated. In this approach, the description of the local Gibbs states, based on the properties of the semi-group $\exp(-t H_\Lambda)$, $t > 0$, is translated into a *"probabilistic language"*, that opens a possibility to apply here corresponding concepts and techniques. In this language, our model is the system of infinite dimensional "spins" ω_ℓ, $\ell \in \mathbb{L}$, being *continuous loops* $\omega_\ell \in C(S_\beta \to \mathbb{R}^\nu)$. Here $\beta := 1/T > 0$ is the inverse (absolute) temperature and $S_\beta \cong [0, \beta]$ is a circle of length β. The distribution of each spin ω_ℓ is given by the *path measure* of the β-periodic Ornstein-Uhlenbeck process corresponding to $H_\ell^{\rm har}$ multiplied by a density obtained from the anharmonic potential with the help of the Feynman-Kac formula. Afterwards, finite subsystems are associated with conditional probability measures, which by the *DLR* equation determine the set of Gibbs measures $\mathcal{G}^{\rm t}$. This approach is called *Euclidean* due to its conceptual analogy with the Euclidean quantum field theory. Its further development was conducted in the papers [5]–[14], [19]–[21], [24, 37, 38, 105, 131], [162]–[164], [215, 214]. Actually, the Euclidean approach remains so far the only method which allows to construct and study Gibbs states for infinite systems of quantum particles described by unbounded operators. Among its impressive achievements one has to mention the settlement in [6, 8, 9] of a long standing problem of the influence of quantum effects on structural phase transitions in quantum anharmonic crystals (which on the physical level was first discussed in [264], see also [105, 214, 278, 279]). In this chapter we give a complete description of the set $\mathcal{G}^{\rm t}$ for the model (3.1) and hence essentially finalize the development of the Euclidean approach for such models.

In Subsection 3.1.1 we introduce the object of our study in the form of a system of interacting quantum oscillators (3.1), (3.2), finite subsystems of which are described by their Schrödinger operators (3.3). The basic elements of the Euclidean approach will be presented in Subsection 3.1.2. Afterwards, in Subsection 3.1.3 we introduce the spaces of temperature loops Ω_Λ and the probability measures $\mu_\Lambda$on these spaces, which give a canonical realization for the β-periodic stochastic processes generated by the corresponding Schrödinger operators H_Λ, $\Lambda \Subset \mathbb{L}$. The notion of temperedness, which is important in all systems with interactions of infinite range, is discussed in Subsection 3.1.4. In Subsection 3.1.5 we describe in detail the corresponding Gibbsian formalism and define the set $\mathcal{G}^{\rm t}$ of all tempered Euclidean Gibbs measures μ on the *"temperature loop lattice"* $\Omega := [C(S_\beta \to \mathbb{R}^\nu)]^{\mathbb{L}}$.

3.1.1 The model and its physical background

The *quantum anharmonic oscillator* is a mathematical model of a localized quantum particle moving in a potential field with sufficient growth at infinity and possibly multiple minima. Infinite systems of interacting quantum anharmonic oscillators possess quite rich properties, connected with the possibility of ordering caused by the interaction as well as with quantum stabilization competing the ordering. Most of the systems of this kind are related with solids, such as ionic crystals containing localized light particles oscillating in the field created by heavy ionic complexes, or quantum crystals consisting entirely of such particles [48, 160, 267, 277]. Quantum anharmonic oscillators also are used as parts of the models describing interaction of vibrating quantum particles with a radiation (photon) field [143, 220] or strong electron-electron correlations caused by the interaction of electrons with vibrating ions responsible for such phenomena as superconductivity, charge density waves, etc., see [113]. Thus, infinite systems of interacting quantum anharmonic oscillators are quite important models and their rigorous description is still a challenging mathematical task.

The infinite system of quantum oscillators we consider has the following *heuristic*

Hamiltonian

$$H = \frac{1}{2}\sum_{\ell,\ell'} W_{\ell\ell'}(q_\ell, q_{\ell'}) + \sum_\ell H_\ell, \tag{3.1}$$

where the displacements q_ℓ are ν-dimensional vectors. The sums run through a countable set $\mathbb{L} \subset \mathbb{R}^d$ which will be equipped with the Euclidean distance $|\ell - \ell'|$. Each Hamiltonian

$$H_\ell = H_\ell^{\rm har} + V_\ell(q_\ell) := \frac{1}{2m}|p_\ell|^2 + \frac{a}{2}|q_\ell|^2 + V_\ell(q_\ell), \quad m, a > 0, \tag{3.2}$$

describes an isolated anharmonic oscillator of the reduced mass $m = m_{\rm ph}/\hbar^2$ and momentum p_ℓ. Its part $H_\ell^{\rm har}$ corresponds to a ν-dimensional quantum *harmonic oscillator* of rigidity a.

The Hamiltonian (3.1) has no direct mathematical meaning and is "represented" by *local Hamiltonians* H_Λ, $\Lambda \Subset \mathbb{L}$, which are

$$\begin{aligned} H_\Lambda &= \sum_{\ell\in\Lambda}\left[H_\ell^{\rm har} + V_\ell(q_\ell)\right] + \frac{1}{2}\sum_{\ell,\ell'\in\Lambda} W_{\ell\ell'}(q_l, q_{\ell'}) \\ &= \frac{1}{2m}\sum_{\ell\in\Lambda}|p_\ell|^2 + W_\Lambda(q_\Lambda), \quad q_\Lambda = (q_\ell)_{\ell\in\Lambda}. \end{aligned} \tag{3.3}$$

In the last line, the first term is the kinetic energy; the potential energy is

$$W_\Lambda(q_\Lambda) = \frac{1}{2}\sum_{\ell,\ell'\in\Lambda} W_{\ell\ell'}(q_\ell, q_{\ell'}) + \sum_{\ell\in\Lambda}\left[\frac{a}{2}|q_\ell|^2 + V_\ell(q_\ell)\right]. \tag{3.4}$$

The *interaction potentials*

$$\begin{aligned} V_\ell \in C(\mathbb{R}^\nu \to \mathbb{R}), \quad V_\ell(0) = 0, \quad \ell \in \mathbb{L}, \\ W_{\ell\ell'} = W_{\ell'\ell} \in C(\mathbb{R}^\nu \times \mathbb{R}^\nu \to \mathbb{R}), \quad W_{\ell\ell} \equiv 0, \quad \ell,\, \ell' \in \mathbb{L}, \end{aligned} \tag{3.5}$$

satisfy the basic hypotheses from Chapter 2, which for the reader's convenience we recall here:

Assumption (W) *There exist constants $R \geq 2$, $C_W \geq 0$ and a symmetric matrix $\mathbf{J} = (J_{\ell\ell'})_{\mathbb{L}\times\mathbb{L}}$ with the non-negative entries and zero diagonal, such that for all $q_\ell, q_{\ell'} \in \mathbb{R}^\nu$*

$$|W_{\ell\ell'}(q_\ell, q_{\ell'})| \leq \frac{1}{2}J_{\ell\ell'}(C_W + |q_\ell|^R + |q_{\ell'}|^R). \tag{3.6}$$

Assumption (V) *There exist a continuous function $V : \mathbb{R}^\nu \to \mathbb{R}$ and constants $P > 2$, $A_V > 0$, and $B_V \in \mathbb{R}$, such that for all $\ell \in \mathbb{L}$ and $q_\ell \in \mathbb{R}^\nu$*

$$A_V|q_\ell|^P + B_V \leq V_\ell(q_\ell) \leq V(q_\ell). \tag{3.7}$$

The *interaction intensities*, which are symmetric and vanish at the diagonal,

$$J_{\ell\ell'} = J_{\ell'\ell} \geq 0, \quad J_{\ell\ell} = 0, \quad \ell,\, \ell' \in \mathbb{L}, \tag{3.8}$$

are subject to the following

Assumption $(\mathbf{J}_0)$ *The dynamical matrix* $\mathbf{J} = (J_{\ell\ell'})_{\mathbb{L}\times\mathbb{L}}$ *generates a bounded operator in the Banach space* $l^\infty(\mathbb{L})$ *(and hence in all* $l^p(\mathbb{L})$*,* $1 \leq p \leq \infty$*), that means*

$$||\mathbf{J}||_0 := ||\mathbf{J}||_{\mathcal{L}(l^\infty(\mathbb{L}))} = \sup_{\ell} \sum_{\ell'} J_{\ell\ell'} < \infty. \tag{3.9}$$

Unlike the earlier Assumption $(\mathbf{J})$ in Chapter 2, we do not yet specify a decay rate of $J_{\ell\ell'}$ as the distance $|\ell - \ell'| \to \infty$. Note that the lower bound in (3.7) is responsible for confining each particle in the vicinity of its equilibrium position, while the upper bound is to guarantee that the oscillations of the particles located far from the origin are not suppressed. An example of $W_{\ell\ell'}$ and V_ℓ to bear in mind is the polynomials

$$W_{\ell\ell'}(q_\ell, q_{\ell'}) := \pm J_{\ell\ell'} \cdot (q_\ell, q_{\ell'}), \tag{3.10}$$
$$V_\ell(x_\ell) = \sum\nolimits_{\mathrm{s}=1}^{\mathrm{p}} b_\ell^{(\mathrm{s})} |q_\ell|^{2\mathrm{s}} - (h, q_\ell), \quad b_\ell^{(\mathrm{s})} \in \mathbb{R}, \quad b_\ell^{(\mathrm{p})} > 0, \quad \mathrm{p} \geq 2,$$

in which $h \in \mathbb{R}^\nu$ is an *external field* and the coefficients $b_\ell^{(s)}$ vary in certain intervals, such that both estimates (3.7) hold. Under Assumptions $(\mathbf{V})$ the *Schrödinger operator* H_Λ is a self-adjoint lower bounded operator in the complex Hilbert space $L^2(\mathbb{R}^{\nu|\Lambda|})$ having discrete spectrum. It generates a positivity preserving C_0-semigroup such that

$$\mathrm{trace}[\exp(-\tau H_\Lambda)] < \infty, \quad \text{for all } \tau > 0. \tag{3.11}$$

We indicate some important special cases of our model:

Definition 3.1 *The model is called the quantum anharmonic crystal, if* $\mathbb{L}$ *is a lattice, e.g.* $\mathbb{Z}^d$*. The model is* ***ferromagnetic*** *(in the physical interpretation, ferroelectric) if* $W_{\ell\ell'}(q_\ell, q_{\ell'}) := -J_{\ell\ell'} \cdot (q_\ell, q_{\ell'})$ *with* $J_{\ell\ell'} \geq 0$ *for all* $\ell, \ell' \in \mathbb{L}$*. The interaction has* ***finite range*** *if there exists* $r > 0$ *such that* $W_{\ell\ell'} \equiv 0$ *whenever* $|\ell - \ell'| > r$*. The model is* ***translation invariant*** *if* $\mathbb{L} = \mathbb{Z}^d$ *and* $V_\ell := V$*,* $W_{\ell\ell'} := W_{\ell-\ell'}$ *for all* ℓ, ℓ'*. The model is rotation invariant if* $V_\ell \circ U = V_\ell$ *and* $W_{\ell\ell'} \circ U \times U' = W_{\ell\ell'}$ *for any pair of orthogonal transformations* U*,* $U' \in O(\nu)$*.*

If $\mathbb{L} = \mathbb{Z}^d$, $V_\ell \equiv 0$, and $W_{\ell\ell'}(q_\ell, q_{\ell'}) := \pm J_{\ell\ell'} \cdot (q_\ell, q_{\ell'})$ for all ℓ, ℓ', the model is exactly solvable and is known as a *quantum harmonic crystal* (cf. [120]). It is stable if $||\mathbf{J}||_0 < a$, see Remark 3.25 below.

Remark 3.2 Afterwords, it would be instructive to compare our results on the quantum systems with the analogous classical ones. In accordance with the *Borg-Heisenberg correspondence principle* in quantum physics, the large-mass limit $m \to \infty$ (or $\hbar \to 0$)

of the model (3.1) gives rise to an infinite system of interacting *classical particles.* Such system is described by the configuration space $\Omega_{\rm cl} := [\mathbb{R}^\nu]^{\mathbb{L}} \ni (q_\ell)_{\ell\in\mathbb{L}} := q$ and the potential energy functional

$$H(q) = \frac{a}{2}\sum_{\ell} |q_\ell|^2 + \sum_{\ell} V_\ell(q_\ell) + \frac{1}{2}\sum_{\ell,\ell'} W_{\ell\ell'}(q_\ell, q_{\ell'}), \tag{3.12}$$

which was our main subject in Chapter 2. For a mathematical justification of the quasi-classical limit for the corresponding Gibbs states see Remark 3.5.

3.1.2 Quantum Gibbs states in the Euclidean approach

A complete description of thermal equilibrium properties of quantum systems might be given in terms of their Gibbs states. As was already mentioned, we take the *Euclidean* (i.e., *path space*) *approach* first implemented to quantum lattice systems by S. Albeverio and R. Høegh-Krohn in [3]. In this subsection we outline the basic elements of the Euclidean approach in the context of our model. For a detailed discussion about the deep intricate connections between quantum states and measures on loop spaces we refer e.g. to [7, 13, 24, 46].

To each $\Lambda \Subset \mathbb{L}$ there corresponds the local Hamiltonian H_Λ defined by (3.3), which acts in the physical Hilbert space $\mathcal{H}_\Lambda := L^2(\mathbb{R}^{\nu|\Lambda|} \to \mathbb{C})$. In view of (3.11) one can introduce the *local Gibbs state*

$$\mathfrak{C}_\Lambda \ni A \mapsto \varrho_\Lambda(A) := \frac{\mathrm{trace}(Ae^{-\beta H_\Lambda})}{\mathrm{trace}(e^{-\beta H_\Lambda})}, \tag{3.13}$$

which is a positive normalized functional on the algebra $\mathfrak{C}_\Lambda$ of all bounded linear operators (observables) on $\mathcal{H}_\Lambda$. The mappings

$$\mathfrak{C}_\Lambda \ni A \mapsto \mathfrak{a}_t^\Lambda(A) := e^{itH_\Lambda} A e^{-itH_\Lambda}, \quad t \in \mathbb{R}, \tag{3.14}$$

constitute the group of time automorphisms which describes the dynamics of the system in Λ. The state ϱ_Λ satisfies the *KMS* (*Kubo-Martin-Schwinger*) thermal equilibrium condition relative to the dynamics $\mathfrak{a}_t^\Lambda$, see Definition 1.1 in [159]. Multiplication operators by bounded continuous functions act as

$$(F\psi)(q_\Lambda) = F(q_\Lambda)\cdot\psi(q_\Lambda), \quad \psi \in \mathcal{H}_\Lambda, \quad F \in C_{\rm b}(\mathbb{R}^{\nu|\Lambda|}), \quad q_\Lambda \in \mathbb{R}^{\nu|\Lambda|}.$$

One can prove that the linear span of the products

$$\mathfrak{a}_{t_1}^\Lambda(F_1)\cdots\mathfrak{a}_{t_n}^\Lambda(F_n), \tag{3.15}$$

with all possible choices of $n \in \mathbb{N}$, $t_1, \ldots, t_n \in \mathbb{R}$ and $F_1, \ldots, F_n \in C_{\rm b}(\mathbb{R}^{\nu|\Lambda|})$, is σ-weakly dense in $\mathfrak{C}_\Lambda$. Therefore, as a σ-weakly continuous functional (see page 65 of the first volume of [66]), the state (3.13) is fully determined by its values on (3.15), that is, by the (real time) *Green functions*

$$G^\Lambda_{F_1,\ldots,F_n}(t_1,\ldots,t_n) := \varrho_\Lambda\left[\mathfrak{a}_{t_1}^\Lambda(F_1)\cdots\mathfrak{a}_{t_n}^\Lambda(F_n)\right]. \tag{3.16}$$

They can be considered as restrictions of functions $G^{\Lambda}_{F_1,\ldots,F_n}(z_1,\ldots,z_n)$, analytic in the tubular domain

$$\mathcal{D}^n_\beta = \{(z_1,\ldots z_n) \in \mathbb{C}^n \mid 0 < \Im(z_1) < \Im(z_2) < \cdots < \Im(z_n) < \beta\}, \tag{3.17}$$

and continuous on its closure $\bar{\mathcal{D}}^n_\beta \subset \mathbb{C}^n$. For every $n \in \mathbb{N}$, the *"imaginary time"* domain

$$\{(z_1,\ldots,z_n) \in \mathcal{D}^n_\beta \mid \Re(z_1) = \cdots = \Re(z_n) = 0\}$$

is an inner set of uniqueness for functions analytic in $\mathcal{D}^n_\beta$ (see pages 101 and 352 of [264]). Therefore, the Green functions (3.16), and hence the states (3.13), are completely determined by the Matsubara (or *Euclidean Green)* functions

$$\begin{aligned} \mathit{\Gamma}^{\Lambda}_{F_1,\ldots,F_n}(\tau_1,\ldots,\tau_n) &:= G^{\Lambda}_{F_1,\ldots,F_n}(\imath\tau_1,\ldots,\imath\tau_n) \\ &= \mathrm{trace}[F_1 e^{-(\tau_2-\tau_1)H_\Lambda} F_2 e^{-(\tau_3-\tau_2)H_\Lambda} \cdots F_n e^{-(\tau_{n+1}-\tau_n)H_\Lambda}]/\mathrm{trace}[e^{-\beta H_\Lambda}] \end{aligned} \tag{3.18}$$

taken at ordered arguments $0 \le \tau_1 \le \cdots \le \tau_n \le \tau_1 + \beta := \tau_{n+1}$, with all possible choices of $n \in \mathbb{N}$ and $F_1,\ldots,F_n \in C_{\mathrm{b}}(\mathbb{R}^{\nu|\Lambda|})$. Their extension to $[0,\beta]^n$ is given by

$$\mathit{\Gamma}^{\Lambda}_{F_1,\ldots,F_n}(\tau_1,\ldots,\tau_n) = \mathit{\Gamma}^{\Lambda}_{F_{\sigma(1)},\ldots,F_{\sigma(n)}}(\tau_{\sigma(1)},\ldots,\tau_{\sigma(n)}),$$

where σ is the permutation of $\{1,2,\ldots,n\}$ such that $\tau_{\sigma(1)} \le \tau_{\sigma(2)} \le \cdots \le \tau_{\sigma(n)}$. One can show that for every $\theta \in [0,\beta]$,

$$\mathit{\Gamma}^{\Lambda}_{F_1,\ldots,F_n}(\tau_1+\theta,\ldots,\tau_n+\theta) = \mathit{\Gamma}^{\Lambda}_{F_1,\ldots,F_n}(\tau_1,\ldots,\tau_n), \tag{3.19}$$

where addition is modulo β. This periodicity along with the analyticity of the Green functions is equivalent to the *KMS* property of the state (3.13).

The *central element* of the Euclidean approach is the representation of the Matsubara functions (3.18) corresponding to $F_1,\ldots,F_n \in C_{\mathrm{b}}(\mathbb{R}^{\nu|\Lambda|})$ in the form of

$$\mathit{\Gamma}^{\Lambda}_{F_1,\ldots,F_n}(\tau_1,\ldots,\tau_n) = \int_{\Omega_\Lambda} F_1(\omega_\Lambda(\tau_1))\ldots F_n(\omega_\Lambda(\tau_n))\mu_\Lambda(\mathrm{d}\omega_\Lambda), \tag{3.20}$$

where μ_Λ is a certain probability measure on the space Ω_Λ, which we construct in Subsection 3.1.3. This measure is called a *local Euclidean Gibbs measure*. By standard arguments, it is uniquely determined by the integrals (3.20). Since the Matsubara functions $\mathit{\Gamma}^{\Lambda}_{F_1,\ldots,F_n}$ uniquely determine the state ϱ_Λ, the representation (3.20) establishes a *one-to-one correspondence* between the local Gibbs states ϱ_Λ and local Euclidean Gibbs measures μ_Λ. In particular, for a multiplication operator by any $F \in C_{\mathrm{b}}(\mathbb{R}^{\nu|\Lambda|})$ and for all $\tau \in S_\beta$,

$$\varrho_\Lambda(F) = \mathit{\Gamma}^{\Lambda}_F(\tau) = \int_{\Omega_\Lambda} F(\omega_\Lambda(\tau))\mu_\Lambda(\mathrm{d}\omega_\Lambda). \tag{3.21}$$

Thermodynamic properties of the model (3.1) are described by the Gibbs states corresponding to the whole set $\mathbb{L}$. Such states should be defined on the C^*-algebra of *quasi-local observables* $\mathfrak{C}$, being the norm-completion of the algebra of local observables $\cup_{\Lambda \Subset \mathbb{L}} \mathfrak{C}_\Lambda$. Here each $\mathfrak{C}_\Lambda$ is considered as a subalgebra of $\mathfrak{C}_{\Lambda'}$ for any Λ' containing Λ. The *dynamics* of the whole system is to be defined by the limits $\Lambda \nearrow \mathbb{L}$ of the time automorphisms (3.14), which would allow one to define the Gibbs states on $\mathfrak{C}$ as *KMS* states. This *"algebraic"* way can successfully be realized for models described by bounded local Hamiltonians H_Λ, e.g. quantum spin models, see Section 6.2 of [66]. For the model considered here, such limiting automorphisms do not exist and hence there is *no canonical way* to specify Gibbs states of the whole infinite system. Therefore, the Euclidean approach based on the one-to-one correspondence between the local states and measures arising from the representation (3.20) seems to be the only way of developing a mathematical theory of the equilibrium thermodynamic properties of such models. Let us note that for some versions of quantum crystals, a possibility of constructing the limiting states $\varrho = \lim_{\Lambda \nearrow \mathbb{L}} \varrho_\Lambda$ in terms of the limiting path measures $\mu = \lim_{\Lambda \nearrow \mathbb{L}} \mu_\Lambda$ was discussed in [27, 215, 214]. The set of Euclidean Gibbs measures $\mathcal{G}^{\mathrm{t}}$ we concern here will *certainly* includes all the limiting points of this type. Furthermore, there exist axiomatic methods, see [46, 125, 126], analogous to the Osterwalder-Schrader reconstruction theorem in the Euclidean field theory [129, 255], which allow to construct KMS states on certain von Neumann algebras starting from a complete set of Matsubara functions. In our case such a set consists of the functions

$$\Gamma^{\mu}_{F_1,\dots,F_n}(\tau_1,\dots,\tau_n) = \int_{\Omega} F_1(\omega(\tau_1)) \cdots F_n(\omega(\tau_n))\mu(\mathrm{d}\omega), \quad \mu \in \mathcal{G}^{\mathrm{t}}, \tag{3.22}$$

corresponding to all local multiplication operators by bounded continuous functions $F_1, \dots, F_n$. Therefore, the theory of Euclidean Gibbs measures presented in this manuscript can be further developed towards identifying such algebras and states, which we leave as a task for the future.

3.1.3 Temperature loops and β-periodic processes

In this subsection we introduce the local Euclidean Gibbs measures, which provide the integral representation (3.20) for the Matsubara functions. They will be constructed via the Feynman-Kac formula as Gibbs modifications (3.50) of the *"free"* loop measure χ corresponding to a single quantum harmonic oscillator.

(i) Loop spaces

The local Euclidean Gibbs measures are supported by the spaces of β-periodic paths, i.e., *temperature loops*. These are continuous functions defined on the interval $[0, \beta]$ and taking equal values at the endpoints. Here $T = \beta^{-1} > 0$ is the absolute temperature. One can consider the loops as functions on the circle $S_\beta \cong [0, \beta]$ being a compact Riemannian manifold with Lebesgue measure $\mathrm{d}\tau$ and distance

$$|\tau - \tau'|_\beta := \min\{|\tau - \tau'| \ ; \ \beta - |\tau - \tau'|\}, \quad \tau, \tau' \in S_\beta. \tag{3.23}$$

In our context, as (Euclidean) single *spin spaces* at given ℓ we shall use the standard Banach spaces

$$\begin{aligned} L_\beta^R &:= \ L^R(S_\beta \to \mathbb{R}^\nu, \mathrm{d}\tau), \quad R \geq 2, \\ C_\beta := C(S_\beta \to \mathbb{R}^\nu), &\quad C_\beta^\sigma := C^\sigma(S_\beta \to \mathbb{R}^\nu), \quad \sigma \in (0,1), \end{aligned} \tag{3.24}$$

of all integrable respectively (Hölder) continuous functions $\upsilon = (\upsilon^i)_{i=1}^\nu : S_\beta \to \mathbb{R}^\nu$ with the norms

$$\begin{aligned} |\upsilon|_{L_\beta^R} &:= \left[\int_{S_\beta} |\upsilon(\tau)|^R \mathrm{d}\tau\right]^{1/R}, \\ |\upsilon|_{C_\beta} := \sup_{\tau \in S_\beta} |\upsilon(\tau)|, &\quad |\upsilon|_{C_\beta^\sigma} := |\upsilon|_{C_\beta} + \sup_{\tau,\tau' \in S_\beta,\ \tau \neq \tau'} \frac{|\upsilon(\tau) - \upsilon(\tau')|}{|\tau - \tau'|_\beta^\sigma} \ . \end{aligned} \tag{3.25}$$

Note that the Lebesgue spaces L_β^R are needed to take care of the interaction between loops. In analytical constructions, the Hilbert space of all square integrable loops L_β^2 will play role of a *tangent space* to C_β; its inner product and norm are denoted by $(\cdot,\cdot)_{L_\beta^2}$ and $|\cdot|_{L_\beta^2}$. One has the *dense continuous embeddings* $C_\beta^\sigma \hookrightarrow C_\beta \hookrightarrow L_\beta^R$, that by the Kuratowski theorem (cf. page 499 of [175]) yields

$$C_\beta \in \mathcal{B}(L_\beta^R) \quad \text{and} \quad \mathcal{B}(C_\beta) = \mathcal{B}(L_\beta^R) \cap C_\beta. \tag{3.26}$$

Furthermore, we crucially shall use the fact that the embeddings

$$C_\beta^\sigma \hookrightarrow C_\beta^{\sigma'} \hookrightarrow C_\beta \ \text{ are } \textit{compact} \text{ whenever } \ 0 < \sigma' < \sigma. \tag{3.27}$$

However, the reader is warned that the above spaces C_β^σ are not separable and the embeddings $C_\beta^\sigma \hookrightarrow C_\beta^{\sigma'}$ are not dense. Nevertheless, each C_β^σ is measurable as a subset in C_β or L_β^R (cf. page 278 of [240]).

Given $\Lambda \subseteq \mathbb{L}$, we define

$$\Omega_\Lambda := \{\omega_\Lambda = (\omega_\ell)_{\ell\in\Lambda} \mid \omega_\ell \in C_\beta\}, \quad \Omega := \Omega_{\mathbb{L}} = \{\omega = (\omega_\ell)_{\ell\in\mathbb{L}} \mid \omega_\ell \in C_\beta\}. \tag{3.28}$$

These spaces are equipped with the product topology and with the Borel σ-algebras $\mathcal{B}(\Omega_\Lambda)$. Thereby, each Ω_Λ is a Polish space; its elements are called configurations in Λ. In particular, Ω is the *configuration space* for the whole system. For $\Lambda \subset \Lambda'$, the decomposition $\omega_{\Lambda'} = \omega_\Lambda \times \omega_{\Lambda'\setminus\Lambda}$ defines an embedding $\Omega_\Lambda \hookrightarrow \Omega_{\Lambda'}$ by identifying $\omega_\Lambda \in \Omega_\Lambda$ with $\omega_\Lambda \times 0_{\Lambda'\setminus\Lambda} \in \Omega_{\Lambda'}$. By $\mathcal{P}(\Omega_\Lambda)$ and $\mathcal{P}(\Omega)$ we denote the sets of all probability measures on $(\Omega_\Lambda, \mathcal{B}(\Omega_\Lambda))$ and $(\Omega, \mathcal{B}(\Omega))$.

(ii) Basic Gaussian measure related to a harmonic oscillator

A ν-dimensional *quantum harmonic oscillator* of mass $m > 0$ and rigidity $a > 0$ is described by the Hamiltonian, cf. (3.2),

$$H_\ell^{\mathrm{har}} := -\frac{1}{2m}\sum_{i=1}^\nu \left(\frac{\partial}{\partial x_\ell^i}\right)^2 + \frac{a}{2}|x_\ell|^2, \tag{3.29}$$

acting in the physical Hilbert space $\mathcal{H}_\ell := L^2(\mathbb{R}^\nu \to \mathbb{C})$. The integral kernels $\aleph^{\mathrm{har}}(t; x_\ell, y_\ell)$ of the semigroup $\exp(-tH_\ell^{\mathrm{har}})_{t\geq 0}$ are represented, for all $t > 0$ and $x_\ell, y_\ell \in \mathbb{R}^\nu$, by *Mehler's formula*

$$\begin{aligned}
&\aleph^{\mathrm{har}}(t; x_\ell, y_\ell) : \qquad (3.30)\\
&= (m\mathfrak{g}_{2t}/\pi)^{\nu/2} \exp\left\{\frac{1}{2}t\sqrt{\frac{a}{m}} - \frac{1}{2}m\mathfrak{g}_{2t}\left[(1+e^{-2t\sqrt{\frac{a}{m}}})(x_\ell^2+y_\ell^2) - 4e^{-t\sqrt{\frac{a}{m}}}x_\ell y_\ell\right]\right\}\\
&= (m\mathfrak{g}_{2t}/\pi)^{\nu/2} \exp\left\{\frac{1}{2}t\sqrt{\frac{a}{m}} - \frac{\sqrt{am}}{2}\coth\left(t\sqrt{\frac{a}{m}}\right)\left[(x_\ell^2+y_\ell^2) - \frac{2x_\ell y_\ell}{\cosh\left(t\sqrt{\frac{a}{m}}\right)}\right]\right\},
\end{aligned}$$

where we introduced the parameter

$$\mathfrak{g}_t := \sqrt{\frac{a}{m}}\left(1 - e^{-t\sqrt{\frac{a}{m}}}\right)^{-1}, \quad t > 0, \tag{3.31}$$

(cf. e.g. Theorem 1.5.10 in [129] or page 299 of [131]). By means of the kernels (3.30) we can generate a Gaussian β-periodic process $\omega_\ell(\tau) \in \mathbb{R}^\nu$, $\tau \in S_\beta$, which is also known as the *periodic Ornstein-Uhlenbeck velocity process*, see [158]. In quantum statistical mechanics it first appeared in the papers of Albeverio and Høegh-Krohn [3, 148]. The canonical realization of this process on $(C_\beta, \mathcal{B}(C_\beta))$ is a path measure χ given by Kolmogorov's extension theorem through its *marginal distributions*

$$\begin{aligned}
&\chi\left(\{\upsilon \in C_\beta \mid \upsilon(\tau_j) \in B_j \in \mathcal{B}(\mathbb{R}^\nu),\ 1 \leq j \leq n\}\right) \qquad (3.32)\\
&= \frac{1}{Z_{\tau_1,\ldots,\tau_n}}\int_{B_1\times\cdots\times B_n}\prod_{1\leq j\leq n}\aleph^{\mathrm{har}}(\tau_{j+1}-\tau_j; \upsilon(\tau_{j+1}), \upsilon(\tau_j)) \times_{j=1}^n \mathrm{d}\omega_\ell(\tau_j),
\end{aligned}$$

taken at all finite sets of ordered points $0 \leq \tau_1 \leq \cdots \leq \tau_n \leq \tau_{n+1} := \tau_1 + \beta$ on the circle S_β. As a Gaussian process, χ is completely determined by its correlation functions for all $\tau, \tau' \in S_\beta$ and $1 \leq i, i' \leq \nu$,

$$\mathbf{E}_\chi[\upsilon^i(\tau)] = 0, \quad \mathbf{E}_\chi[\upsilon^i(\tau)\upsilon^{i'}(\tau')] = \delta_{ii'}\mathfrak{G}(\tau, \tau'), \tag{3.33}$$

$$\text{with} \qquad \mathfrak{G}(\tau,\tau') := \frac{\left(e^{-\sqrt{\frac{a}{m}}(\beta-|\tau-\tau'|_\beta)} + e^{-\sqrt{\frac{a}{m}}|\tau-\tau'|_\beta}\right)}{2\sqrt{am}\left(1-e^{-\sqrt{\frac{a}{m}}\beta}\right)}. \tag{3.34}$$

Its higher moments for $r \in \mathbb{N}$ can be estimated as

$$\mathbf{E}_\chi|\upsilon^i(\tau)|^{2r} = \frac{(2r)!}{2^r r!}\left[\mathbf{E}_\chi|\upsilon^i(\tau)|^2\right]^r \leq \frac{(2r)!}{2^r r!}\mathfrak{G}^r(\tau,\tau) \leq \frac{(2r)!}{(2a)^r r!}\mathfrak{g}_\beta^r, \tag{3.35}$$

$$\begin{aligned}
&\mathbf{E}_\chi|\upsilon^i(\tau) - \upsilon^i(\tau')|^{2r} = \frac{(2r)!}{2^r r!}\left[\mathbf{E}_\chi|\upsilon^i(\tau)-\upsilon^i(\tau')|^2\right]^r\\
&= \frac{(2r)!}{r!}\left[\mathfrak{G}(\tau,\tau) - \mathfrak{G}(\tau,\tau')\right]^r \leq \frac{(2r)!}{(\sqrt{am})^r r!}\mathfrak{g}_\beta^r|\tau-\tau'|_\beta^r, \qquad (3.36)
\end{aligned}$$

where $\mathfrak{g}_\beta$ is defined by (3.31) with $t := \beta$. In particular, (3.36) yields by Kolmogorov's lemma (page 237 in [128] or page 43 of [257]) that χ is supported by Hölder continuous loops, i.e.,

$$\chi(C_\beta^\sigma) = 1, \quad \text{for all } \sigma \in (0, 1/2). \tag{3.37}$$

The measure (3.32) can be equivalently described in terms of its Fourier transform as follows. In complexification of the Hilbert space L^2_β, let us consider the self-adjoint *Laplace-Beltrami* type operator

$$A := \left(-m \frac{\mathrm{d}^2}{\mathrm{d}\tau^2} + a \right) \otimes \mathbf{Id}_\nu, \tag{3.38}$$

where $\mathbf{Id}_\nu$ is the identity matrix in $\mathbb{R}^\nu$. It has discrete spectrum consisting of the *eigenvalues* (each of them of multiplicity ν)

$$\lambda_k = 2m(\pi k/\beta)^2 + a, \quad k \in \mathbb{Z}, \tag{3.39}$$

which correspond to the *eigenvectors* $\varphi_k \otimes e_i$. Here

$$\varphi_k(\tau) = \begin{cases} \left(\frac{1}{\beta}\right)^{\frac{1}{2}}, & k = 0\ , \\ (2/\beta)^{1/2} \cos(2\pi k \tau/\beta), & k = 1, 2, \dots\ , \\ -\,(2/\beta)^{1/2} \sin(2\pi k \tau/\beta), & k = -1, -2, \dots\ . \end{cases} \tag{3.40}$$

is the complete orthonormal system of trigonometric functions on S_β and $(e^i)_{i=1}^\nu$ is the canonical base of the Euclidean space $\mathbb{R}^\nu$. Thereby, the resolvent A^{-1} is of trace class and the Fourier transform

$$\int_{L^2_\beta} \exp[\imath(\phi, \upsilon)_{L^2_\beta}]\chi(\mathrm{d}\upsilon) := \exp\left\{ -\frac{1}{2}(A^{-1}\phi, \phi)_{L^2_\beta} \right\}, \quad \phi \in L^2_\beta, \tag{3.41}$$

uniquely defines a Gaussian measure χ on $(L^2_\beta, \mathcal{B}(L^2_\beta))$. The corresponding *Green function* (i.e., integral kernel of A^{-1}) is given by

$$(A^{-1}\delta_\tau)(\tau') := \mathfrak{G}(\tau, \tau') \otimes \mathbf{Id}_\nu \in \mathbb{R}^\nu, \quad \tau, \tau' \in S_\beta, \tag{3.42}$$

where

$$\mathfrak{G}(\tau, \tau') := \sum_{k \in \mathbb{Z}} \lambda_k^{-1} \varphi_k(\tau)\varphi_k(\tau') = \frac{2}{\beta a^2} + \frac{4}{\beta} \sum_{k \in \mathbb{N}}^{\infty} \frac{\cos\{2\pi k(\tau - \tau')/\beta\}}{4m\,(\pi k/\beta)^2\, m + a}. \tag{3.43}$$

Respectively, the kernels of the semigroup $\exp(-tA)$, $t > 0$, can be represented by $\mathfrak{K}(t; \tau, \tau') \otimes \mathbf{Id}_\nu$, where

$$\mathfrak{K}(t; \tau, \tau') := \sum_{k \in \mathbb{Z}} e^{-t\lambda_k} \varphi_k(\tau)\varphi_k(\tau') = \frac{2}{\beta} e^{-ta} \sum_{k \in \mathbb{N}}^{\infty} \frac{\cos\{2\pi k(\tau - \tau')/\beta\}}{\exp\{2tm(\pi k/\beta)^2\}}. \tag{3.44}$$

Note that the latter series is known as the classical Jacobi $\vartheta_3(u,r)$-function with parameters $u := (\tau - \tau')/\beta$ and $r = \exp\{-2tm(\pi/\beta)^2\} < 1$ (cf. page 463 of [283]).

One observes that the expressions in (3.34) and (3.43) coincide, which means that both definitions (3.32) and (3.41) give rise to the same measure χ. An account of the properties of χ may be found in [7, 13]. One of them, which plays a crucial role in the sequel, follows directly from (3.37) and *Fernique's theorem* (see Theorem 1.3.24 in [88]).

Proposition 3.3 *For every $\sigma \in (0, 1/2)$, there exists $\lambda_\sigma > 0$ such that*

$$\int_{L^2_\beta} \exp\left(\lambda_\sigma |\upsilon|^2_{C^\sigma_\beta}\right) \chi(\mathrm{d}\upsilon) < \infty. \tag{3.45}$$

Remark 3.4 Actually, a maximal possible value of λ_σ can be calculated in terms of the parameters a, m, and β. To this end we borrow such a powerful tool from the theory of stochastic processes as the *Garsia-Rodemich-Rumsey lemma* (see Inequalities (3.b) and (3.d) on pages 203–204 in [39]). In our notation, it says that (3.36) implies the estimate

$$\mathbf{E}_\chi \left\{ \sup_{\tau \neq \tau'} \frac{|\upsilon^i(\tau) - \upsilon^i(\tau')|}{|\tau - \tau'|^\sigma_\beta} \right\}^{2r} \leq C_{3.46}(1+\sigma^{-1}) \frac{(2r)!}{(\sqrt{am})^r r!} \mathfrak{g}^r_\beta 2^{7r} \beta^{(1-2\sigma)r}, \tag{3.46}$$

which holds, for all $\sigma \in (0, 1/2)$ and large enough $r > 2\,(1-2\sigma)^{-1}$, with some absolute constant $C_{3.46} \geq 1$. Then, by (3.35) and (3.46)

$$\begin{aligned} \mathbf{E}_\chi |\upsilon|^{2r}_{C^\sigma_\beta} &\leq 3^{2r-1} \left[\mathbf{E}_\chi |\upsilon(0)|^{2r} + (1+\beta^{2r\sigma}) \mathbf{E}_\chi \left\{ \sup_{\tau \neq \tau'} \frac{|\upsilon^i(\tau) - \upsilon^i(\tau')|}{|\tau - \tau'|^\sigma_\beta} \right\}^{2r} \right] \\ &\leq C_{3.46}(1+\sigma^{-1}) r! \nu^r 6^{2r} (1+\beta^r) \mathfrak{g}^r_\beta \left[\frac{1}{(2a)^r} + \frac{2^{7r}}{(\sqrt{am})^r} \right]. \end{aligned} \tag{3.47}$$

This allows us to find a small enough $\lambda > 0$, which depends on a, m and β only, such that the corresponding series $\sum_{r=0}^\infty (\lambda^r/r!)\, \mathbf{E}_\chi |\upsilon|^{2r}_{C^\sigma_\beta}$ in the left-hand side in (3.45) is convergent. An important observation resulting from the estimate (3.47) is that this λ can be chosen *the same for all* $\sigma \in (0, 1/2)$, but the value of the integral in (3.45) is certainly growing to the infinity as $\sigma \nearrow 1/2$.

(iii) Local Euclidean Gibbs measures

The above defined χ is the Euclidean Gibbs measure for a single harmonic oscillator. The measure $\mu_\Lambda \in \mathcal{P}(\Omega_\Lambda)$, which corresponds to the system of interacting anharmonic oscillators located in $\Lambda \Subset \mathbb{L}$, is associated with a stationary *β-periodic Markov process* generated by the semigroup $\exp(-\tau H_\Lambda)_{\tau \geq 0}$. The marginal distributions of μ_Λ are given by the integral kernels $\aleph_\tau^\Lambda(x_\Lambda, y_\Lambda)$, $x_\Lambda, y_\Lambda \in \mathbb{R}^{\nu|\Lambda|}$, of the operators $\exp(-\tau H_\Lambda)$, $\tau \in [0, \beta]$. Similarly to (3.32), this means that

$$\mu_\Lambda\left(\{\omega_\Lambda \in \Omega_\Lambda \mid \omega_\Lambda(\tau_j) \in B_j,\ 1 \le j \le n\}\right) \tag{3.48}$$
$$= \frac{1}{Z_{\tau_1,\dots,\tau_n}} \int_{B_1\times\cdots\times B_n} \prod_{1\le j\le n} \aleph^\Lambda_{|\tau_{j+1}-\tau_j|_\beta}(\omega_\Lambda(\tau_{j+1}), \omega_\Lambda(\tau_j)) \times_{j=1}^n \mathrm{d}\omega_\Lambda(\tau_j),$$

for all $n \in \mathbb{N}$ and Borel sets $B_1, \dots, B_n \in \mathcal{B}(\mathbb{R}^{\nu|\Lambda|})$. Herefrom the *basic relation* in the Euclidean approach is coming out (cf. (3.18) and (3.20)):

$$\mathrm{trace}[F_1 e^{-(\tau_2-\tau_1)H_\Lambda} F_2 e^{-(\tau_3-\tau_2)H_\Lambda} \cdots F_n e^{-(\tau_{n+1}-\tau_n)H_\Lambda}]/\mathrm{trace}[e^{-\beta H_\Lambda}] \tag{3.49}$$
$$= \int_{\Omega_\Lambda} F_1(\omega_\Lambda(\tau_1) \cdots F_n(\omega_\Lambda(\tau_n))\mu_\Lambda(\mathrm{d}\omega_\Lambda),$$

for all bounded functions $F_1, \dots, F_n \in L^\infty(\mathbb{R}^{\nu|\Lambda|})$. Note that both (3.48) and (3.49) are taken at ordered points $0 \le \tau_1 \le \cdots \le \tau_n \le \tau_{n+1} := \tau_1 + \beta$ on the circle S_β. And vice verse, the representation (3.49) uniquely, up to equivalence, defines H_Λ (see [159]). By means of the Feynman-Kac formula the measure μ_Λ can be viewed as a *Gibbs modification*

$$\mu_\Lambda(\mathrm{d}\omega_\Lambda) = (1/Z_\Lambda) \exp\{-I_\Lambda(\omega_\Lambda)\}\, \chi_\Lambda(\mathrm{d}\omega_\Lambda), \tag{3.50}$$

of the *"free"* measure $\chi_\Lambda(\mathrm{d}\omega_\Lambda) = \prod_{\ell\in\Lambda} \chi(\mathrm{d}\omega_\ell)$. Here

$$I_\Lambda(\omega_\Lambda) = \frac{1}{2} \sum_{\ell,\ell'\in\Lambda} \int_0^\beta W_{\ell\ell'}(\omega_\ell(\tau), \omega_{\ell'}(\tau))\mathrm{d}\tau + \sum_{\ell\in\Lambda} \int_0^\beta V_\ell(\omega_\ell(\tau))\mathrm{d}\tau \tag{3.51}$$

is the *Euclidean energy functional* (with empty boundary condition $\xi = \varnothing$) describing the system of interacting paths ω_ℓ, $\ell \in \Lambda$, whereas

$$Z_\Lambda = \int_{\Omega_\Lambda} \exp\{-I_\Lambda(\omega_\Lambda)\}\, \chi_\Lambda(\mathrm{d}\omega_\Lambda), \tag{3.52}$$

is the partition function. As mentioned above, μ_Λ is the local Gibbs measure, where *local* means corresponding to a $\Lambda \Subset \mathbb{L}$.

Remark 3.5 Let us briefly analyze what happens in the *quasiclassical limit* $m \to \infty$. First we observe that the pair covariances $\mathfrak{G}_m(\tau, \tau') := \mathfrak{G}(\tau, \tau')$ in (3.33) converge to $(a\beta)^{-1}$. The constants in the right-hand side (3.35) and (3.36) are then uniformly bounded for m taking values in any compact interval in $\mathbb{R}_+$. By Kolmogorov's lemma (cf. Theorem 2, page 485 of [128]) this implies the *tightness* of the family of measures $\chi_m := \chi$ defined by (3.32). We claim that they weakly converge in C_β to a Gaussian measure g, which is supported by the subspace $\mathcal{H}_0 \cong \mathbb{R}^\nu$ of constant loops $\omega(\tau) \equiv q \in \mathbb{R}^\nu$ and coincides with the *normal distribution* $g(\mathrm{d}x) := (a\beta/2\pi)^{\nu/2} \exp\{-a\beta|x|^2/2\}\,\mathrm{d}x$. Employing the eigenvalues (3.39), it is easy to see that covariance operators A^{-1} converge in the trace norm in $L^2_\beta = \mathcal{H}_0 \oplus \mathcal{H}_0^\perp$ to the operator $(a\beta)^{-1}\mathbf{Id}_\nu \oplus 0$. Thus, by

Lemma 5.1, page 182 of [226], we have the *weak convergence* of the Gaussian measures $\chi_m \to g$ in L^2_β, and hence g is the unique limit point also in the weak topology on C_β. Since the mappings $\Omega_\Lambda \ni \omega_\Lambda \to \exp\{-I_\Lambda(\omega_\Lambda)\}$ are continuous and bounded, this immediately implies the weak convergence of the local Gibbs measures $\mu_\Lambda(\mathrm{d}\omega_\Lambda)$ to the classical Gibbs distributions $(1/Z_\Lambda)\exp\{-\beta W_\Lambda(q_\Lambda)\}\,dq_\Lambda$ with the potential energy $W_\Lambda(q_\Lambda)$ given by (3.4).

3.1.4 Weights and tempered configurations of loops

According to the original paper [3], the Euclidean Gibbs measures on the whole $\mathbb{L}$ we are interested in will have a *heuristic representation*

$$\mu(\mathrm{d}\omega) := Z^{-1}\exp\{-I(\omega)\}\prod_{\ell\in\mathbb{L}}\chi(\mathrm{d}\omega_\ell), \tag{3.53}$$

where $I(\omega)$ is the Euclidean action functional associated with the Hamiltonian (3.1), (3.2) and formally written as the infinite sum over all $\ell, \ell' \in \mathbb{L}$ in (3.51). In full analogy with classical statistical mechanics, a rigorous meaning can be given to the measure μ by the *DLR formalism* as a Gibbsian random field on $\mathbb{L}$, but now with the *infinite-dimensional* single spin spaces. As compared to the classical (non-quantum) systems with vector spin $x_\ell \in \mathbb{R}^\nu$, this leads to a more sophisticated realization of the *DLR* scheme to be performed below. Namely, we shall need a variety of the spin spaces like C_β, C^σ_β,or L^R_β to describe the local properties of the Gibbs measures μ in volumes $\Lambda \Subset \mathbb{L}$. Of course, the topological features of these functional spaces should be taken into account carefully.

So, in Subsections 3.1.4, 3.1.5 we shall first study the corresponding *local specification* $\Pi = \{\pi_\Lambda\}_{\Lambda\Subset\mathbb{L}}$. Its kernels $\pi_\Lambda(\mathrm{d}\omega|\xi)$ can be explicitly written (cf. (3.78)) in terms of the *energy functional* $I_\Lambda(\cdot|\xi)$ describing the interaction with a configuration $\xi \in \Omega$ fixed outside of Λ. In accordance with (3.3) it is

$$I_\Lambda(\omega_\Lambda|\xi) = I_\Lambda(\omega_\Lambda) + \sum_{\ell\in\Lambda,\ \ell'\in\Lambda^c}\int_0^\beta W_{\ell\ell'}(\omega_\ell(\tau), \xi_{\ell'}(\tau))\mathrm{d}\tau, \tag{3.54}$$

where I_Λ is given by (3.51). Clearly, the second term in (3.54) makes sense for all $\xi \in \Omega$ only if the interaction has finite range. Otherwise, we have to confirm ourselves to reasonable subsets of $\xi \in \Omega$, whose elements fulfill some natural restrictions on the growth of $\{|\xi_\ell|_{L^R_\beta}\}_{\ell\in\mathbb{L}}$ being in accordance with the decay of $J_{\ell\ell'}$. As is commonly accepted, configurations with controlled growth are called *tempered*. Because of a rather general character of the set $\mathbb{L}$ on which our system lives, to impose such growth restrictions we shall use families of weights $(w_\alpha)_{\alpha\in\mathcal{I}}$.

Definition 3.6 ***Weights*** *are the symmetric mappings* $w_\alpha : \mathbb{L}\times\mathbb{L} \to (0,1]$, *indexed by an interval*

$$\alpha \in \mathcal{I}=(\underline{\alpha}, \overline{\alpha}), \quad 0 \le \underline{\alpha} < \overline{\alpha} \le \infty, \tag{3.55}$$

which satisfy the following conditions:

(a) *for any* $\alpha \in \mathcal{I}$ *and* ℓ, $w_\alpha(\ell, \ell) = 1$;

(b) *for any* $\alpha \in \mathcal{I}$ *and* ℓ_1, ℓ_2, ℓ_3,

$$w_\alpha(\ell_1, \ell_2) \cdot w_\alpha(\ell_2, \ell_3) \leq w_\alpha(\ell_1, \ell_3) \quad (\textit{triangle inequality}), \tag{3.56}$$

(c) *for any* $\alpha, \alpha' \in \mathcal{I}$, *such that* $\alpha < \alpha'$, *and arbitrary* ℓ, ℓ',

$$w_{\alpha'}(\ell, \ell') \leq w_\alpha(\ell, \ell'), \qquad \lim_{|\ell - \ell'| \to \infty} w_{\alpha'}(\ell, \ell')/w_\alpha(\ell, \ell') = 0. \tag{3.57}$$

The *optimal* choice of $(w_\alpha)_{\alpha \in \mathcal{I}}$ depends on the indexing set $\mathbb{L}$ and on the decay of $J_{\ell\ell'}$, which thus will be subject to the following

Assumption $(\mathbf{L}_\alpha)$ *For all* $\alpha \in \mathcal{I}$, *it holds*

$$\sup_\ell \sum_{\ell'} \log(1 + |\ell - \ell'|) \cdot w_\alpha(\ell, \ell') < \infty. \tag{3.58}$$

Assumption $(\mathbf{J}_\alpha)$ *For all* $\alpha \in \mathcal{I}$, *it holds*

$$||\mathbf{J}||_\alpha := \sup_\ell \sum_{\ell'} J_{\ell\ell'} \left[w_\alpha(\ell, \ell')\right]^{-1} < \infty. \tag{3.59}$$

Given $\iota > 0$, *which is a parameter of the theory, there exists* $\alpha \in \mathcal{I}$ *such that*

$$||\mathbf{J}||_\alpha - ||\mathbf{J}||_0 < \iota. \tag{3.60}$$

The precise meaning of this ι, depending on the other parameters of the model, will be specified later. We observes that the conditions (3.58) and (3.59) are mutually competitive. One easily finds examples of $J_{\ell\ell'}$ obeying (3.9), but such that (3.58) and (3.59) cannot be fulfilled simultaneously by any system of weights w_α.

Example 3.7 Let us recall some typical situations, cf. Assumptions $(\mathbf{J})$ and $(\mathbf{J}_\delta)$ respectively in Subsections 2.1.1 and 2.1.2. Suppose first that

$$\sup_\ell \sum_{\ell'} J_{\ell\ell'} \exp\left(\alpha|\ell - \ell'|\right) < \infty, \quad \text{for a } \textit{certain } \alpha > 0. \tag{3.61}$$

Furthermore, let the series in (3.61) *converge uniform*: for any $\epsilon > 0$ one finds $N(\alpha, \epsilon) \in \mathbb{N}$ such that for all $\ell \in \mathbb{L}$ and $N \geq N(\alpha, \epsilon)$

$$\sum_{\ell' :\ |\ell' - \ell| > N} J_{\ell\ell'} \exp\left(\alpha|\ell - \ell'|\right) < \epsilon. \tag{3.62}$$

Let $\overline{\alpha}$ denote the (possibly infinite) supremum of such α, then we set

$$\mathcal{I} = (0, \overline{\alpha}), \quad w_\alpha(\ell, \ell') := \exp\left(-\alpha|\ell - \ell'|\right). \tag{3.63}$$

The second class of dynamical matrices $(J_{\ell\ell'})_{\mathbb{L}\times\mathbb{L}}$ we shall consider fulfills a (weaker than 3.61)) condition

$$\sup_{\ell}\sum_{\ell'} J_{\ell\ell'}\left(1+|\ell-\ell'|\right)^{\alpha d}<\infty, \quad \text{for a certain } \alpha>1. \tag{3.64}$$

Again we assume the property similar to (3.62) but with

$$\sum_{\ell' :\ |\ell'-\ell|>N} J_{\ell\ell'}\left(1+|\ell-\ell'|\right)^{\alpha d}<\epsilon. \tag{3.65}$$

Taking $\overline{\alpha}$ as the supremum of such α, we set

$$\mathcal{I}=(1,\overline{\alpha}), \qquad w_\alpha(\ell,\ell'):=\left(1+\varepsilon|\ell-\ell'|\right)^{-\alpha d}, \quad \varepsilon>0. \tag{3.66}$$

In both these cases, for any value of the parameter $\iota>0$ one finds small enough $\alpha,\varepsilon>0$, such that (3.60) is satisfied. Recall that for the latter family this claim has been already checked by (2.68)–(2.70). The first family of weights is considered in a perfect analogy, namely by (3.64), (3.65)

$$||\mathbf{J}||_\alpha\le\exp\left(\alpha N\right)||\mathbf{J}||_0+\sup_{\ell}\sum_{\ell' :\ |\ell'-\ell|>N} J_{\ell\ell'}\exp\left(\alpha|\ell-\ell'|\right)\to||\mathbf{J}||_0, \tag{3.67}$$

as $N\to\infty$ and $\alpha:=\alpha(N)\to 0$.

Let $u=(u_\ell)_{\ell\in\mathbb{L}}\in\mathbb{R}^{\mathbb{L}}$ be configurations of real numbers. Fixed some initial point $\ell_0\in\mathbb{L}$, we define the norms

$$|u|_{l^p_{\alpha,\ell_0}}:=\sum_{\ell}|u_\ell|w_\alpha(\ell_0,\ell), \qquad |u|_{l^\infty_{\alpha,\ell_0}}:=\sup_{\ell}\left\{|u_\ell|w_\alpha(\ell_0,\ell)\right\},$$

and introduce the Banach spaces

$$l^p(w_\alpha):=\left\{u\in\mathbb{R}^{\mathbb{L}}\ \middle|\ |u|_{l^p_{\alpha,\ell_0}}<\infty\right\}, \quad \alpha\in\mathcal{I}, \quad p=1,+\infty. \tag{3.68}$$

Remark 3.8 By (3.57) the embedding $l^1(w_\alpha)\hookrightarrow l^1(w_{\alpha'})$ is *compact* for any $\alpha<\alpha'$. By (3.59) the linear operator defined as $(\mathbf{J}u)_\ell=\sum_{\ell'}J_{\ell\ell'}u_{\ell'}$, $\ell\in\mathbb{L}$, is *bounded* in all spaces $l^p(w_\alpha)$ with $p=1,+\infty$. Its norm does not exceed $||\mathbf{J}||_\alpha$.

For $\alpha\in\mathcal{I}$, we define

$$\Omega_\alpha:=\left\{\omega\in\Omega\ \mid\ \|\omega\|_{\alpha,\ell_0}:=\left[\sum_{\ell}|\omega_\ell|^R_{L^R_\beta}w_\alpha(\ell_0,\ell)\right]^{1/R}<\infty\right\}, \tag{3.69}$$

which is a locally convex *Polish space* with the topology induced by the system of seminorms $||\omega||_\alpha$ and $|\omega_\ell|_{C_\beta}$, $\ell \in \mathbb{L}$. A possible choice of the *consistent metric* is

$$\rho_\alpha(\omega, \omega') = \|\omega - \omega'\|_{\alpha,\ell_0} + \sum_\ell 2^{-|\ell|} \cdot \frac{|\omega_\ell - \omega'_\ell|_{C_\beta}}{1 + |\omega_\ell - \omega'_\ell|_{C_\beta}}. \tag{3.70}$$

In view of (2.1), the constant configuration ($\omega_\ell(\tau) \equiv 1$ for all ℓ, τ) belongs to each Ω_α. Recall that the parameter $R \geq 2$ describes the (largest possible) order of polynomial growth allowed for $W_{\ell\ell'}$ by Assumption (**W**).

Remark 3.9 The topology of each of the spaces $l^p(w_\alpha)$ and Ω_α is *independent* of the particular choice of ℓ_0. This follows from the properties of the weights w_α assumed in Definition 3.6. Without loss of generality we may always suppose that $\ell_0 := 0 \in \mathbb{L}$ and respectively adopt the notation $\|\omega\|_\alpha := \|\omega\|_{\alpha,\ell_0=0.}$

There are at least two natural sets of *tempered configurations* defined as

$$\Omega^{\rm t}_{\rm pr} = \bigcap_{\alpha \in \mathcal{I}} \Omega_\alpha, \quad \Omega^{\rm t}_{\rm ind} = \bigcup_{\alpha \in \mathcal{I}} \Omega_\alpha, \quad \Omega^{\rm t}_{\rm pr} \subset \Omega^{\rm t}_{\rm ind}. \tag{3.71}$$

Equipped with the *projective limit* topology, $\Omega^{\rm t}_{\rm pr}$ becomes a Polish space as well. In contrast, the bigger space $\Omega^{\rm t}_{\rm ind}$ which will be equipped with the *inductive limit* topology is not metrizable at all. For any $\alpha \in \mathcal{I}$, we have continuous dense embeddings $\Omega^{\rm t}_{\rm pr} \hookrightarrow \Omega_\alpha \hookrightarrow \Omega^{\rm t}_{\rm ind} \hookrightarrow \Omega$. Then by the Kuratowski theorem it follows that $\Omega_\alpha \in \mathcal{B}(\Omega)$ and the Borel σ-algebras of all these spaces coincide with the ones induced on them by $\mathcal{B}(\Omega)$. The above notion of temperedness, which is based on the weight families and the corresponding projective or inductive limits, is the *most general* one. In Chapter 2 we actually made a particular choice $\Omega^{\rm t}_{\rm cl} := \bigcup_{\alpha>1} \Omega_\alpha = \bigcup_{p>d} \Omega_p$ with $w_\alpha(\ell, \ell') := (1 + \varepsilon|\ell - \ell'|)^{-\alpha d}$ and $\alpha = p/d \in \mathcal{I} = (1, \infty)$. As a rule, if it does not lead to the reader's confusion, in each concrete model we shall use the *standard notation* $\Omega^{\rm t}$ and $\mathcal{G}^{\rm t}$, by omitting all additional subscripts.

Now we are at a position to complete the definition of the function (3.54).

Proposition 3.10 *For every $\alpha \in \mathcal{I}$ and $\Lambda \Subset \mathbb{L}$, the mapping $\Omega_\alpha \times \Omega_\alpha \ni (\omega, \xi) \mapsto I_\Lambda(\omega_\Lambda|\xi)$ is continuous. Furthermore, for every ball $B_\alpha(r) = \{\omega \in \Omega_\alpha \mid \rho_\alpha(0, \omega) < r\}$ with a finite radius $r > 0$, it follows that*

$$-\infty < \inf_{\omega \in \Omega,\ \xi \in B_\alpha(r)} I_\Lambda(\omega_\Lambda|\xi) \leq \sup_{\omega, \xi \in B_\alpha(r)} |I_\Lambda(\omega_\Lambda|\xi)| < \infty. \tag{3.72}$$

Proof. We modify to the quantum case the arguments used for proving claim (i) in Proposition 2.3 (i). Consider any cofinal sequence of volumes $\mathcal{L} = \{\Lambda_N\}_{N \in \mathbb{N}}$ containing Λ. As the potentials $V_\ell : \mathbb{R}^\nu \to \mathbb{R}$ and $W_{\ell\ell'} : \mathbb{R}^{2\nu} \to \mathbb{R}$ are continuous, the mappings

$(\omega,\xi) \mapsto I_\Lambda(\omega_\Lambda|\xi_{\Lambda_N^c})$ are uniformly continuous and bounded on the sets $B_\alpha(r)\times B_\alpha(r)$. Furthermore,

$$\sum_{\ell'\in\Lambda^c}\int_0^\beta |W_{\ell\ell'}(\omega_\ell,\xi_{\ell'})|\mathrm{d}\tau \leq \frac{1}{2}||\mathbf{J}||_0\left(\beta C_W + |\omega_\ell|^R_{L^R_\beta}\right)$$
$$+\frac{1}{2}\sum_{\ell'\in\Lambda^c} J_{\ell\ell'}\left[w_\alpha(\ell_0,\ell')\right]^{-1}\cdot|\xi_{\ell'}|^R_{L^R_\beta} w_\alpha(\ell_0,\ell')$$
$$\leq \frac{1}{2}||\mathbf{J}||_\alpha\left(\beta C_W + |\omega_\ell|^R_{L^R_\beta} + \|\xi_{\Lambda^c}\|^R_{\alpha,\ell_0}[w_\alpha(\ell_0,\ell)]^{-1}\right) \tag{3.73}$$

where we used the triangle inequality (3.56). Thus the convergence in (3.54) is uniform on $B_\alpha(r)\times B_\alpha(r)$. For the limit mapping $I_\Lambda(\omega_\Lambda|\xi) := \lim_{\mathcal{L}} I_\Lambda(\omega_\Lambda|\xi_{\Lambda_N^c})$ this yields the continuity stated and the upper bound in (3.72). To prove the lower bound we employ the superquadratic growth of V_ℓ assumed in (2.7). Then for each $\varkappa > 0$ and $\alpha\in\mathcal{I}$, one finds $C_{3.74} > 0$ such that for any $\omega\in\Omega$ and $\xi\in\Omega_\alpha$,

$$I_\Lambda(\omega_\Lambda|\xi) \geq B_V\beta|\Lambda| + A_V\beta^{1-R/P}\sum_{\ell\in\Lambda}|\omega_\ell|^R_{L^R_\beta} \tag{3.74}$$
$$-\frac{1}{2}\sum_{\ell,\ell'\in\Lambda}\int_0^\beta|W_{\ell\ell'}(\omega_\ell,\omega_{\ell'})|\mathrm{d}\tau - \sum_{\ell\in\Lambda,\ \ell'\in\Lambda^c}\int_0^\beta |W_{\ell\ell'}(\omega_\ell,\xi_{\ell'})|\mathrm{d}\tau$$
$$\geq -C_{3.74}|\Lambda| + \varkappa\sum_{\ell\in\Lambda}|\omega_\ell|^R_{L^R_\beta} - ||\mathbf{J}||_\alpha\|\xi_{\Lambda^c}\|^R_{\alpha,\ell_0}\sum_{\ell\in\Lambda}[w_\alpha(\ell_0,\ell)]^{-1}.$$

To get the this estimate we used (3.73) and Hölder's inequality. ■

Now for $\Lambda \Subset \mathbb{L}$ and $\xi\in\Omega^{\mathrm{t}}$, we introduce the *partition function*

$$Z_\Lambda(\xi) = \int_{\Omega_\Lambda}\exp\left[-I_\Lambda(\omega_\Lambda|\xi)\right]\chi_\Lambda(\mathrm{d}\omega_\Lambda). \tag{3.75}$$

An immediate corollary of the estimates (3.45) and (3.74) is the following

Proposition 3.11 *For every $\Lambda\Subset\mathbb{L}$, the function $\Omega^{\mathrm{t}}\ni\xi\mapsto Z_\Lambda(\xi)\in(0,+\infty)$ is continuous. Moreover, for any $r>0$,*

$$0 < \inf_{\xi\in B_\alpha(r)} Z_\Lambda(\xi) \leq \sup_{\xi\in B_\alpha(r)} Z_\Lambda(\xi) < \infty. \tag{3.76}$$

3.1.5 Local specification and the DLR equation

In the remainder of this chapter we decide (cf. (3.71)) for the following choice of the subset of *tempered configurations*

$$\Omega^{\mathrm{t}} := \Omega^{\mathrm{t}}_{\mathrm{pr}} = \bigcap_{\alpha\in\mathcal{I}}\Omega_\alpha \tag{3.77}$$

and respectively will be concerned with the Euclidean Gibbs measures $\mu \in \mathcal{G}^{\rm t}$ supported by this $\Omega^{\rm t}$. We start with the *local Gibbs specification* $\Pi = \{\pi_\Lambda\}_{\Lambda \Subset \mathbb{L}}$, which is a family of the measure kernels

$$\mathcal{B}(\Omega) \times \Omega \ni (B, \xi) \mapsto \pi_\Lambda(B|\xi) \in [0, 1]$$

defined as follows. For $\xi \in \Omega^{\rm t}$, $\Lambda \Subset \mathbb{L}$, and $B \in \mathcal{B}(\Omega)$, we set

$$\pi_\Lambda(B|\xi) = \frac{1}{Z_\Lambda(\xi)} \int_{\Omega_\Lambda} \exp\{-I_\Lambda(\omega_\Lambda|\xi)\} \mathbf{1}_B(\omega_\Lambda \times \xi_{\Lambda^c})\chi_\Lambda(\mathrm{d}\omega_\Lambda), \tag{3.78}$$

where $\mathbf{1}_B$ stands for the indicator on $B \in \mathcal{B}(\Omega)$. We also set

$$\pi_\Lambda(\cdot|\xi) \equiv 0, \quad \text{for } \ \xi \in \Omega \setminus \Omega^{\rm t}. \tag{3.79}$$

From these definitions one readily derives a *consistency property*

$$\int_\Omega \pi_\Lambda(B|\omega)\pi_{\Lambda'}(\mathrm{d}\omega|\xi) = \pi_{\Lambda'}(B|\xi), \quad \Lambda \subseteq \Lambda', \tag{3.80}$$

which holds for all $B \in \mathcal{B}(\Omega)$ and $\xi \in \Omega$. Furthermore, by (3.74) it follows that for any $\xi \in \Omega$, $\sigma \in (0, 1/2)$, and $\varkappa > 0$,

$$\int_\Omega \exp\left\{\sum_{\ell \in \Lambda} \left(\lambda_\sigma |\omega_\ell|^2_{C^\sigma_\beta} + \varkappa |\omega_\ell|^R_{L^R_\beta}\right)\right\} \pi_\Lambda(\mathrm{d}\omega|\xi) < \infty, \tag{3.81}$$

where λ_σ is the same as in Proposition 3.3.

By $C_{\rm b}(\Omega)$ (respectively, $C_{\rm b}(\Omega_\alpha)$ and $C_{\rm b}(\Omega^{\rm t})$) we denote the Banach spaces of all *bounded continuous functions* $f : \Omega \to \mathbb{R}$ (respectively, $f : \Omega_\alpha \to \mathbb{R}$ and $f : \Omega^{\rm t} \to \mathbb{R}$) equipped with the supremum norm. For every $\alpha \in \mathcal{I}$, one has a natural embedding $C_{\rm b}(\Omega) \hookrightarrow C_{\rm b}(\Omega_\alpha) \hookrightarrow C_{\rm b}(\Omega^{\rm t})$.

Proposition 3.12 ***(Feller property)*** *For each $\alpha \in \mathcal{I}$, $\Lambda \Subset \mathbb{L}$ and any $f \in C_{\rm b}(\Omega_\alpha)$, the function*

$$\begin{aligned} \Omega_\alpha \ni \xi \mapsto & (\pi_\Lambda f)(\xi) \\ & := \frac{1}{Z_\Lambda(\xi)} \int_{\Omega_\Lambda} f(\omega_\Lambda \times \xi_{\Lambda^c}) \exp\{-I_\Lambda(\omega_\Lambda|\xi)\} \chi_\Lambda(\mathrm{d}\omega_\Lambda), \end{aligned} \tag{3.82}$$

belongs to $C_{\rm b}(\Omega_\alpha)$. The linear operator $f \mapsto \pi_\Lambda f$ is a contraction in $C_{\rm b}(\Omega_\alpha)$.

Proof. By Lemma 3.10 and Proposition 3.11 the integrand

$$F_\Lambda(\omega_\Lambda|\xi) := f(\omega_\Lambda \times \xi_{\Lambda^c}) \exp\{-I_\Lambda(\omega_\Lambda|\xi)\} / Z_\Lambda(\xi)$$

is continuous in both variables. Moreover, by (3.72) and (3.76) the map

$$\Omega_\alpha \ni \xi \mapsto \sup_{\omega_\Lambda \in \Omega_\Lambda} |F_\Lambda(\omega_\Lambda|\xi)|$$

is bounded on every ball $B_\alpha(r)$. Thus Lebesgue's dominated convergence theorem yields the continuity stated. Finally,

$$\sup_{\xi\in\Omega_\alpha} |(\pi_\Lambda f)(\xi)| \leq \sup_{\xi\in\Omega_\alpha} |f(\xi)|. \tag{3.83}$$

■

Note that by (3.78), for $\xi \in \Omega^{\mathrm{t}}$, $\alpha \in \mathcal{I}$, and $f \in C_{\mathrm{b}}(\Omega_\alpha)$, we just have

$$(\pi_\Lambda f)(\xi) = \int_\Omega f(\omega)\pi_\Lambda(\mathrm{d}\omega|\xi). \tag{3.84}$$

Recall that the particular cases of our model were specified by Definition 3.1. For $B \in \mathcal{B}(\Omega)$, $U \in O(\nu)$, and $\ell_0 \in \mathbb{L}$ we set

$$U\omega := (U\omega_\ell)_{\ell\in\mathbb{L}} \qquad UB := \{U\omega \mid \omega \in B\},$$

and in the lattice case

$$t_{\ell_0}(\omega) := (\omega_{\ell-\ell_0})_{\ell\in\mathbb{L}}, \qquad t_{\ell_0}(B) := \{t_{\ell_0}(\omega) \mid \omega \in B\}.$$

If the model possesses the corresponding symmetry, then one has

$$\pi_\Lambda(UB|U\xi) = \pi_\Lambda(B|\xi), \qquad \pi_{\Lambda+\ell}(t_\ell(B)|t_\ell(\xi)) = \pi_\Lambda(B|\xi), \tag{3.85}$$

which ought to hold for all U, ℓ, B, and ξ.

Definition 3.13 *A measure $\mu \in \mathcal{P}(\Omega)$ is called a **tempered Euclidean Gibbs measure** if it satisfies the DLR equilibrium equation*

$$(\pi_\Lambda\mu)(B) := \int_\Omega \pi_\Lambda(B|\omega)\mu(\mathrm{d}\omega) = \mu(B), \tag{3.86}$$

for all $\Lambda \Subset \mathbb{L}$ and $B \in \mathcal{B}(\Omega)$. By $\mathcal{G}^{\mathrm{t}}$ we denote the set of all tempered Euclidean Gibbs measures of our model existing at a given β.

Remark 3.14 (i) So far we do not know whether $\mathcal{G}^{\mathrm{t}}$ is non-void, but all its elements μ, provided such exist, *should* be supported by Ω^{t}. Indeed, by (3.78) and (3.79), $\pi_\Lambda(\Omega \setminus \Omega^{\mathrm{t}}|\xi) = 0$ for every $\Lambda \Subset \mathbb{L}$ and $\xi \in \Omega$. Then by (3.86)

$$\mu(\Omega \setminus \Omega^{\mathrm{t}}) = 0 \quad \Longrightarrow \quad \mu(\Omega^{\mathrm{t}}) = 1. \tag{3.87}$$

Along with (3.81), this yields that all $\mu \in \mathcal{G}^{\mathrm{t}}$ are supported by *Hölder continuous loops*, i.e.,

$$\mu\left(\left\{\omega \in \Omega^{\mathrm{t}} \mid \forall \ell \in \mathbb{L}: \ \omega_\ell \in C^\sigma_\beta\right\}\right) = 1. \tag{3.88}$$

(ii) Another possibility (cf. (3.71), which actually was realized in Chapter 2, is to fix everywhere

$$\Omega^{\mathrm{t}} := \Omega^{\mathrm{t}}_{\mathrm{ind}} = \bigcup_{\alpha\in\mathcal{I}} \Omega_\alpha. \tag{3.89}$$

Then, we define the tempered Gibbs measures $\mu \in \mathcal{G}^{\mathrm{t}}$ as those solutions of the *DLR* equation (3.86) which are supported by some Ω_α with $\alpha = \alpha(\mu) > 0$. Actually it does not a matter from which definition we decided to start, because *à-posteori* (cf. Propositions 2.19 and 3.21) one may show that in our model the both sets $\mathcal{G}^{\mathrm{t}}$ ought to coincide.

(ii) If the model is translation and/or rotation invariant, then the corresponding transformations preserve $\mathcal{G}^{\mathrm{t}}$. That is, for any $\mu \in \mathcal{G}^{\mathrm{t}}$, $U \in O(\nu)$, and $\ell \in \mathbb{L}$,

$$\Theta_U(\mu) := \mu \circ U^{-1} \in \mathcal{G}^{\mathrm{t}}, \qquad \theta_\ell(\mu) := \mu \circ t_\ell^{-1} \in \mathcal{G}^{\mathrm{t}}. \tag{3.90}$$

In particular, if $\mathcal{G}^{\mathrm{t}}$ is a singleton, its unique element should be invariant in the same sense as the model.

One more invariance of the Euclidean Gibbs measures is related with the dependence of their Matsubara functions on τ's.

Definition 3.15 *A measure $\mu \in \mathcal{G}^{\mathrm{t}}$ is called τ-shift invariant if its Matsubara functions (3.22) have the property (3.19).*

The τ-shift invariance is crucial for *reconstructing* quantum Gibbs states on von Neumann algebras, see [46, 125, 126]. Actually, only the elements of $\mathcal{G}^{\mathrm{t}}$ which have this property are of physical relevance.

By $\mathcal{W}$ (respectively, $\mathcal{W}_\alpha$ and $\mathcal{W}^{\mathrm{t}}$) we denote the usual *weak topology* on the set of all probability measures $\mathcal{P}(\Omega)$ (respectively, $\mathcal{P}(\Omega_\alpha)$ and $\mathcal{P}(\Omega^{\mathrm{t}})$), which is defined by means of bounded continuous functions. With this topology, each of the sets $\mathcal{P}(\Omega)$, $\mathcal{P}(\Omega_\alpha)$, and $\mathcal{P}(\Omega^{\mathrm{t}})$ becomes a *Polish space* (Theorem 6.5, page 46 of [226]).

The proof of the existence of Euclidean Gibbs measures will be based on the following statement (analogous to Proposition 2.7 in the classical case) .

Proposition 3.16 *For each $\alpha \in \mathcal{I}$, every $\mathcal{W}_\alpha$-accumulation point $\mu \in \mathcal{P}(\Omega^{\mathrm{t}})$ of the family $\{\pi_\Lambda(\cdot|\xi) \mid \Lambda \Subset \mathbb{L},\ \xi \in \Omega^{\mathrm{t}}\}$, as $\Lambda \nearrow \mathbb{L}$, is an element of $\mathcal{G}^{\mathrm{t}}$.*

Proof. Note that each $C_{\mathrm{b}}(\Omega_\alpha)$ is a measure defining class for $\mathcal{P}(\Omega^{\mathrm{t}})$. Then a measure $\mu \in \mathcal{P}(\Omega^{\mathrm{t}})$ solves (3.86) iff for all $f \in C_{\mathrm{b}}(\Omega_\alpha)$ and $\Lambda \Subset \mathbb{L}$,

$$\int_{\Omega^{\mathrm{t}}} f(\omega)\mu(\mathrm{d}\omega) = \int_{\Omega^{\mathrm{t}}} (\pi_\Lambda f)(\omega)\mu(\mathrm{d}\omega). \tag{3.91}$$

Let $\{\pi_{\Lambda_N}(\cdot|\xi_N)\}_{N\in\mathbb{N}}$ converge in $\mathcal{W}_\alpha$ to some $\mu \in \mathcal{P}(\Omega^{\mathrm{t}})$. For every $\Lambda \Subset \mathbb{L}$, one finds $N_\Lambda \in \mathbb{N}$ such that $\Lambda \subset \Lambda_N$ for all $N > N_\Lambda$. Then by (3.80), one has

$$\int_{\Omega^{\mathrm{t}}} f(\omega)\pi_{\Lambda_N}(\mathrm{d}\omega|\xi_N) = \int_{\Omega^{\mathrm{t}}} (\pi_\Lambda f)(\omega)\pi_{\Lambda_N}(\mathrm{d}\omega|\xi_N).$$

Now by Proposition 3.12, one can pass to the limit $N \to \infty$ and get (3.91). ■

Let us stress that in the statement above we suppose that the accumulation point is a probability measure on Ω^{t}. In general, the convergence of $\{\mu_N\}_{N\in\mathbb{N}} \subset \mathcal{P}(\Omega^{\mathrm{t}})$ in every $\mathcal{W}_\alpha$, $\alpha \in \mathcal{I}$, does *not yet* imply its $\mathcal{W}^{\mathrm{t}}$-convergence. However, in Lemma 3.30 and Corollary 3.32 below we show that the topologies induced by $\mathcal{W}_\alpha$ and $\mathcal{W}^{\mathrm{t}}$ on a certain subset of $\mathcal{P}(\Omega)$, which includes $\mathcal{G}^{\mathrm{t}}$ and all $\pi_\Lambda(\cdot|\xi)$, do coincide.

Remark 3.17 (i) The realization of the Euclidean approach with the help of the *"temperature loop lattice"* $\Omega_\beta := [C(S_\beta)]^{\mathbb{L}}$ and the Gibbs measures on it, which we have constructed above, is not the only possible one. Instead of the temperature dependent spin space $C(S_\beta)$, in certain situations (e.g., for varying β) it would be more convenient to use a *"standard" loop space* $C(S_1)$ consisting of all continuous functions on the circle $S_1 \cong [0,1]$. Consider the following unitary operator of dilatation in the physical Hilbert space $\mathcal{H}_\Lambda := L^2(\mathbb{R}^{\nu|\Lambda|} \to \mathbb{C})$

$$(U_\lambda \psi)(q_\ell) := \lambda^{\nu|\Lambda|/2}\psi(\lambda q_\ell), \quad \ell \in \Lambda.$$

Under the so-called *Symanzik transform* (cf. e.g. page 986 of [214])

$$U_\lambda q_\ell U_\lambda^{-1} = \lambda q_\ell, \quad U_\lambda p_\ell U_\lambda^{-1} = \lambda^{-1} p_\ell, \quad p_\ell = -i\mathrm{d}/\mathrm{d}q_\ell,$$

the local Hamiltonian (3.3), (3.4)

$$H_\Lambda := \frac{1}{2m}\sum_\ell |p_\ell|^2 + W_\Lambda(q_\Lambda), \tag{3.92}$$
$$W_\Lambda(q_\Lambda) := \frac{1}{2}\sum_{\ell,\ell'\in\Lambda} W_{\ell\ell'}(q_\ell, q_{\ell'}) + \sum_{\ell\in\Lambda}\left[\frac{a}{2}|q_\ell|^2 + V_\ell(q_\ell)\right],$$

is *unitary equivalent* to a new one

$$\tilde{H}_\Lambda := \frac{1}{2m\lambda^2}\sum_\ell |p_\ell|^2 + \tilde{W}_\Lambda(q_\Lambda), \tag{3.93}$$
$$\tilde{W}_\Lambda(q_\Lambda) := \frac{1}{2}\sum_{\ell,\ell'\in\Lambda} W_{\ell\ell'}(\lambda q_\ell, \lambda q_{\ell'}) + \sum_{\ell\in\Lambda}\left[\frac{a\lambda^2}{2}|q_\ell|^2 + V_\ell(\lambda q_\ell)\right].$$

Then, for a fixed $\beta > 0$, by choosing the scaling parameter $\lambda := \sqrt{\beta/m}$ we can reduce the former problem to studying the local quantum state (3.13), but at the temperature $\tilde{\beta} = 1$ and with the modified Hamiltonian (3.93). The corresponding local Gibbs measure $\tilde{\mu}_\Lambda$, which is now supported by $\Omega_\Lambda := [C(S_1)]^\Lambda$, has the representation

$$\tilde{\mu}_\Lambda(\mathrm{d}\omega_\Lambda) = \left(1/\tilde{Z}_\Lambda\right)\exp\left\{-\tilde{I}_\Lambda(\omega_\Lambda)\right\}\prod_{\ell\in\Lambda}\tilde{\chi}(\mathrm{d}\omega_\ell), \tag{3.94}$$

$$\tilde{I}_\Lambda(\omega_\Lambda) = \frac{1}{2}\sum_{\ell,\ell'\in\Lambda}\int_0^1 W_{\ell\ell'}\left(\sqrt{\frac{\beta}{m}}\omega_\ell(\tau), \sqrt{\frac{\beta}{m}}\omega_{\ell'}(\tau)\right)\mathrm{d}\tau + \sum_{\ell\in\Lambda}\int_0^1 V_\ell\left(\sqrt{\frac{\beta}{m}}\omega_\ell(\tau)\right)\mathrm{d}\tau. \tag{3.95}$$

Here $\tilde{\chi}$ is the Gaussian measure on $C(S_1)$, which is given by its Fourier transform (3.41) with the operator

$$\tilde{A} = \left(-\frac{\mathrm{d}^2}{\mathrm{d}\tau^2} + a\frac{\beta^2}{m}\right)\otimes \mathbf{Id}_\nu. \tag{3.96}$$

Following the *DLR* scheme, we then define the associated Euclidean Gibbs measures $\mu \in \mathcal{G}^{\rm t}$ on the universal configuration space $\Omega := [C(S_1)]^{\mathbb{L}}$, but with the interaction potentials *depending* on β in their own. In particular, such realization would be helpful in studying the uniqueness of $\mu \in \mathcal{G}^{\rm t}$ simultaneously for all β's taking values from some interval $\mathcal{I} \subset \mathbb{R}_+$, see Remark 3.35.

(ii) In the habilitation thesis we do not touch the case of *zero absolute temperature*, i.e., $\beta = \infty$. The corresponding Gibbs measures $\mu \in \mathcal{G}^{\rm gr}$ on the *"path lattice"* $[C(\mathbb{R} \to \mathbb{R}^\nu)]^{\mathbb{L}}$, so-called *Euclidean ground states*, also allow the *DLR* description, but through a family of local kernels $\pi_{\mathcal{I}\times\Lambda}$ indexed by *"time-space"* windows $\mathcal{I}\times\Lambda$ with $I \Subset \mathbb{R}$, $\Lambda \Subset \mathbb{L}$, cf. [216]. A principal difference with the case of finite β is the *absence* of a (independent from boundary conditions ξ) reference measure χ such that all $\pi_{\mathcal{I}\times\Lambda}(\mathrm{d}\omega|\xi)$ could be defined as its Gibbs modifications. So far, there are few rigorous results about the Gibbs measures on infinite volume path spaces, which all are mainly related to the existence problem via cluster expansions [105, 216, 214]. The stochastic dynamics for such models has been constructed in the early papers [273]–[276] of the author. Recall that, for each $\Lambda \Subset \mathbb{L}$, the corresponding Gibbs measure $\mu_\Lambda^{\rm gr}$ on $[C(\mathbb{R} \to \mathbb{R}^\nu)]^\Lambda$ is well-known as the $P(\varphi)_1$*-processes* and can be looked upon as a special case of the Euclidean field theory in space-dimension zero (cf. [153, 155]). On the other hand, the latter processes is a trivial example of the so-called Gibbs measures relative to a *Brownian motion* on the path space $C(\mathbb{R} \to \mathbb{R}^\nu)$, whose study has been initiated in [220] (see also further contributions [138, 195]).

3.2 Euclidean Gibbs measures in the general case

In this section we perform the study of the set $\mathcal{G}^{\rm t}$ in the general case, where we do not suppose that our system is translation invariant or "ferromagnetic", i.e., attractive.

In Subsection 3.2.1 we formulate our main results on existence, support properties, and uniqueness for the tempered Euclidean Gibbs measures $\mu \in \mathcal{G}^{\rm t}$. The proof of these properties relies essentially on the moment estimates in the spaces of Hölder continuous loops $\omega_\ell \in C_\beta^\sigma$, $\sigma \in (0, 1/2)$, which will be established for the probability kernels $\pi_\Lambda(\mathrm{d}\omega|\xi)$ of the local specification $\Pi = \{\pi_\Lambda\}_{\Lambda\Subset\mathbb{L}}$ in Subsection 3.2.2. In turn, Subsection 3.2.3 contains supporting results about the weak convergence of tempered probability measures on loop spaces. Afterwards, in Subsection 3.2.4 we shall verify that the set $\mathcal{G}^{\rm t}$ is *nonvoid* (Theorem 3.18) and establish *à-priori estimates* on its elements in terms of parameters of the interaction (Theorem 3.19). Subsection 3.2.5 is devoted to the proof of uniqueness criteria for $\mu \in \mathcal{G}^{\rm t}$ (Theorems 3.22 and 3.23). In Subsection 3.2.6 we discuss some possible generalizations of the model.

3.2.1 Main statements

The theorems below provide us with *basic information* for any further investigation of the Euclidean Gibbs measures. We suppose that Assumptions $(\mathbf{W})$, $(\mathbf{J}_\alpha)$, and $(\mathbf{V})$ are fulfilled without mentioning this again in the formulations of our statements. We begin by establishing *existence* of tempered Gibbs measures and *compactness* of their set $\mathcal{G}^{\rm t}$. For models with non-compact spins, here they are even infinite-dimensional, such a property is not evident at all.

Theorem 3.18 *For all values of $\beta > 0$, the set of tempered Euclidean Gibbs measures $\mathcal{G}^{\rm t}$ is nonempty and $\mathcal{W}^{\rm t}$- compact.*

The next theorem says that the tempered Euclidean Gibbs measures satisfy an *exponential moment estimate* in the Hölder spaces C^σ_β, similar to the one (3.45) valid for the free loop measure χ. Recall that the norm $|\cdot|_{C^\sigma_\beta}$ was defined by (3.25).

Theorem 3.19 *For every $\sigma \in (0, 1/2)$ and $\kappa > 0$, there exists a positive constant $C_{3.97} := C_{3.97}(\sigma, \kappa)$ such that for $\mu \in \mathcal{G}^{\rm t}$*

$$\sup_{\ell} \int_{\Omega} \exp\left(\lambda_\sigma |\omega_\ell|^2_{C^\sigma_\beta} + \kappa |\omega_\ell|^R_{L^R_\beta}\right) \mu({\rm d}\omega) \leq C_{3.97} \tag{3.97}$$

where λ_σ is the same as in (3.45).

This bound is called *à-priori*, since it holds *independently* of the existence result. The constant $C_{3.97}$, which surely depends on σ and κ, can be calculated explicitly in terms of parameters of the interaction. The estimate (3.97) plays a crucial role in the theory of the set $\mathcal{G}^{\rm t}$ and yields helpful information about the regularity and support properties of its elements. So, like as in the classical case (see Proposition 2.17), all finite-volume projections of $\mu \in \mathcal{G}^{\rm t}$ are of *sub-Gaussian growth.*

Proposition 3.20 *For each $\mu \in \mathcal{G}^{\rm t}$, its projections $\mu_\Lambda := \mu \circ \mathbb{P}_\Lambda^{-1}$ under the mappings $\mathbb{P}_\Lambda : \omega \mapsto \omega_\Lambda$, $\Lambda \Subset \mathbb{L}$, are absolutely continuous with respect to the Gaussian measures $\chi_\Lambda({\rm d}\omega_\Lambda) := \prod_{\ell\in\Lambda} \chi({\rm d}\omega_\ell)$ on $(\Omega_\Lambda, \mathcal{B}(\Omega_\Lambda))$. The corresponding Radon–Nikodym derivatives obey the Ruelle-type bound*

$$\frac{{\rm d}\mu_\Lambda}{{\rm d}\chi_\Lambda}(\omega_\Lambda) =: \rho_{\mu,\Lambda}(\omega_\Lambda) \leq \exp\left(-\varUpsilon \sum\nolimits_{\ell\in\Lambda} |\omega_\ell|^R_{L^R_\beta} + \mathcal{K}_\Lambda\right) \tag{3.98}$$

with an arbitrary $\varUpsilon \in (0, \infty)$ and a certain $\mathcal{K}_\Lambda := \mathcal{K}_\Lambda(\varUpsilon) \in \mathbb{R}$, which can be chosen the same for all such μ.

The set of tempered configurations $\Omega^{\rm t}$ was introduced in (3.69) and (3.77) by means of rather slack restrictions imposed on the L^R_β-norms of ω_ℓ. By construction, the elements of $\mathcal{G}^{\rm t}$ are supported by this set, see (3.71). It turns out that they have a much *smaller support* (a kind of the *Lebowitz-Presutti* one, see Proposition 2.19), which is described in terms of the Hölder norms $|\omega_\ell|_{C^\sigma_\beta}$ and does not depend on the particular choice of weights $(w_\alpha)_{\alpha\in\mathcal{I}}$.

Given $b > 0$ and $\sigma \in (0, 1/2)$, let us consider

$$\begin{aligned} \varXi(b, \sigma) := \Big\{\omega \in \Omega \mid (\forall \ell_0 \in \mathbb{L})\ (\exists \Lambda_{\omega,\ell_0} \Subset \mathbb{L})\ (\forall \ell \in \Lambda^c_{\omega,\ell_0}) : \\ |\omega_\ell|^2_{C^\sigma_\beta} \leq b \log(1 + |\ell - \ell_0|)\Big\}, \end{aligned} \tag{3.99}$$

which in view of (3.58) is a Borel subset of $\Omega^{\rm t}$.

Proposition 3.21 *For every $\sigma \in (0, 1/2)$ there exists $b > 0$, which depends on σ and on the parameters of the model only, such that for all $\mu \in \mathcal{G}^{\rm t}$*

$$\mu(\Xi(b, \sigma)) = 1. \tag{3.100}$$

The last results in this group are sufficient conditions for $\mathcal{G}^{\rm t}$ to be *a singleton*. The first of them says that, fixed all other parameters, the set of tempered Gibbs measures consists of exactly one point provided the strength of the interaction, i.e., the constant $||\mathbf{J}||_0$, is *small*. We give a *simple analytical proof*, which covers the case of superquadratic interactions and universally applies both to the quantum and to the classical systems. For this purpose we shall use the uniqueness criterion for lattice systems with non-compact spin spaces and *finite range* interactions, which was suggested by R. Dobrushin and E. Pechersky (see Subsection 2.3.1).

Theorem 3.22 *Consider the spin system (2.1) on the lattice $\mathbb{L} := \mathbb{Z}^d$. Let the matrix $(J_{\ell\ell'})_{\mathbb{L}\times\mathbb{L}}$ be translation invariant and have finite range, see Assumption $(\mathbf{J}_{\rm fin})$ in Subsection 2.3.1. Then, for any $\beta > 0$ one finds a proper $\mathcal{J}(\beta) > 0$, such that for all values of $||\mathbf{J}||_0 \leq \mathcal{J}(\beta)$ the corresponding set $\mathcal{G}^{\rm t}$ is singleton.*

The second uniqueness result holds in particular for *high temperatures*, i.e., small β. It relies on the renown Dobrushin criterion (see Subsection 2.3.4), which allows a straightforward modification to the interactions having infinite range (see Theorem 4.46) but is restricted to the pair potentials $W_{\ell\ell'}$ growing not fastly than quadratic. The uniqueness is obtained by controlling the *"non-convexity"* of the potential energy (3.4). Like as in its classical counterpart, Theorem 2.34, we suppose that the following holds additionally to the basic Assumptions $(\mathbf{W})$, $(\mathbf{J}_\alpha)$, and $(\mathbf{V})$:

Let

$$V_\ell = U_\ell + Q_\ell, \tag{3.101}$$

where $U_\ell \in C^2(\mathbb{R}^\nu)$ and $Q_\ell \in C_{\rm b}(\mathbb{R}^\nu)$ are such that *uniformly* for all $\ell \in \mathbb{L}$ and $x_\ell \in \mathbb{R}^\nu$

$$U''_\ell(q_\ell) \geq a_U \mathbf{Id}_\nu \quad \text{with} \quad a_U > -\infty, \tag{3.102}$$

$$\mathbf{Osc}\ Q_\ell := \sup_{\mathbb{R}^\nu} Q_\ell - \inf_{\mathbb{R}^\nu} Q_\ell \ \leq \delta_Q < \infty. \tag{3.103}$$

The functions $W_{\ell\ell'} \in C^2(\mathbb{R}^{2\nu})$ fulfill the following estimate on their second derivatives: for all ℓ,$\ell' \in \mathbb{L}$ and $x_\ell, x_{\ell'} \in \mathbb{R}^\nu$

$$\partial^2_{q_\ell} W_{\ell\ell}(q_\ell, q_{\ell'}) \geq J_{\ell\ell'} a_W \mathbf{Id}_\nu, \quad |\partial^2_{q_\ell q_{\ell'}} W_{\ell\ell'}(q_\ell, q_{\ell'})|_{\mathcal{L}(\mathbb{R}^\nu)} \leq J_{\ell\ell'} b_W, \tag{3.104}$$

with some a_W, $b_W \in \mathbb{R}$. The matrix $(J_{\ell\ell'})_{\mathbb{L}\times\mathbb{L}}$ has possibly infinite range. The constants in (3.102) and (3.104) are connected by the relation

$$a + a_U + a_W ||\mathbf{J}||_0 > 0, \tag{3.105}$$

where $a > 0$ is the same as in (3.2). Clearly, the decomposition (3.101) is not unique; its optimal realization for certain types of V_ℓ is discussed in Subsection 3.2.4.

Theorem 3.23 *Under the above assumptions, the set* $\mathcal{G}^{\rm t}$ *is singleton if*

$$(a + a_U)||\mathbf{J}||_0^{-1} + a_W > b_W e^{2\beta\delta_Q}. \tag{3.106}$$

Corollary 3.24 *Let the pair interaction be harmonic, i.e.,* $W_{\ell\ell'}(x_\ell, x_{\ell'}) = \pm J_{\ell\ell'}(x_\ell, x_{\ell'})_{\mathbb{R}^\nu}$ *for all* ℓ, ℓ'*. Then, the set* $\mathcal{G}^{\rm t}$ *is a singleton if*

$$e^{2\beta\delta_Q} < (a + a_U)/||\mathbf{J}||_0. \tag{3.107}$$

Remark 3.25 The latter condition is surely fulfilled at all $\beta > 0$ if

$$\delta_Q = 0 \quad \text{and} \quad ||\mathbf{J}||_0 < a + a_U. \tag{3.108}$$

In this case the potential energy W_Λ given by (3.4) is convex. If the oscillators are harmonic and thus $a_U = \delta_Q = 0$, this yields the *stability condition* $||\mathbf{J}||_0 < a$. The sufficient condition (3.106) does not contain the particle mass m. Hence, the property stated holds also in the *quasiclassical limit* $m \to \infty$, which gives rise to Theorem 2.33 for the classical system (2.1).

3.2.2 Moment estimates on loop spaces

Moment estimates for the kernels (3.78) we are going to derive will allow for proving the $\mathcal{W}^{\rm t}$*-relative compactness* of the set $\{\pi_\Lambda(\cdot|\xi)\}_{\Lambda \Subset \mathbb{L}}$, which by Lemma 3.16 will yield $\mathcal{G}^{\rm t} \neq \varnothing$. Integrating them over $\xi \in \Omega^{\rm t}$ we shall get by the *DLR* equation (3.86) the corresponding estimates for the elements of $\mathcal{G}^{\rm t}$. The basic ideas are similar to that employed for the classical spin systems in Subsection 2.2.2, however now some technicalities are much more involved. To shorten notation we write π_ℓ instead of $\pi_{\{\ell\}}$.

Lemma 3.26 *For every* $\sigma \in (0, 1/2)$ *and* $\kappa > 0$*, there exists a corresponding* $\Upsilon = \Upsilon_\sigma(\beta, \kappa) > 0$ *such that for all* $\ell \in \mathbb{L}$ *and* $\xi \in \Omega^{\rm t}$

$$\int_\Omega \exp\left\{\lambda_\sigma|\omega_\ell|^2_{C^\sigma_\beta} + \kappa|\omega_\ell|^R_{L^R_\beta}\right\}\pi_\ell(\mathrm{d}\omega|\xi) \le \exp\left\{\Upsilon + \sum_{\ell'(\neq\ell)} J_{\ell\ell'}|\xi_{\ell'}|^R_{L^R_\beta}\right\}. \tag{3.109}$$

Here $\lambda_\sigma > 0$ *is the same as in (3.45).*

Proof. Integrating over $\tau \in S_\beta$ the underlying estimate (2.52) for the classical spins, we get for all ω, $\xi \in \Omega^{\rm t}$

$$\sum_{\ell'(\neq\ell)} \int_0^\beta |W_{\ell\ell'}(\omega_\ell, \xi_{\ell'})|\mathrm{d}\tau \le \frac{||\mathbf{J}||_0}{2}\left(\beta C_W + |\omega_\ell|^R_{L^R_\beta}\right) + \frac{1}{2}\sum_{\ell'(\neq\ell)} J_{\ell\ell'}|\xi_{\ell'}|^R_{L^R_\beta}. \tag{3.110}$$

Combined with (3.51), (3.54), (3.75), and (3.78), this gives us that

$$\text{LHS}\,(3.109) \le [X_\ell/Y_\ell]\cdot\exp\left\{\sum_{\ell'(\neq\ell)} J_{\ell\ell'}|\xi_{\ell'}|^R_{L^R_\beta} + \beta||\mathbf{J}||_0\right\},$$

where we set

$$X_\ell := \int_\Omega \exp\left\{\lambda_\sigma|\omega_\ell|^2_{C^\sigma_\beta} + (\kappa + ||\mathbf{J}||_0/2)\,|\omega_\ell|^R_{L^R_\beta} - \int_{S_\beta} V_\ell(\omega_\ell(\tau))\mathrm{d}\tau\right\}\chi(\mathrm{d}\omega_\ell), \quad (3.111)$$

$$Y_\ell := \int_\Omega \exp\left\{-\frac{||\mathbf{J}||_0}{2}|\omega_\ell|^R_{L^R_\beta} - \int_{S_\beta} V_\ell(\omega_\ell(\tau))\mathrm{d}\tau\right\}\chi(\mathrm{d}\omega_\ell). \quad (3.112)$$

By Proposition 3.3 and the upper/lower bounds in (2.9) we see that

$$X := \sup_\ell X_\ell < \infty, \quad Y := \inf_\ell Y_\ell > 0. \quad (3.113)$$

Thus, the required estimate (3.109) holds with the constant

$$\Upsilon := \Upsilon(\beta, \kappa) := \log\{X/Y\} + \beta C_W ||\mathbf{J}||_0 < \infty. \quad (3.114)$$

■

By Jensen's inequality we readily get from (3.109) the following

Corollary 3.27 ***(Dobrushin's bound)*** *For all $\ell \in \mathbb{L}$ and $\xi \in \Omega^{\mathrm{t}}$*

$$\int_\Omega h(\omega_\ell)\pi_\ell(\mathrm{d}\omega|\xi) \leq \mathcal{C} + \sum_{\ell'} I_{\ell\ell'}|\xi_{\ell'}|^R_{L^R_\beta} \leq \mathcal{C} + \sum_{\ell'} I_{\ell\ell'} h(\xi_{\ell'}), \quad (3.115)$$

with $\mathcal{C} := \Upsilon/\kappa$, $I_{\ell\ell'} := J_{\ell\ell'}/\kappa$, and a compact function $h : C_\beta \to \mathbb{R}$,

$$h(\omega_\ell) := \lambda_\sigma \kappa^{-1}|\omega_\ell|^2_{C^\sigma_\beta} + |\omega_\ell|^R_{L^2_\beta}. \quad (3.116)$$

Note that the function $h(\omega_\ell)$ is a sum of two *nonlinear terms*, the first of which guarantees the *compactness* on the spin space C_β, whereas the second one controls the *growth* in L^R_β of the pair interaction $W_{\ell\ell'}$. Contrary to the known results in the classical case (cf. e.g. [42, 71, 259] and the discussion in Subsection 2.2.1), the validity of the Dobrushin existence criterion for quantum systems with (infinite dimensional) spin spaces like C_β, L^R_β was not covered by any previous work. So, (3.116) seems to be the first *explicit example* of a compact function satisfying Dobrushin's condition on loop spaces.

The next step is to get similar moment estimates for the kernels $\pi_\Lambda(\mathrm{d}\omega|\xi)$ with *arbitrary* $\Lambda \Subset \mathbb{L}$. Let the parameters σ, κ, and λ_σ be the same as in (3.109). We set for $\ell \in \Lambda$

$$n_\ell(\Lambda|\xi) := \log\left\{\int_\Omega \exp\left(\lambda_\sigma|\omega_\ell|^2_{C^\sigma_\beta} + \kappa|\omega_\ell|^R_{L^R_\beta}\right)\pi_\Lambda(\mathrm{d}\omega|\xi)\right\}, \quad (3.117)$$

which is nonnegative and makes sense by (3.81).

Lemma 3.28 *For every $\alpha \in \mathcal{I}$, there exists a finite $\Upsilon_{\sigma,\alpha} := \Upsilon_{\sigma,\alpha}(\beta,\kappa)$ such that*

$$\limsup_{\Lambda \nearrow \mathbb{L}} \left[\sum_{\ell \in \Lambda} n_\ell(\Lambda|\xi) w_\alpha(\ell_0,\ell) \right] \leq \Upsilon_{\sigma,\alpha}, \tag{3.118}$$

uniformly for all $\ell_0 \in \mathbb{L}$ and $\xi \in \Omega_\alpha$. This implies, in particular, that for all $\ell \in \mathbb{L}$ and $\xi \in \Omega_\alpha$

$$\limsup_{\Lambda \nearrow \mathbb{L}} \int_\Omega \exp\left(\lambda_\sigma |\omega_\ell|^2_{C^\sigma_\beta} + \kappa |\omega_\ell|^R_{L^R_\beta} \right) \pi_\Lambda(\mathrm{d}\omega|\xi) = \exp \Upsilon_{\sigma,\alpha}. \tag{3.119}$$

Proof. Integrating both sides of (3.109) with respect to the measure $\pi_\Lambda(\mathrm{d}\omega|\xi)$ and taking into account (3.80), we arrive at

$$\begin{aligned} n_\ell(\Lambda|\xi) &\leq \Upsilon + \sum_{\ell' \in \Lambda^c} |J_{\ell\ell'}| \cdot |\xi_{\ell'}|^R_{L^R_\beta} + \log \left\{ \int_\Omega \exp\left(\sum_{\ell' \in \Lambda} J_{\ell\ell'} |\omega_{\ell'}|^R_{L^R_\beta} \right) \pi_\Lambda(\mathrm{d}\omega|\xi) \right\} \\ &\leq \Upsilon + \sum_{\ell' \in \Lambda^c} J_{\ell\ell'} |\xi_{\ell'}|^R_{L^R_\beta} + \kappa^{-1} \sum_{\ell' \in \Lambda} J_{\ell\ell'} n_{\ell'}(\Lambda|\xi). \end{aligned} \tag{3.120}$$

Note that in the last line we have used the multiple Hölder's inequality (2.73), which was possible due to the choice of $\kappa > ||\mathbf{J}||_\alpha$. After summing in (3.120) over $\ell \in \Lambda$, we get that

$$\begin{aligned} n_{\ell_0}(\Lambda|\xi) &\leq \sum_{\ell \in \Lambda} n_\ell(\Lambda|\xi) w_\alpha(\ell_0,\ell) \\ &\leq \frac{1}{1-\kappa^{-1}||\mathbf{J}||_\alpha} \left[\Upsilon \sum_{\ell' \in \Lambda} w_\alpha(\ell_0,\ell') + ||\mathbf{J}||_\alpha \sum_{\ell' \in \Lambda^c} |\xi_{\ell'}|^R_{L^R_\beta} w_\alpha(\ell_0,\ell') \right]. \end{aligned} \tag{3.121}$$

For all $\xi \in \Omega_\alpha$, this immediately yields the result

$$\begin{aligned} \limsup_{\Lambda \nearrow \mathbb{L}} n_{\ell_0}(\Lambda|\xi) &\leq \limsup_{\Lambda \nearrow \mathbb{L}} \left[\sum_{\ell \in \Lambda} n_\ell(\Lambda|\xi) w_\alpha(\ell_0,\ell) \right] \\ &\leq \frac{\Upsilon}{1-\kappa^{-1}||\mathbf{J}||_\alpha} \sup_{\ell_0} \left[\sum_\ell w_\alpha(\ell_0,\ell) \right] =: \Upsilon_{\sigma,\alpha}. \end{aligned} \tag{3.122}$$

■

Recall that the norm $\|\cdot\|_\alpha$ was defined by (3.69). Given $\alpha \in \mathcal{I}$ and $\sigma \in (0,1/2)$, by analogy we set

$$\|\xi\|_{\alpha,\sigma} := \left[\sum_\ell |\xi_\ell|^2_{C^\sigma_\beta} w_\alpha(\ell_0,\ell) \right]^{1/2}, \tag{3.123}$$

where a choice of the initial point $\ell_0 \in \mathbb{L}$ (it may be e.g. $\ell_0 = 0$) is of no principal importance, see Remark 3.9.

Lemma 3.29 *Let the assumptions of Lemma 3.26 be satisfied. Then for every $\alpha \in \mathcal{I}$ and $\xi \in \Omega^{\rm t}$, one finds a positive $C_\alpha(\xi)$ such that*

$$\sup_{\Lambda \Subset \mathbb{L}} \int_\Omega \|\omega\|_\alpha^2 \pi_\Lambda(\mathrm{d}\omega|\xi) \leq C_\alpha(\xi). \tag{3.124}$$

Furthermore, for every $\sigma \in (0, 1/2)$ and those ξ for which the norm (3.123) is finite, one finds a $\mathcal{C}_{\alpha,\sigma}(\xi) > 0$ such that for all $\Lambda \Subset \mathbb{L}$

$$\int_\Omega \|\omega\|_{\alpha,\sigma}^2 \pi_\Lambda(\mathrm{d}\omega|\xi) \leq \mathcal{C}_{\alpha,\sigma}(\xi). \tag{3.125}$$

Proof. By the Jensen inequality and (3.121) one has for any $\xi \in \Omega^{\rm t}$

$$\begin{aligned} \limsup_{\Lambda \nearrow \mathbb{L}} \int_\Omega \|\omega\|_\alpha^2 \pi_\Lambda(\mathrm{d}\omega|\xi) &\leq \lambda_\sigma^{-1} \limsup_{\Lambda \nearrow \mathbb{L}} \left[\sum_{\ell \in \Lambda} n_\ell(\Lambda|\xi) w_\alpha(\ell_0, \ell) \right] \\ + \limsup_{\Lambda \nearrow \mathbb{L}} \left[\sum_{\ell \in \Lambda^c} |\xi_\ell|_{L_\beta^R}^R w_\alpha(\ell_0, \ell) \right] &\leq \lambda_\sigma^{-1} \Upsilon_{\sigma,\alpha}(\beta, \kappa). \end{aligned} \tag{3.126}$$

Hence, the set consisting of the left-hand sides of (3.124) indexed by $\Lambda \Subset \mathbb{L}$ is bounded in $\mathbb{R}$. The proof of (3.125) is analogous. ■

3.2.3 Weak convergence of tempered measures

First, we describe a typical subset of $\mathcal{P}(\Omega^{\rm t})$ on which all the topologies $\mathcal{W}^{\rm t}$, $\mathcal{W}$, and $\mathcal{W}_\alpha$, $\alpha \in \mathcal{I}$, *coincide*, see Lemma 3.30. Then, in Corollary 3.32 we check that this certainly will be the case for the subset $\mathcal{G}^{\rm t}$ of all tempered Euclidean Gibbs measures associated with the model (3.1). Finally, in Lemma 3.33 we use the corresponding weak convergence to identify the *extreme* elements in $\mathcal{G}^{\rm t}$.

Recall that $f : \Omega \to \mathbb{R}$ is a *local function* if it is measurable with respect to $\mathcal{B}(\Omega_\Lambda)$ for a certain $\Lambda := \Lambda_f \Subset \mathbb{L}$. Let $\mathcal{F}C_{\rm b}(\Omega^{\rm t})$ be the subset of all local functions which are continuous and bounded on $\Omega^{\rm t}$ (see the notation in Subsection 2.3.5 (ii)).

Lemma 3.30 *Let a sequence $\{\mu_n\}_{n\in\mathbb{N}} \subset \mathcal{P}(\Omega^{\rm t})$ have the following properties:*

(a) For some $\theta > 0$ and each $\alpha \in \mathcal{I}$, its obeys the uniform integrability estimate

$$\sup_n \int_\Omega \|\omega\|_\alpha^{1+\theta} \mu_n(\mathrm{d}\omega) \leq C_{\theta.\alpha} < \infty; \tag{3.127}$$

(b) $\{\mu_n(f)\}_{n\in\mathbb{N}}$ is a Cauchy sequence in $\mathbb{R}$ for every $f \in \mathcal{F}C_{\rm b}(\Omega^{\rm t})$.

Then $\{\mu_n\}_{n\in\mathbb{N}}$ converges in $\mathcal{W}^{\rm t}$ to a certain $\mu \in \mathcal{P}(\Omega^{\rm t})$.

Proof. The topology of the Polish space $\Omega^{\rm t}$ is consistent with the following *metric* (cf. (3.69), (3.70) for $\ell_0 := 0$)

$$\rho(\omega, \tilde{\omega}) = \sum_{k=1}^\infty 2^{-k} \frac{\|\omega - \tilde{\omega}\|_{\alpha_k}}{1 + \|\omega - \tilde{\omega}\|_{\alpha_k}} + \sum_\ell 2^{-|\ell_0 - \ell|} \frac{|\omega_\ell - \tilde{\omega}_\ell|_{C_\beta}}{1 + |\omega_\ell - \tilde{\omega}_\ell|_{C_\beta}}, \tag{3.128}$$

where $\{\alpha_k\}_{k\in\mathbb{N}} \subset \mathcal{I} = (\underline{\alpha}, \overline{\alpha})$ is a monotone strictly decreasing sequence converging to $\underline{\alpha}$. Let us denote by $C^{\rm u}_{\rm b}(\Omega^{\rm t};\rho)$ the set of all bounded functions $f : \Omega^{\rm t} \to \mathbb{R}$ which are *uniformly continuous* with respect to (3.128). Thus in accordance with a known fact (see e.g. Theorem 2.1.1, page 19 of [62]), to prove the lemma it suffices to show that under the above conditions $\{\mu_n(f)\}_{n\in\mathbb{N}}$ is a Cauchy sequence for every $f \in C^{\rm u}_{\rm b}(\Omega^{\rm t};\rho)$. For this f and any given $\varepsilon > 0$, let us fix some $\delta > 0$ such that for all $\omega, \tilde{\omega} \in \Omega^{\rm t}$,

$$|f(\omega) - f(\tilde{\omega})| < \varepsilon/6, \quad \text{whenever} \quad \rho(\omega, \tilde{\omega}) < \delta. \tag{3.129}$$

Next, we choose $\Lambda_\delta \Subset \mathbb{L}$ and $k_\delta \in \mathbb{N}$ such that

$$\sum_{\ell\in\Lambda_\delta^c} 2^{-|\ell_0-\ell|} < \delta/3, \qquad \sum_{k=k_\delta}^{\infty} 2^{-k} = 2^{-k_\delta+1} < \delta/3. \tag{3.130}$$

For $\alpha \in \mathcal{I}$ and $R > 0$, we consider the balls

$$B_\alpha(R) = \{\omega \in \Omega^{\rm t} \mid \|\omega\|_\alpha \le R\}.$$

By (3.127) and Chebyshev's inequality, one has uniformly for all $n \in \mathbb{N}$

$$\mu_n\left(\Omega^{\rm t} \setminus B_\alpha(R)\right) \le C_{\theta,\alpha}/R^{1+\theta}. \tag{3.131}$$

For these ε, δ, and $k := k_\delta$ from (3.130), we pick up a corresponding $R := R_{\varepsilon,\delta}$ such that

$$C_{\theta,\alpha_k}\|f\|_\infty/R^{1+\theta} < \frac{\varepsilon}{12}. \tag{3.132}$$

Thereafter, we choose a larger domain $\Lambda := \Lambda_\delta(R) \Subset \mathbb{L}$ which containing Λ_δ, which obeys

$$\sup_{\ell\in\Lambda^c}\left\{w_{\alpha_{k-1}}(0,\ell)/w_{\alpha_k}(0,\ell)\right\} < \frac{\delta}{3R^{1+\theta}}, \tag{3.133}$$

which is possible in view of (3.57). For this $\Lambda \Subset \mathbb{L}$ and arbitrary $n, m \in \mathbb{N}$, we now estimate

$$\begin{aligned} |\mu_n(f) - \mu_m(f)| &\le |\mu_n(f_\Lambda) - \mu_m(f_\Lambda)| \\ &+ 2\max\{\mu_n(|f - f_\Lambda|); \mu_m(|f - f_\Lambda|)\}, \end{aligned} \tag{3.134}$$

where $f_\Lambda(\omega) := f(\omega_\Lambda)$. Furthermore, by (3.131)

$$\begin{aligned} \mu_n(|f - f_\Lambda|) &\le 2C_{\theta,\alpha_k}\|f\|_\infty/R^{1+\theta} \\ &+ \int_{B_{\alpha_k}(R)} |f(\omega) - f(\omega_\Lambda)|\,\mu_n({\rm d}\omega). \end{aligned} \tag{3.135}$$

For $\omega \in B_{\alpha_k}(R)$ and $k' = 1, 2, \ldots, k-1$, one has

$$\begin{aligned} \|\omega - \omega_\Lambda\|^2_{\alpha_{k'}} &= \sum_{\ell\in\Lambda^c} |\omega_\ell|^2_{L^2_\beta} w_{\alpha_k}(0,\ell)\cdot\left[w_{\alpha_{k'}}(0,\ell)/w_{\alpha_k}(0,\ell)\right] \\ &\le \frac{\delta}{3R^{1+\theta}}\sum_{\ell\in\Lambda^c} |\omega_\ell|^2_{L^2_\beta} w_{\alpha_k}(0,\ell) < \frac{\delta}{3}, \end{aligned} \tag{3.136}$$

where (3.133) has been used. Then by (3.128), (3.130), it follows that

$$\forall \omega \in B_{\alpha_k}(R) : \quad \rho(\omega, \omega_\Lambda \times 0_{\Lambda^c}) < \delta, \tag{3.137}$$

which together with (3.129), (3.132) yields in (3.135)

$$\mu_n(|f - f_\Lambda|) < \frac{\varepsilon}{6} + \frac{\varepsilon}{6}\mu_n\left(B_{\alpha_k}(R)\right) \leq \frac{\varepsilon}{3}.$$

By assumption (b) of the lemma, one finds N_ε such that for all $n, m > N_\varepsilon$,

$$|\mu_n(f_\Lambda) - \mu_m(f_\Lambda)| < \frac{\varepsilon}{3}.$$

Plugging the latter two estimates back into in (3.134), we conclude that $\{\mu_n\}_{n\in\mathbb{N}}$ is a Cauchy sequence in the complete metric space $(\mathcal{P}(\Omega^{\mathrm{t}}), \mathcal{W}^{\mathrm{t}})$. Thus this sequence $\mathcal{W}^{\mathrm{t}}$-converges to some $\mu \in \mathcal{P}(\Omega^{\mathrm{t}})$. Moreover, from (3.127) and Fatou's lemma, it follows that the limit point μ satisfies the same estimate

$$\int_\Omega \|\omega\|_\alpha^2 \mu(\mathrm{d}\omega) \leq C_\alpha. \tag{3.138}$$

■

Corollary 3.31 *Let $\mathcal{P}_0$ be a subset of $\mathcal{P}(\Omega^{\mathrm{t}})$ such that for some $\theta > 0$ and each $\alpha \in \mathcal{I}$*

$$\sup_{\mu\in\mathcal{P}_0} \int_\Omega \|\omega\|_\alpha^{1+\theta} \mu(\mathrm{d}\omega) < \infty. \tag{3.139}$$

Then, being restricted to $\mathcal{P}_0$, the (metrizable) topologies $\mathcal{W}^{\mathrm{t}}$, $\mathcal{W}$, and all $\mathcal{W}_\alpha$, $\alpha \in \mathcal{I}$, coincide.

Corollary 3.32 *The topologies induced on $\mathcal{G}^{\mathrm{t}}$ by $\mathcal{W}^{\mathrm{t}}$, $\mathcal{W}$, and $\mathcal{W}_\alpha$ coincide.*

Proof. Follows immediately from Corollary 3.31 and the à-priori bound (3.97). ■

The above characterization of the $\mathcal{W}^{\mathrm{t}}$-convergence can be used to recognize the *extreme elements* (or *pure tempered states*) $\mu \in \mathrm{ex}(\mathcal{G}^{\mathrm{t}})$. Let $\mathcal{F}C_{\mathrm{exp}}(\Omega^{\mathrm{t}})$ be the set of all continuous local functions $f : \Omega^{\mathrm{t}} \to \mathbb{R}$, for which there exist $\Lambda_f \Subset \mathbb{L}$, $\sigma \in (0, 1/2)$, and $C_{\sigma,f} > 0$, such that for all $\omega \in \Omega^{\mathrm{t}}$

$$f(\omega) = f(\omega_\Lambda) \;\text{ and }\; |f(\omega)|^2 \leq C_{\sigma,f} \sum_{\ell\in\Lambda_f} \exp\left(\lambda_\sigma |\omega_\ell|^2_{C^\sigma_\beta}\right), \tag{3.140}$$

where λ_σ is the same as in (3.45) and (3.109).

Lemma 3.33 *For every $\mu \in \mathrm{ex}(\mathcal{G}^{\mathrm{t}})$ and any cofinal sequence $\mathcal{L}$, the following assertions hold:*

(a) For μ-almost all $\xi \in \Omega^{\mathrm{t}}$, the sequence $\{\pi_\Lambda(\cdot|\xi)\}_{\Lambda\in\mathcal{L}}$ converges to this μ in the topology $\mathcal{W}^{\mathrm{t}}$;

(b) For every $f \in \mathcal{F}C_{\mathrm{exp}}(\Omega^{\mathrm{t}})$ and μ-almost all $\xi \in \Omega^{\mathrm{t}}$, one has $\lim_{\mathcal{L}}(\pi_\Lambda f)(\xi) = \mu(f)$.

Proof. Claim (c) of Theorem 7.12 in [122] (or claim (a) there and Corollary 2.1.1 (i) in [62]), implies that for μ-almost all $\xi \in \Omega^{\rm t}$

$$\lim_{\mathcal{L}} (\pi_\Lambda f)(\xi) = \mu(f), \quad \text{for any local } f \in \mathcal{F}C_{\rm b}(\Omega^{\rm t}). \tag{3.141}$$

Then the convergence stated in our claim (a) follows from Lemmas 3.29 and 3.30. To prove claim (b), for a given function $f \in \mathcal{F}C_{\rm exp}(\Omega^{\rm t})$ let us construct its approximation by a sequence $\{f_N\}_{N\in\mathbb{N}} \subset C_{\rm b}(\Omega^{\rm t})$ defined by

$$f_N(\omega) = \begin{cases} f(\omega), & \text{if } \ |f(\omega)| \leq N; \\ N f(\omega)/|f(\omega)|, & \text{otherwise.} \end{cases}$$

Then by (3.141) there exists a Borel set $\Xi_f \subset \Omega^{\rm t}$, such that $\mu(\Xi_\mu) = 1$ and for every $N \in \mathbb{N}$,

$$\lim_{\mathcal{L}} (\pi_\Lambda f_N)(\xi) = \mu(f_N), \quad \text{for all } \xi \in \Xi_f. \tag{3.142}$$

Note that by (3.117), (3.122), and (3.140), for any $\xi \in \Xi_f$ one finds a positive $C_{3.143}(f,\xi)$ such that, for all sets $\Lambda \Subset \mathbb{L}$ containing Λ_f,

$$\int_\Omega |f(\omega)|^2 \pi_\Lambda(\mathrm{d}\omega|\xi) \leq C_{3.143}(f,\xi). \tag{3.143}$$

Hence

$$\begin{aligned} |(\pi_\Lambda f)(\xi) - (\pi_\Lambda f_N)(\xi)| &\leq 2 \int_{\{\omega \ | \ |f(\omega)|>N\}} |f(\omega)| \pi_\Lambda(\mathrm{d}\omega|\xi) \\ &\leq \frac{2}{N} \int_\Omega |f(\omega)|^2 \pi_\Lambda(\mathrm{d}\omega|\xi) \leq \frac{2}{N} C_{3.143}(f,\xi). \end{aligned}$$

Similarly, by means of (3.140) and Theorem 3.19, one gets

$$|\mu(f) - \mu(f_N)| \leq \frac{2}{N} C_f C_{3.97}.$$

The latter two inequalities and (3.142) allow us to estimate $|\pi_\Lambda f - \mu(f)|$ and thereby to complete the proof. ■

3.2.4 Existence and à-priori estimates

Here we demonstrate elementary proofs of Theorems 3.18, 3.19 and Propositions 3.20, 3.21 resulting from them, based on the key estimates (3.109), (3.118), and (3.119). The existence of Euclidean Gibbs measures and the à-priori bound (3.97) can be proven *independently*. To establish the compactness of $\mathcal{G}^{\rm t}$ we will need (3.97), thus we first prove Theorem 3.19.

Proof of Theorem 3.19. Let us show that every $\mu \in \mathcal{P}(\Omega)$ which solves the *DLR* equation (3.86) ought to obey (3.97) with one and the same $C_{3.97}$. To this end we apply the bounds for the kernels $\pi_\Lambda(\cdot|\xi)$ obtained above. Consider the cut-off functions

$$G_N(\omega_\ell) := \exp\left(\min\left\{\lambda_\sigma |\omega_\ell|^2_{C^\sigma_\beta} + \kappa |\omega_\ell|^R_{L^R_\beta}; N\right\}\right), \quad N \in \mathbb{N}.$$

By the *DLR* equation (3.86), Fatou's lemma, and the estimate (3.119) with an arbitrarily chosen $\alpha \in \mathcal{I}$, we get that

$$\begin{aligned}\int_{\Omega} G_N(\omega_\ell)\mu(\mathrm{d}\omega) &= \limsup_{\Lambda \nearrow \mathbb{L}} \int_{\Omega} \left[\int_{\Omega} G_N(\omega_\ell)\pi_\Lambda(\mathrm{d}\omega|\xi)\right]\mu(\mathrm{d}\xi) \\ &\leq \limsup_{\Lambda \nearrow \mathbb{L}} \int_{\Omega} \left[\int_{\Omega} \exp\left\{\lambda_\sigma |\omega_\ell|^2_{C^\sigma_\beta} + \kappa |\omega_\ell|^R_{L^R_\beta}\right\}\pi_\Lambda(\mathrm{d}\omega|\xi)\right]\mu(\mathrm{d}\xi) \\ &\leq \int_{\Omega} \left[\limsup_{\Lambda \nearrow \mathbb{L}} \int_{\Omega} \exp\left\{\lambda_\sigma |\omega_\ell|^2_{C^\sigma_\beta} + \kappa |\omega_\ell|^R_{L^R_\beta}\right\}\pi_\Lambda(\mathrm{d}\omega|\xi)\right]\mu(\mathrm{d}\xi) \\ &\leq \exp C_{3.118}(\alpha) =: C_{3.97}.\end{aligned}$$

In view of the support property (3.88) of any measure solving (3.86), we can pass here to the limit $N \to \infty$ and get (3.97). ■

Now, we show that the set $\mathcal{G}^{\mathrm{t}}$ is nonempty, since it surely contains limit points for$\pi_\Lambda(\mathrm{d}\omega|\xi)$ as $\Lambda \nearrow \mathbb{L}$.

Proof of Theorem 3.18. Let us introduce the next scale of Banach spaces

$$\Omega_{\alpha,\sigma} = \{\omega \in \Omega \mid \|\omega\|_{\alpha,\sigma} < \infty\}, \quad \sigma \in (0, 1/2), \quad \alpha \in \mathcal{I}, \tag{3.144}$$

where the norm $\|\cdot\|_{\alpha,\sigma}$ was defined by (3.123). For any pair $\alpha, \alpha' \in \mathcal{I}$ such that $\alpha < \alpha'$, the embedding $\Omega_{\alpha,\sigma} \hookrightarrow \Omega_{\alpha'}$ is compact, see (3.69), (3.27), and Remark 2.1. This fact and the estimate (3.125) which holds for any $\xi \in \Omega_{\alpha,\sigma}$, imply by Prokhorov's criterion the relative compactness of the set $\{\pi_\Lambda(\cdot|\xi)\}_{\Lambda \Subset \mathbb{L}}$ in $\mathcal{W}_{\alpha'}$. Therefore, the sequence $\{\pi_\Lambda(\mathrm{d}\omega|0)\}_{\Lambda \Subset \mathbb{L}}$ is relatively compact in every $\mathcal{W}_\alpha$, $\alpha \in \mathcal{I}$. Then Proposition 3.16 yields $\mathcal{G}^{\mathrm{t}} \neq \varnothing$. By the same Prokhorov criterion and the estimate (3.97), we get the $\mathcal{W}_\alpha$-relative compactness of $\mathcal{G}^{\mathrm{t}}$. Then in view of the Feller property (Proposition 3.12), the set $\mathcal{G}^{\mathrm{t}}$ is closed and hence compact in every $\mathcal{W}_\alpha$, $\alpha \in \mathcal{I}$, which by Corollary 3.32 completes the proof. ■

Proof of Proposition 3.20. We follow the same pattern as that used to prove Proposition 2.17 in Subsection 2.2.4. From (3.78), (3.86) it is clear that the Radon–Nikodym derivatives would have the form

$$\begin{aligned}\rho_{\mu,\Lambda}(\omega_\Lambda) := \exp&\left\{-\sum_{\ell\in\Lambda}\int_0^\beta V_\ell(\omega_\ell)d\tau - \frac{1}{2}\sum_{\ell,\ell'\in\Lambda}\int_0^\beta W_{\ell\ell'}(\omega_\ell,\omega_{\ell'})\mathrm{d}\tau\right\} \\ &\times \int_{\Omega}[1/Z_\Lambda(\xi)]\exp\left\{-\sum_{\ell\in\Lambda,\ \ell'\in\Lambda^c}\int_0^\beta W_{\ell\ell'}(\omega_\ell,\xi_{\ell'})\mathrm{d}\tau\right\}\mu(\mathrm{d}\xi).\end{aligned} \tag{3.145}$$

By the calculations similar to (3.110)–(3.112), we find that

$$\begin{aligned}\mathrm{RHS}\,(3.145) \leq (1/Y)^{|\Lambda|}&\int_{\Omega}\exp\left\{\sum_{\ell\in\Lambda,\ \ell'\in\Lambda^c} J_{\ell\ell'}|\xi_{\ell'}|^R_{L^R_\beta}\right\}\mu(\mathrm{d}\xi) \\ \times \exp&\left\{-\sum_{\ell\in\Lambda}\int_0^\beta V_\ell(\omega_\ell)\mathrm{d}\tau + \frac{1}{2}||\mathbf{J}||_0\sum_{\ell\in\Lambda}|\omega_\ell|^R_{L^R_\beta} + ||\mathbf{J}||_0|\Lambda|\right\},\end{aligned} \tag{3.146}$$

with a constant $Y > 0$ defined by (3.113). The integral in the first line in (3.146) can be estimated by Hölder's inequality and Theorem 3.19. We observe that its value does not exceed $C_{3.97}(\sigma, \kappa)$, which corresponds to any choice of $\kappa \geq ||\mathbf{J}||_0|\Lambda|$ in (3.97). Combined with the lower bound (2.7) on the growth of V_ℓ, this yields us the required estimate (3.98) on $\rho_{\mu,\Lambda}$and hence implies $\rho_{\mu,\Lambda} \in L^1(\mu)$. ■

Proof of Proposition 3.21 We proceed similarly to the proof of Proposition 2.17 in Subsection 2.2.4. Note that the set (3.99) can be written as

$$\Xi(b,\sigma) = \bigcap_{\ell_0 \in \mathbb{L}} \Xi(\ell_0, b, \sigma) = \bigcap_{\ell_0 \in \mathbb{L}} \bigcup_{\Lambda \Subset \mathbb{L}} \bigcap_{\ell \in \Lambda^c} \Xi_\ell(\ell_0, b, \sigma), \tag{3.147}$$

where

$$\Xi_\ell(\ell_0, b, \sigma) := \left\{ x \in \Omega \;\middle|\; |\omega_\ell|^2_{C^\sigma_\beta} \leq b \log(1 + |\ell - \ell_0|) \right\}, \tag{3.148}$$
$$\Xi(\ell_0, b, \sigma) := \bigcup_{\Lambda \Subset \mathbb{L}} \bigcap_{\ell \in \Lambda^c} [\Xi_\ell(\ell_0, b, \sigma)].$$

By Chebyshev's inequality and the estimate (3.97), we have

$$\mu\left([\Xi_\ell(\ell_0, b, \sigma)]^c\right) \leq C_{3.97}(\sigma, \kappa) \cdot (1 + |\ell - \ell_0|)^{-b\lambda_\sigma}. \tag{3.149}$$

Therefore, for any cofinal sequence $\mathcal{L}$ it holds by (3.148), (3.149)

$$\mu\left([\Xi(\ell_0, b, \sigma)]^c\right) \leq C_{3.97}(\sigma, \kappa) \cdot \lim_{\Lambda \in \mathcal{L}} \sum\nolimits_{\ell \in \Lambda^c} (1 + |\ell - \ell_0|)^{-b\lambda_\sigma}. \tag{3.150}$$

In view of Assumption $(\mathbf{L}_d)$, cf. (2.2), the latter series converges for any $b > d/\lambda_\sigma$. In this case $\mu\left([\Xi(\ell_0, b, \sigma)]^c\right) = 0$, which by (3.147) yields the result $\mu\left([\Xi(b, \sigma)]^c\right) = 0$. ■

Remark 3.34 According to its definition (cf. (3.77) or (3.89)), the set $\mathcal{G}^{\mathrm{t}}$ in advance contains the so-called *Ruelle type "superstable"* Gibbs measures $\mu \in \mathcal{G}^{\mathrm{st}}$. For translation invariant quantum systems, such measures were introduced in [224] by the support condition

$$\sup_{N \in \mathbb{N}} \left\{ (1 + 2N)^{-d} \sum\nolimits_{|\ell| \leq N} |\omega_\ell|^2_{L^2_\beta} \right\} \leq C(\omega) < \infty, \quad \forall \omega \in \Omega \quad (\mu - \text{a.e.}). \tag{3.151}$$

It is worth noting that by the Birkhoff-Khinchin ergodic theorem, for any translation invariant measure $\mu \in \mathcal{P}(\Omega^{\mathrm{t}})$ obeying (3.97), it follows a much *stronger support* property: for every $\sigma \in (0, 1/2)$, $\kappa > 0$ and for μ-almost all ω

$$\sup_{N \in \mathbb{N}} \left\{ (1 + 2N)^{-d} \sum_{\ell : |\ell| \leq N} \exp\left(\lambda_\sigma |\omega_\ell|^2_{C^\sigma_\beta} + \kappa |\omega_\ell|^2_{L^2_\beta} \right) \right\} \leq C(\sigma, \varkappa, \omega). \tag{3.152}$$

In particular, every *periodic* Euclidean Gibbs measure to be constructed in Subsection 3.3.7 (i) will have this property. It would allow us to refine the statement of Theorem 3.18 by claiming that $\mathcal{G}^{\mathrm{st}} \neq \varnothing$. In this respect see also Remark 2.21 concerning with the classical case.

3.2.5 Proof of the uniqueness results

The proof of Theorem 3.22 will be based on the *Dobrushin-Pechersky criterion* for the uniqueness of lattice Gibbs fields. For the precise formulation of this criterion we refer to Subsection 2.3.1, where it has been applied to the classical systems of unbounded spins.

Proof of Theorem 3.22. In the present framework, a continuous function $h : C_\beta \to \mathbb{R}$ can be chosen as

$$h(\omega_\ell) := \lambda_\sigma \kappa^{-1} |\omega_\ell|^2_{C_\beta} + |\omega_\ell|^R_{L^R_\beta}, \tag{3.153}$$

where κ, σ, and λ_σ are the same as Lemma 3.26. Note that, unlike Dobrushin's existence criterion, this h needs not to be compact, cf. Remark 2.22. Let us pick up some constants $\mathcal{I} < 1/\mathrm{a}^{\mathrm{b}} < 1$, $\mathcal{K} < 1$, and $\mathcal{C} \geq 0$; recall that $\mathrm{a}, \mathrm{b} \geq 1$ are the parameters of the lattice $\mathbb{Z}^d$ defined in (2.109), (2.110). Our aim is to show that, for *any large* but fixed $\mathcal{R} > 0$, one finds a small enough $\mathcal{J} := \mathcal{J}(\beta, \mathcal{R}) > 0$ such that, at all values $||\mathbf{J}||_0 \leq \mathcal{J}$ the next two conditions are fulfilled for the probability kernels $\pi_\ell(\mathrm{d}\omega|\xi)$ and their projections $\mu_{\ell,\xi}(\mathrm{d}\omega_\ell) := \pi_\ell(\mathrm{d}\omega|\xi) \circ \mathbb{P}_\ell^{-1}$:

Condition $(\mathbf{DP}_1)$ *For each $\ell \in \mathbb{L}$ and all configurations $\xi \in \Omega$, it holds*

$$\int_\Omega h(x_\ell)\pi_\ell(\mathrm{d}\omega|\xi) \leq \mathcal{C} + \sum_{\ell' \in \partial_r(\ell)} I_{\ell-\ell'} h(\xi_{\ell'}), \tag{3.154}$$

where the sequence $(I_\ell \geq 0)_{\ell \in \partial_r(0)}$ is such that $||\mathbf{I}||_0 := \sum_{\ell \in \partial_r(0)} I_\ell \leq \mathcal{I}$.

Condition $(\mathbf{DP}_2)$ *For each pair of distinct points $\ell, \ell' \in \mathbb{L}$, $|\ell - \ell'| \leq r$, and for all configurations $\xi, \tilde{\xi} \in \Omega$ differing only at ℓ' and satisfying*

$$\sup_{j \in \mathbb{L}} \left\{ h(\xi_j), h(\tilde{\xi}_j) \right\} \leq \mathcal{R}, \tag{3.155}$$

the following estimate in the (half) total variation probability distance in the spin space C_β holds

$$\mathbf{D}_{\mathrm{var}} \left(\mu_{\ell,\xi}, \mu_{\ell,\tilde{\xi}} \right) \leq K_{\ell-\ell'}. \tag{3.156}$$

The sequence $(K_\ell \geq 0)_{\ell \in \partial_r(0)}$ here is such that $||\mathbf{K}||_0 := \sum_{\ell \in \partial_r(0)} K_\ell \leq \mathcal{K}$.

The Dobrushin-Pechersky theorem then says that there exists *at most one* measure $\mu \in \mathcal{P}(\Omega)$ solving the *DLR* equation (3.86) and satisfying the *à-priori* bound

$$\sup_\ell \int_\Omega h(\omega_\ell)\mu(\mathrm{d}\omega) < \infty. \tag{3.157}$$

To verify $(\mathbf{DP}_1)$ and $(\mathbf{DP}_2)$ we shall use the same arguments as those in the proof of Theorem 2.25. The validity of the first condition for each value of $||\mathbf{J}||_0 < \mathcal{J}_0 := \kappa\mathcal{I}$ is obvious by Corollary 3.27. We may set in (3.154) $I_{\ell\ell'} := J_{\ell\ell'}/\kappa$ and $\mathcal{C} := \Upsilon/\kappa$, whereby the parameter $\Upsilon := \Upsilon(||\mathbf{J}||_0)$, which is determined by (3.111)–(3.114), attains its maximum at the endpoint $||\mathbf{J}||_0 := \mathcal{J}_0$. To check the second condition, let us consider any $\xi, \tilde{\xi}$ coinciding off ℓ' and satisfying (3.155). By elementary calculations similar to (2.134)–(2.140), the variation distance can be estimated as

$$\mathbf{D}_{\mathrm{var}}\left(\pi_\ell(\mathrm{d}\omega|\xi), \pi_\ell(\mathrm{d}\omega|\tilde{\xi})\right) := \sup_{B\in\mathcal{B}(C_\beta)} |\mu_\ell(B|y) - \mu_\ell(B|\tilde{y})|$$
$$= \frac{1}{2}\int_\Omega \left|1 - \frac{Z_\ell(\xi)}{Z_\ell(\tilde{\xi})}\exp\left\{\Delta W_{\ell\ell'}(\omega_\ell)\right\}\right| \pi_\ell(\mathrm{d}\omega|\xi) \le \frac{1}{2}\mathcal{I}_{\ell\ell'}\left(4 + \mathcal{I}_{\ell\ell'}\right), \tag{3.158}$$

where we set

$$\Delta W_{\ell\ell'}(\omega_\ell) := \int_0^\beta \left[W_{\ell\ell'}(\omega_\ell, \xi_{\ell'}) - W_{\ell\ell'}(\omega_\ell, \tilde{\xi}_{\ell'})\right] \mathrm{d}\tau, \tag{3.159}$$

$$\mathcal{I}_{\ell\ell'} := \sup_{\substack{\xi,\tilde{\xi}\in\Omega \\ \forall j:\ h(\xi_j),\, h(\tilde{\xi}_j)\le\mathcal{R}}} \int_\Omega |\Delta W_{\ell\ell'}(\omega_\ell)| \cdot \exp|\Delta W_{\ell\ell'}(\omega_\ell)|\pi_\ell(\mathrm{d}\omega|\xi). \tag{3.160}$$

To estimate the right-hand side in (3.160), let us fix some positive $\epsilon < \kappa - \mathcal{J}_0$. Taking into account (2.7), (3.109), and (3.159), we find that

$$\begin{aligned} \mathcal{I}_{\ell\ell'} &\le (J_{\ell-\ell'}/\epsilon)\exp\left\{(\mathcal{J}_0+\epsilon)\left(\beta C_W + \mathcal{R}\right)\right\} \\ &\quad \times \sup_{\substack{\xi\in\Omega \\ \forall j:\ h(\xi_j)\le\mathcal{R}}} \int_\Omega \exp\left\{(\mathcal{J}_0+\epsilon)|\omega_\ell|^R_{L^R_\beta}\right\}\pi_\ell(\mathrm{d}\omega|\xi) \\ &\le (J_{\ell-\ell'}/\epsilon)\exp\left\{\Upsilon + \mathcal{J}_0\mathcal{R} + (\mathcal{J}_0+\epsilon)\left(\beta C_W + \mathcal{R}\right)\right\}. \end{aligned} \tag{3.161}$$

Hence,

$$\mathbf{D}_{\mathrm{var}}\left(\pi_\ell(\mathrm{d}\omega|\xi), \pi_\ell(\mathrm{d}\omega|\tilde{\xi})\right) \le K_{\ell-\ell'} := J_{\ell-\ell'}\mathcal{C}(\beta, \mathcal{J}_0, \mathcal{R}), \tag{3.162}$$

where the constant $\mathcal{C}(\beta, \mathcal{J}_0, \mathcal{R})$ can be written explicitly from (3.158) and (3.161). Finally, choosing $||\mathbf{J}||_0 \le \mathcal{J} < \mathcal{J}_0$ small enough, one gets the required property $||\mathbf{K}||_0 < \mathcal{K}$. ■

Remark 3.35 In order to apply the Dobrushin-Pechersky criterion when β is varying in some interval in $\mathbb{R}_+$, we should be more careful and first pass to the equivalent realization of the Euclidean Gibbs measures on some universal loop space, see (3.94)–(3.96). It is expected that for $\beta \to +0$ the results similar to Theorems 2.25 and 2.26 would also hold in the quantum case.

To prove Theorem 3.23 we shall use the general *Dobrushin uniqueness criterion*, see Subsection 2.3.4.(i). In so far, our arguments will be an infinite dimensional extension (to the loop spin spaces) of those used in the classical case for proving Theorem 2.34. To some extend we shall follow the earlier joint papers [19, 21], where a similar uniqueness statement was obtained for a certain subclass of the quantum lattice models (3.1), (3.2) with the interactions of finite range only.

Proof of Theorem 3.23 Indeed, we shall establish even a stronger fact that there exists at most one Gibbs measure μ on the enlarged configuration space $\tilde{\Omega}^{\mathrm{t}} \supset \Omega^{\mathrm{t}}$,

$$\tilde{\Omega}^{\mathrm{t}} = \left\{ \omega \in \left[L^2_\beta\right]^{\mathbb{L}} \mid (\forall \alpha \in \mathcal{I}) : \|\omega\|^2_\alpha := \sum_\ell |\omega_\ell|^2_{L^2_\beta} w_\alpha(0,\ell) < \infty \right\},$$

which satisfies the moment estimate

$$\sup_\ell \int_\Omega |\omega_\ell|^2_{L_\beta} \mu(\mathrm{d}\omega) < \infty$$

and corresponds to the local specification $\{\pi_\Lambda\}_{\Lambda \Subset \mathbb{L}}$ with the definition (3.54) of the interaction $I_\Lambda(\omega|\xi)$ extended to all $\xi \in \tilde{\Omega}^{\mathrm{t}}$ (cf. Proposition 3.10). The Dobrushin coefficients are given by

$$D_{\ell\ell'} = \sup \left\{ \frac{\mathbf{W}_{L^2_\beta}\left(\mu_{\ell,\xi}, \mu_{\ell,\xi'}\right)}{|\xi_\ell - \xi'_\ell|_{L^2_\beta}} \right\}, \quad \ell \neq \ell', \tag{3.163}$$

where the upper bound is taken over all pairs of *tempered* boundary conditions ξ, $\xi' \in \tilde{\Omega}^{\mathrm{t}}$ which differ only at the site ℓ'. Here $\mathbf{W}_{L^2_\beta}$ denotes the Wasserstein distance between probabilities in L^2_β, cf. (2.169), that is in our setting

$$\mathbf{W}_{L^2_\beta}\left(\mu_{\ell,\xi}, \mu_{\ell,\xi'}\right) = \sup_{f \in \mathrm{Lip}_1} \left| \int_{L^2_\beta} f(\omega_\ell) \, \left[\mu_{\ell,\xi}(\mathrm{d}\omega_\ell) - \mu_{\ell,\xi'}(\mathrm{d}\omega_\ell)\right]) \right|, \tag{3.164}$$

where Lip_1 is the unit ball in the space of real-valued Lipschitz functions on L^2_β. According to Dobrushin's criterion (more precisely, its modification in Theorem 4.46 for the interactions of possibly *infinite range*) the uniqueness stated will follow if (respectively, the *stronger* version of) Contraction Condition ($\mathbf{D_2}$) holds, that is for some $\alpha \in \mathcal{I}$

$$||\mathbf{D}||_0 < ||\mathbf{D}||_\alpha := \sup_\ell \sum_{\ell'} D_{\ell\ell'} \left[w_\alpha(\ell,\ell')\right]^{-1} < 1. \tag{3.165}$$

For $\ell \neq \ell'$, $\xi \in \tilde{\Omega}^{\mathrm{t}}$, and $f \in \mathrm{Lip}_1(L^2_\beta)$, consider the mapping

$$L^2_\beta \ni \xi_{\ell'} \mapsto F_\ell(\xi) := \int_{L^2_\beta} f(\omega_\ell) \mu_{\ell,\xi}(\mathrm{d}\omega_\ell).$$

Because of (3.59), (3.81), and (3.104), this mapping is Fréchet differentiable and its partial derivative in direction $\varphi \in L^2_\beta$ can be written as

$$\left(\nabla_{\xi_{\ell'}} F_\ell(\xi), \varphi\right)_{L^2_\beta} = -\mathbf{Cov}_{\mu_{\ell,\xi}} \left\{ f(\omega_\ell) \, ; \, \left(\partial_{y_{\ell'}} W_{\ell\ell'}(\omega_\ell, \xi_{\ell'}), \varphi\right)_{L^2_\beta} \right\}.$$

The latter is estimated by

$$\left| \left(\nabla_{\xi_{\ell'}} F_\ell(\xi), \varphi\right)_{L^2_\beta} \right| \leq \mathbf{Var}^{1/2}_{\mu_{\ell,\xi}} f \cdot \mathbf{Var}^{1/2}_{\mu_{\ell,\xi}} \left(\partial_{y_{\ell'}} W_{\ell\ell'}(\omega_\ell, \xi_{\ell'}), \varphi\right)_{L^2_\beta}, \tag{3.166}$$

where $\mathbf{Cov}_{\mu_{\ell,\xi}}$ and $\mathbf{Var}_{\mu_{\ell,\xi}}$ denote the covariance and variance with respect to the measure $\mu_{\ell,\xi}$. Now we are at the crucial point in the proof. Namely, we use Theorem 4.52 and its Corollary 4.54, saying that the measures $\mu_{\ell,\xi}$ satisfy the log-Sobolev inequality and, as a sequel, the variance estimate

$$\mathbf{Var}_{\mu_{\ell,\xi}} f \leq \frac{1}{C_{\rm LS}}, \quad \text{for all} \quad f \in \mathrm{Lip}_1(L^2_\beta), \tag{3.167}$$

with the log-Sobolev coefficient

$$C_{\rm LS} := (a + a_U + a_W ||\mathbf{J}||_0)\, e^{-2\beta\delta_Q}, \tag{3.168}$$

which is independent of ξ. Hence, by the upper bound in (3.104),

$$\mathrm{Var}_{\mu_{\ell,\xi}} \left(\partial_{y_{\ell'}} W_{\ell\ell'}(\omega_\ell, \xi_{\ell'}), \varphi \right)_{L^2_\beta} \leq \frac{1}{C_{\rm LS}} J^2_{\ell\ell'} b^2_W |\varphi|^2_{L^2_\beta}. \tag{3.169}$$

Plugging (3.167)–(3.169) into (3.166), we thus get

$$\left| \left(\nabla_{\xi_{\ell'}} F(\xi_{\ell'}), \varphi \right)_{L^2_\beta} \right| \leq \frac{1}{C_{\rm LS}} J_{\ell\ell'} b_W |\varphi|_{L^2_\beta},$$

which by the mean-value theorem implies

$$\left| \int_{L^2_\beta} f(\omega_\ell) \mu_{\ell,\xi}(\mathrm{d}\omega_\ell) - \int_{L^2_\beta} f(\omega_\ell) \mu_{\ell,\xi'}(\mathrm{d}\omega_\ell) \right| \leq \frac{1}{C_{\rm LS}} J_{\ell\ell'} b_W |\xi_\ell - \xi_{\ell'}|_{L^2_\beta},$$

or in terms of the Wasserstein distance

$$\mathbf{W}_{L^2_\beta} \left(\mu_{\ell,\xi}, \mu_{\ell,\xi'} \right) \leq \frac{1}{C_{\rm LS}} J_{\ell\ell'} b_W |\xi_\ell - \xi_{\ell'}|_{L^2_\beta}. \tag{3.170}$$

Hence, like as in (2.193) for the classical case,

$$||\mathbf{D}||_0 \leq ||\mathbf{D}||_\alpha \leq e^{2\beta\delta_Q} \frac{b_W ||\mathbf{J}||_\alpha}{(a_U + a_W ||\mathbf{J}||_0)}, \tag{3.171}$$

and the fulfilment of the Dobrushin condition (3.165) is ensured by Assumption $(\mathbf{J}_\alpha)$ and (3.106). ■

Remark 3.36 As already was discussed in Subsection 2.3.5, the validity of Dobrushin's contraction condition ensures certain *mixing properties* of the specification $\{\pi_\Lambda\}_{\Lambda \Subset \mathbb{L}}$ and the corresponding (unique) Gibbs measure $\mu \in \mathcal{G}^{\rm t}$. In particular, the following condition holding with some $\alpha \in \mathcal{I}$

$$||\mathbf{D}||_\alpha := \sup_\ell \sum_{\ell'} |D_{\ell\ell'}| \cdot [w_\alpha(\ell, \ell')]^{-1} < 1$$

implies (cf. (2.200)–(2.204)) that for all $f,\, g \in C_b^1(\mathbb{R})$, $\varphi,\, \varphi' \in L^2_\beta$, and $\ell, \ell' \in \mathbb{L}$,

$$\begin{aligned} &\mathbf{Cov}_\mu \left[f\left((\omega_k, \varphi)_{L^2_\beta} \right) ;\ g\left((\omega_{k'}, \varphi')_{L^2_\beta} \right) \right] \\ \leq\ & C_{3.172} w_\alpha(\ell, \ell') \cdot ||f'||_{L^\infty} ||g'||_{L^\infty} ||\varphi||_{L^2_\beta} ||\varphi'||_{L^2_\beta}, \end{aligned} \tag{3.172}$$

with the constant.

$$C_{3.172} = (1 - ||\mathbf{D}||_\alpha)^{-1} \sup_\ell \mathbf{E}_\mu |\omega_\ell|^2_{L^2_\beta} > 0.$$

Note that the uniqueness result and its consequences *essentially* depend on the choice of a single spin space (in our case, L^2_β), with respect to which the Dobrushin coefficients are calculated.

Example 3.37 Here we demonstrate how to produce the *optimal* decomposition in (3.101)–(3.103). For $\nu = 1$ let us analyze a typical example of the φ^4-potential

$$V(q) = \kappa q^4 - \lambda q^2, \qquad q \in \mathbb{R}, \qquad \kappa, \lambda > 0. \tag{3.173}$$

This potential has a two-well shape with minima at the points $q_{\min} = \pm\sqrt{\lambda/2\kappa}$. Let us decompose V as follows

$$\begin{aligned} U(q) := V(q) \quad &\textit{and} \quad Q(q) := 0, \quad \text{for} \quad |q| \geq \sqrt{\lambda/k}, \\ U(q) := \frac{2k}{3\lambda}&\left(q^{26} - (\lambda/\kappa)^3\right) - \kappa q^4 + \lambda q^2, \quad \text{for} \quad |q| \leq \sqrt{\lambda/k}, \\ Q(q) := -\frac{2k}{3\lambda}&\left(q^{26} - (\lambda/\kappa)^3\right) + 2\kappa q^4 - 2\lambda q^2, \quad \text{for} \quad |q| \leq \sqrt{\lambda/k}. \end{aligned}$$

It is easy to calculate that

$$\min_{\mathbb{R}} U'' \geq \frac{1}{5}\lambda, \quad \mathbf{Osc}\, Q \leq \frac{5}{2}\lambda^2/\kappa.$$

The uniqueness condition (3.107) then can be rewritten as

$$\exp\left(5\beta\lambda^2/\kappa\right) < (a + \lambda/5)\, ||\mathbf{J}||_0^{-1}, \tag{3.174}$$

which, fixed all other parameters, can always be achieved by the *small depth* of wells $|\min_{\mathbb{R}} V| = \lambda^2/4\kappa$. Let us also consider the *space scaling* of this potential, that is $V^\varepsilon(q) := V(\varepsilon^{-1}q)$ with $\varepsilon > 0$. The uniqueness condition (3.174) now takes the form

$$\exp\left(5\beta\lambda^2/\kappa\right) < \left(a + \lambda/5\varepsilon^2\right) ||\mathbf{J}||_0^{-1},$$

which always to be valid for $\varepsilon \ll 1$. Making ε small we draw together the wells of the potential (3.173) while keeping the depth of the wells fixed. This corresponds to applying pressure to the system considered. So, as a result we provide a rigorous justifucation of the well known *physical phenomenon* that the pressure can remove the critical behavior of the system, see [67]. Actually, we can generalize this example by taking for V any polynomial of even degree with a positive leading coefficient, like that in (3.10).

3.2.6 Further generalizations

Here we briefly discuss how to modify the previous setting in order to include the *many-particle interactions* or the indexing sets represented by *graphs* (cf. Subsection 2.2.5 for the classical case).

(i) The case of P = R

Suppose that Assumptions (**W**), ($\mathbf{J}_0$), and (**V**) are fulfilled with $P = R \geq 2$. For the classical spin systems such situation was analyzed in all details in Subsections 2.2.2–2.2.4. So, for fixed $\beta > 0$, the statements of Theorems 3.18 and 3.19 are still true if Assumption ($\mathbf{V}_1$) from Subsection 2.2.2 holds. In our notation this means

$$\iota := (2/3)A_V - ||\mathbf{J}||_0 > 0, \qquad (3.175)$$

where $A_V > 0$ is the (largest possible) constant in (3.7). Respectively, in the corresponding moment estimates (3.97) and (3.109) one may take any $0 < \kappa < A_V - ||\mathbf{J}||_0/2$. To this end, let us choose some $\alpha_0 \in \mathcal{I}$ such that $||\mathbf{J}||_{\alpha_0} - ||\mathbf{J}||_0 < \iota$. For the above value of the parameter (3.175), this is possible by the assumption (3.60). Thereafter, we go through the above proofs in Subsections 3.2.2, 3.2.4 and use everywhere the system of weights $(w_\alpha)_{\alpha \in \mathcal{I}_0}$ indexed by the smaller interval $\mathcal{I}_0 \subseteq \mathcal{I}$ and satisfying the basic relation $||\mathbf{J}||_\alpha < (2/3)A_V$.

(ii) Many-particle interactions

Our results extend to quantum systems with *many-particle* interactions of possibly *infinite range* and *unbounded order*. Such systems are described by the heuristic Hamiltonian

$$H = \sum_{\ell} \left[H_\ell^{\mathrm{har}} + V_\ell(q_\ell) \right] + \sum_{N=2}^{\leq \infty} \sum_{\{\ell_1, \dots, \ell_N\}} W_{\ell_1 \dots \ell_N}(q_{\ell_l}, \dots, q_{\ell_N}), \qquad (3.176)$$

where the N-particle interaction potentials (taken over unordered finite sets $\{\ell_1, \dots, \ell_N\}$ consisting of $N \geq 2$ distinct points) are given by continuous symmetric functions $W_{\ell_1 \dots \ell_n} : \mathbb{R}^{\nu N} \to \mathbb{R}$. Then Theorems 3.18 and 3.19 are true under Assumptions (**V**), ($\mathbf{W}_\infty$), and ($\mathbf{J}_{\infty,\alpha}$), where the later two are respectively the following modification of (**W**) and ($\mathbf{J}_\alpha$):

Assumption ($\mathbf{W}_\infty$) There exist $R \geq 2$ and $J_{\ell_1 \dots \ell_N} \geq 0$, such that for all $\{\ell_1, \dots, \ell_N\} \subset \mathbb{L}$, $q_1, \dots, q_N \in \mathbb{R}^\nu$ and $N \geq 2$

$$|W_{\ell_1 \dots \ell_N}(q_1, \dots, q_N)| \leq \frac{1}{2} J_{\ell_1 \dots \ell_N} \left(C_W + \sum_{n=1}^{N} |q_n|^R \right). \qquad (3.177)$$

Assumption $(\mathbf{J}_{\infty,\alpha})$ The family of symmetric matrices $(J_{\ell_1\ldots\ell_N})_{\mathbb{L}^N}$ is fastly decreasing, that is for all $\alpha \in \mathcal{I}$

$$||\mathbf{J}||_\alpha := \sum_{N=2}^{\leq\infty} N^2 \sup_{\ell_1} \left\{ \sum_{\{\ell_2,\ldots,\ell_N\}} J_{\ell_1\ldots\ell_N} \left(1 + \sum\nolimits_{n=1}^{N} [w_\alpha(\ell_1,\ell_n)]^{-1}\right) \right\} < \infty. \tag{3.178}$$

The proofs of Theorems 3.18 and 3.19 are similar to those carried before. With obvious modifications we can also state the uniqueness results of Theorems 3.22 and 3.24.

(iii) Quantum systems on graphs

Instead of $\mathbb{L}$ one can consider more general indexing sets. In particular, this could be an *infinite graph* $\mathbb{G}(\mathbb{V},\mathbb{E})$ with the properties as described in Subsection 2.2.5 (iii). Under Assumption $(\mathbf{G}_\delta)$ holding with some $\delta_0 > 0$, we make a natural choice of the weights

$$w_\delta(v,v') := \exp\{-\delta\rho(\nu,v')\}, \quad \alpha := \delta \in \mathcal{I} := (\delta_0,\infty) \subset \mathbb{R}. \tag{3.179}$$

Setting $\Omega := [C_\beta]^{\mathbb{G}}$, we define the subset of tempered loop configurations

$$\Omega^{\mathrm{t}} := \bigcap_{v\in\mathbb{V},\ \delta\in\mathcal{I}} \Omega_{v,\delta}\ , \tag{3.180}$$

$$\Omega_\delta := \left\{ \omega \in \Omega \,\middle|\, ||\omega||_\delta := \left[\sum_v |\omega_{v'}|^R_{L^R_\beta} w_\alpha(v,o)\right]^{1/R} < \infty \right\}.$$

On the graph $\mathbb{G}$, we now consider an interacting quantum system with the formal Hamiltonian

$$H = \sum_v \left[H_v^{\mathrm{har}} + V_v(q_v)\right] + \frac{1}{2}\sum_{v\sim v'} W_{vv'}(q_v,q_{v'}), \tag{3.181}$$

where the potentials $W_{vv'} : \mathbb{R}^{2\nu} \to \mathbb{R}$ and $V_v : \mathbb{R}^\nu \to \mathbb{R}$ fulfill the former Assumptions $(\mathbf{W})$ and $(\mathbf{V})$. The matrix $\mathbf{J} := (J_{vv'})_{\mathbb{V}\times\mathbb{V}}$ in Assumption $(\mathbf{J}_0)$ has the entries $J_{vv'} = J > 0$ if $v \sim v'$ and $J_{vv'} = 0$ otherwise. Fixed an inverse temperature $\beta > 0$, one defines the local specification $\{\pi_\Lambda\}_{\Lambda\Subset\mathbb{V}}$. We confirm ourselves to the subset of tempered Euclidean Gibbs measures $\mu \in \mathcal{G}^{\mathrm{t}}$ which are supported by Ω^{t}. Modifying the proofs of Theorems 3.18 and 3.19 for the system of weights (3.179), one can conclude that the set $\mathcal{G}^{\mathrm{t}}$ is not empty and that all its elements obey the à-priori bound (3.97). Actually, the existence of some $\mu \in \mathcal{G}$ can be proved on any graph of bounded degree $m(\mathbb{G}) \leq \delta_0 < \infty$. Dobrushin's uniqueness criterion on graphs will be discussed in Subsection 4.4.2 below.

3.3 Ferromagnetic scalar models

In this section we consider in more detail the special case of $\nu = 1$ and *attractive harmonic* interactions

$$W_{\ell\ell'}(q_\ell,q_{\ell'}) := -J_{\ell\ell'}q_\ell q_{\ell'} \quad \text{with} \quad J_{\ell\ell'} \geq 0. \tag{3.182}$$

So, the model we focused on is described by the formal Hamiltonian

$$H = \sum_{\ell} \left[H_{\ell}^{\rm har} + V_{\ell}(q_{\ell}) - hq_{\ell} \right] - \frac{1}{2} \sum_{\ell,\ell' \in \mathbb{L}} J_{\ell\ell'} q_{\ell} q_{\ell'}, \tag{3.183}$$

where $h \in \mathbb{R}$ is an *external field* and the one-particle potentials V_{ℓ} and the *dynamical matrix* $\mathbf{J} := (J_{\ell\ell'})_{\mathbb{L}\times\mathbb{L}}$ satisfy respectively Assumptions $(\mathbf{V})$ and $(\mathbf{J}_{\alpha})$. The emphasis will be placed on the study of *critical behavior*. Among other things, we shall prove that: (a) $|\mathcal{G}^{\rm t}| > 1$ at low temperatures ($\beta^{-1} \ll 1$); (b) $|\mathcal{G}^{\rm t}| = 1$ due to small masses ($m \ll 1$) and at a nonzero external field ($h \neq 0$). This allows us to develop a qualitative theory of phase transitions and quantum effects, which interprets the most important experimental data known for the corresponding physical objects (see Subsection 3.1.1). In Subsection 3.3.1 we present the main theorems, whereas their proofs and the supporting material will be distributed through Subsections 3.3.2–3.3.9.

3.3.1 Discussion of results

A general observation is that the system (3.182) is best suited for the study of critical behavior, since now one can use a variety of *correlation inequalities* (*FKG*, *GKS*, *Lebowitz*, and the other listed in Subsection 3.3.4) relying on the additional properties of the interaction. We stress that all the results described below are specific for the ferromagnetic systems and cannot be obtained by general methods which were developed in the foregoing sections. Of particular interest is the *new comparison criterion* for even ferromagnets (applicable both in the classical and in the quantum cases), which is stated by Proposition 3.64 and allows to *extend* considerably the classes of interactions in Theorems 3.45 and 3.46 describing respectively phase transitions or a uniqueness regime.

(i) Maximal and minimal elements in $\mathcal{G}^{\rm t}$

Let us first introduce a *partial order* on the set $\mathcal{G}^{\rm t}$, which is a universal tool in all ferromagnetic models. As the components of the configurations $\omega \in \Omega$ are continuous functions $\omega_{\ell} : S_{\beta} \to \mathbb{R}^{\nu}$, we may write $\omega \leq \tilde{\omega}$ if $\omega_{\ell}(\tau) \leq \tilde{\omega}_{\ell}(\tau)$ for all ℓ and τ. Thereby, we define the following set of increasing functions

$$K_{+}(\Omega^{\rm t}) = \{ f \in C_{\rm b}(\Omega^{\rm t}) \mid f(\omega) \leq f(\tilde{\omega}), \quad \text{if} \quad \omega \leq \tilde{\omega} \}, \tag{3.184}$$

which is a proper cone.

Lemma 3.38 *If for given $\mu, \tilde{\mu} \in \mathcal{G}^{\rm t}$, one has*

$$\mu(f) = \tilde{\mu}(f) \qquad \textit{for all} \quad f \in K_{+}(\Omega^{\rm t}), \tag{3.185}$$

then $\mu = \tilde{\mu}$.

The proof of this lemma will be done below in Subsection 3.3.2. We use it to establish a dual order on $\mathcal{G}^{\rm t}$, which is also called the *stochastic domination*.

Definition 3.39 *For $\mu, \tilde{\mu} \in \mathcal{G}^{\rm t}$ we say that $\mu \preceq \tilde{\mu}$, if*

$$\mu(f) \leq \tilde{\mu}(f), \qquad \text{for all} \quad f \in K_+(\Omega^{\rm t}). \tag{3.186}$$

Remark 3.40 (i) An important feature of the model (3.183) is that its local specification $\Pi = \{\pi_\Lambda\}_{\Lambda \Subset \mathbb{L}}$ is *attractive* in the sense of C. Preston [233]. This means that each π_Λ leaves the cone $K_+(\Omega^{\rm t})$ invariant, i.e.,

$$\pi_\Lambda f \in K_+(\Omega^{\rm t}) \quad \text{for any } \Lambda \Subset \mathbb{L} \text{ and } f \in K_+(\Omega^{\rm t}). \tag{3.187}$$

As was shown in Propositions IV.2 in [42], this property also holds for *general* ferromagnetic pair interactions of the form

$$W_{\ell\ell'}(q_\ell, q_{\ell'}) := J_{\ell\ell'} w(q_\ell - q_{\ell'}) \geq 0, \tag{3.188}$$

where $w \in C^2(\mathbb{R} \to \mathbb{R}_+)$ is an even convex function with $w'' \geq 0$.

(ii) Sometimes (see e.g. the proof of Lemma 3.53) it will be more convenient to relate our model to the formal Hamiltonian

$$\tilde{H} = \sum_\ell \left[H_\ell^{\rm har} + \tilde{V}_\ell(q_\ell) \right] + \frac{1}{4} \sum_{\ell,\ell' \in \Lambda} J_{\ell\ell'}(q_\ell - q_{\ell'})^2, \tag{3.189}$$

where we set

$$\tilde{V}_\ell(q_\ell) := V_\ell(q_\ell) - \frac{||\mathbf{J}||_0}{2} q_\ell^2 - h q_\ell. \tag{3.190}$$

Both models (3.183) and (3.189) are *completely equivalent* in the Euclidean approach, since they possess the same local specification $\Pi = \{\pi_\Lambda\}_{\Lambda \Subset \mathbb{L}}$.

We claim that now the set $\mathcal{G}^{\rm t}$ has unique *maximal* and *minimal* elements – an analogous fact is well known for classical spin systems (cf. Theorem IV.3 and Propositions IV.4, V.1, V.3 in [42]). The *extreme* elements $\mu_\pm$ will play a crucial role in proving Theorems 3.45, 3.46, and 3.48.

Theorem 3.41 *The set of all Euclidean Gibbs measures $\mathcal{G}^{\rm t}$ of the scalar ferromagnetic model possesses maximal μ_+ and minimal μ_- elements in the sense of Definition 3.39. These elements are extreme and τ-shift invariant; they are also translation invariant if the model is translation invariant. If $V_\ell(-x) = V_\ell(x)$ for all ℓ, then $\mu_+(B) = \mu_-(-B)$ for all $B \in \mathcal{B}(\Omega)$.*

(ii) The pressure and its applications

Now let the model be translation invariant, which in particular means $\mathbb{L} := \mathbb{Z}^d$. We are going to study the *limiting pressure* which contains important information about the thermodynamic properties of the model A special attention will be paid to the dependence of the pressure on the *external field* $h \in \mathbb{R}$, cf. (3.183). The corresponding analytic properties will be applied to the uniqueness problem, see e.g. Proposition 3.43.

First, we define the pressure for the local state in a volume $\Lambda \Subset \mathbb{L}$ as

$$p_\Lambda(\xi) := \frac{1}{|\Lambda|} \log Z_\Lambda(\xi), \quad \xi \in \Omega^{\rm t}, \tag{3.191}$$

where the partition function $Z_\Lambda(\xi)$ is given by

$$\begin{aligned} Z_\Lambda(\xi) \ : \ &= \int_{\Omega_\Lambda} \exp\Bigg\{ -\sum_{\ell\in\Lambda} \int_0^\beta V(\omega_\ell){\rm d}\tau \\ &+ \frac{1}{2}\sum_{\ell,\ell'\in\Lambda} J_{\ell\ell'}(\omega_\ell,\omega_{\ell'})_{L^2_\beta} + \sum_{\ell\in\Lambda,\ \ell'\in\Lambda^c} J_{\ell\ell'}(\omega_\ell,\xi_{\ell'})_{L^2_\beta}\Bigg\} \chi_\Lambda({\rm d}\omega_\Lambda). \end{aligned} \tag{3.192}$$

For $\mu \in \mathcal{G}^{\rm t}$, we then set

$$p^\mu_\Lambda := \int_\Omega p_\Lambda(\xi)\mu({\rm d}\xi). \tag{3.193}$$

If for a cofinal sequence $\mathcal{L}$ the limit

$$p^\mu = \lim_{\mathcal{L}} p^\mu_\Lambda \tag{3.194}$$

exists, we shall call it the *pressure in the state* μ. To obtain these limits we impose a certain condition on the sequences $\mathcal{L}$. Given $l = (l_1, \dots l_d)$, $l' = (l'_1, \dots l'_d) \in \mathbb{L} := \mathbb{Z}^d$, such that $l_j < l'_j$ for all $j = 1, \dots, d$, we define

$$\Gamma := \{\ell \in \mathbb{L} \mid l_j \le \ell_j \le l'_j, \ \text{ for all } j = 1, \dots, d\}. \tag{3.195}$$

For this parallelepiped, let $\mathfrak{G}(\Gamma)$ be the family of all pairwise disjoint translates of Γ which cover $\mathbb{L}$. Then for any $\Lambda \Subset \mathbb{L}$, by $N_-(\Lambda|\Gamma)$ (respectively, $N_+(\Lambda|\Gamma)$) we denote the number of the elements of $\mathfrak{G}(\Gamma)$ which are contained in Λ (respectively, have non-void intersections with Λ). Then (similarly to Section 2.1 in [251] or Appendix C in [136]) we introduce

Definition 3.42 *A cofinal sequence $\mathcal{L}$ is a **van Hove sequence** if for every Γ,*

$$(a) \quad \lim_{\mathcal{L}} N_-(\Lambda|\Gamma) = +\infty; \qquad (b) \quad \lim_{\mathcal{L}} \left(N_-(\Lambda|\Gamma)/N_+(\Lambda|\Gamma)\right) = 1. \tag{3.196}$$

Theorem 3.43 *For each Gibbs measure $\mu \in \mathcal{G}^{\rm t}$ and for any van Hove sequence $\mathcal{L}$, the limit (3.194) exists. Its value does not depend on the particular choice of μ and $\mathcal{L}$, and is equal to the free energy (3.233) constructed in Lemma 3.53 below.*

For classical ferromagnetic spin models, similar statements about the limiting pressure can be found in [42, 184]. The following quantum analog of the result of J. Lebowitz and A. Martin-Löf [181], which will be proven in Subsection 3.3.3, is a standard sequel of Theorems 3.41 and 3.43. Here we write $p := p(h)$ to emphasize the dependence of the pressure on the *magnetic field* $h \in \mathbb{R}$.

Proposition 3.44 *If $p(h)$ is differentiable in some neighborhood of a given $h_0 \in \mathbb{R}$, then $\mathcal{G}^{\rm t}$ is a singleton at this h_0.*

The above proposition will be applied e.g. in proving the uniqueness result of Theorem 3.48.

(iii) Phase transitions at $\beta \gg 1$

In the *DLR* approach the multiplicity of the Gibbs states corresponds to *phase transitions*. In physical systems structural phase transitions manifest themselves in the macroscopic displacements of particles from their equilibrium positions (a *long-range order*). For translation invariant ferroelectric models with $V_\ell = V$ obeying certain conditions, the appearance of such macroscopic displacements in low temperature and heavy mass regime was proven in [38, 100, 140, 162, 228]. Thus, in our model it is reasonable to expect that $|\mathcal{G}^{\rm t}| > 1$ at large β and m. The latter fact would imply the appearance of macroscopic displacements, but the converse need not to be true in general. To avoid technical complications we concentrate at the case of $\mathbb{L} := \mathbb{Z}^d$ with $d \geq 3$; by means of correlation inequalities the results obtained can be extended to irregular $\mathbb{L} \subset \mathbb{R}^d$.

Let us impose further conditions on $J_{\ell\ell'}$ and V_ℓ. The first one presumes that an intensity of the interaction between the nearest neighbors is *uniformly nonzero*, i.e.,

$$\inf_{\ell,\ell':\ |\ell-\ell'|=1} J_{\ell\ell'} := \mathcal{J} > 0. \tag{3.197}$$

Next we suppose that V_ℓ are *even* continuous functions and hence can be written in the form

$$V_\ell(x) = V_\ell(-x) = v_\ell(x^2), \quad x \in \mathbb{R}. \tag{3.198}$$

Typically the long-range order destroys $\mathbb{Z}_2$-symmetry in a way that there appear Gibbs measures $\mu \in \mathcal{G}^{\rm t}$ obeying $\mathbf{E}_\mu \omega_\ell \neq 0$, which immediately would imply $|\mathcal{G}^{\rm t}| > 1$. Let the upper bound in (3.7) can be chosen in the form

$$V(x) = \sum_{\rm s=1}^{\rm p} b^{(\rm s)} x^{2\rm s}; \quad 2b^{(1)} < -a; \quad b^{(\rm s)} \geq 0 \ \text{ for } \ {\rm s} \geq 2, \tag{3.199}$$

where a is the same as in (3.4), and $\rm p \geq 2$ is either a positive integer or the infinity. If $\rm p < \infty$, the polynomials (3.199) belong to the so-called *EMN* (*Ellis-Monroe-Newman*, see [104, 107]) class. If $\rm p = \infty$, we assume that the series

$$\varPhi(t) := \sum_{\rm s=2}^{\infty} \frac{(2\rm s)!}{2^{\rm s-1}(\rm s-1)!} b^{(\rm s)} t^{\rm s-1} \tag{3.200}$$

converges at some $t > 0$. Since $2b^{(1)} + a < 0$, the equation

$$a + 2b^{(1)} + \varPhi(t) = 0, \tag{3.201}$$

has a unique solution $t_* > 0$. Finally, we suppose that for each ℓ the function $V - V_\ell$ is *increasing* on $\mathbb{R}_+$, i.e.,

$$V(x) - V_\ell(x) \leq V(\tilde{x}) - V_\ell(\tilde{x}), \quad \text{whenever} \ \ x^2 \leq \tilde{x}^2. \tag{3.202}$$

If the above v_ℓ in (3.198) are differentiable, the latter condition may be formulated as the upper bound on their derivatives

$$v'_\ell(t) \leq \sum_{\mathrm{s}=2}^{\mathrm{p}} \mathrm{s} b^{(\mathrm{s})} t^{\mathrm{s}-1}, \quad t \geq 0. \tag{3.203}$$

In particular, if V_ℓ themselves are polynomials having the form (3.10) with $h = 0$ and $\mathrm{p} < \infty$, we may put

$$b^{(1)} = \sup_\ell b_\ell^{(1)}; \qquad b^{(\mathrm{s})} = \sup_\ell |b_\ell^{(\mathrm{s})}|, \ \mathrm{s} \geq 2. \tag{3.204}$$

One observes that according to the assumptions (3.199)–(3.202) all V_ℓ have a uniform *double-well shape* in the neighborhood of zero.

For $d \geq 3$, we set

$$\theta_d := \frac{1}{(2\pi)^d} \int_{(-\pi,\pi]^d} \frac{\mathrm{d}p}{E(p)}, \tag{3.205}$$

$$E(p) := \sum_{j=1}^{d} (1 - \cos p^j), \quad p = (p^j)_{j=1}^d \in (-\pi, \pi]^d.$$

Let also $f : [0, \infty) \to [0, 1)$ be the function defined *implicitly* by

$$f(0) = 1, \quad f(t \cdot \tanh t) := t^{-1} \cdot \tanh t, \quad \text{for} \ \ t > 0. \tag{3.206}$$

It is convex and monotone decreasing on $(0, \infty)$. For an account of its properties see [102], where it was introduced. By (3.206) one readily observes that for every fixed $t > 0$, the function

$$(0, \infty) \ni \beta \mapsto \phi(\beta, t) := \beta t f(\beta/t), \tag{3.207}$$

is monotone increasing to t^2 as $\beta \to \infty$.

Theorem 3.45 *Let $d \geq 3$ and the above assumptions hold. Then under the condition*

$$\mathcal{J} > \theta_d / 8mt_*^2 \tag{3.208}$$

(which always holds e.g. for a mass m large enough), there exists a critical temperature $\beta_* > 0$ such that $|\mathcal{G}^{\mathrm{t}}| > 1$ whenever $\beta > \beta_*$. The bound β_* is the unique solution of the equation

$$2\theta_d m/J = \phi(\beta, 4mt_*). \tag{3.209}$$

(iv) Uniqueness at $m \ll 1$ or $h \neq 0$.

On the other hand as was shown in [5, 9, 278, 279], *quantum effects* occurring in particular at small values of the particle mass m can suppress abnormal fluctuations. Thus, in this case one might expect that $|\mathcal{G}^{\mathrm{t}}| = 1$ holding simultaneously *at all temperatures*.

Turning to the probabilistic interpretation, our model describes a system of interacting diffusion processes, in which strong quantum effects correspond to *large diffusion intensity*. The most advanced result in this domain – the uniqueness at all β due to quantum effects – was proven in [8]. However, it was essentially restricted to the models with nearest neighbor interaction and V being certain polynomials (of the *EMN* type, cf. (3.199)). In Theorem 3.46 below we extend this result in two directions. We consider a substantially *larger* class of anharmonic potentials and make *precise* the bounds of the uniqueness regime. Furthermore, unlike to the mentioned papers, we do not assume that the interaction has finite range or that $\mathbb{L}$ is regular.

Regarding the anharmonic potentials we again suppose that each V_ℓ is *even* and hence can be written in the form (3.198). Now we look at the lower bound for V_ℓ in (3.7). Our main assumption is that it can be chosen as

$$V(x) = v(x^2), \quad x \in \mathbb{R}, \tag{3.210}$$
$$\text{where} \quad v : [0, \infty) \to \mathbb{R} \ \text{ is } \ \textit{convex}.$$

Such V are usually called to be of the *BFS* (*Brydges-Fröchlich-Spencer*, see [69]) type. Furthermore, we suppose that the function $V_\ell - V$ is *increasing* on $\mathbb{R}_+$, i.e.,

$$v_\ell(t) - v(t) \le v_\ell(\tilde{t}) - v(\tilde{t}) \quad \text{whenever} \ \ t \le \tilde{t}. \tag{3.211}$$

In typical cases of V_ℓ like (3.10), as such v one may take a convex polynomial of degree r ≥ 2. Since the coefficient $b^{(1)} \in \mathbb{R}$ can be a large negative number, the functions V and V_ℓ may have *double wells* of an arbitrary depth.

In $L^2(\mathbb{R}, \mathrm{d}x)$ we introduce the following one-particle Hamiltonian (cf. (3.3), (3.29))

$$\tilde{H} := -\frac{1}{2m}\frac{\partial^2}{\partial x^2} + \frac{a}{2}x^2 + v(x^2), \quad x \in \mathbb{R}. \tag{3.212}$$

It has purely discrete non-degenerate spectrum $\{E_n\}_{n\in\mathbb{N}_0}$. Thus, one can define the *parameter*

$$\varDelta_m := \min_{n\in\mathbb{N}} \left(E_n - E_{n-1}\right), \tag{3.213}$$

which is positive and depends on the model parameters m, a, and on the choice of v. Recall, that $||\mathbf{J}||_0$ was defined by (3.9).

Theorem 3.46 *Let the anharmonic potentials V_ℓ be as above. Then the set of Euclidean Gibbs measures is a singleton if*

$$m\varDelta_m^2 > ||\mathbf{J}||_0. \tag{3.214}$$

Remark 3.47 (i) An important issue is that the above result is *independent of* $\beta > 0$. The relation (3.214) can be looked upon as a stability condition like (3.108), where the parameter $m\varDelta_m^2$ appears as the *oscillator rigidity*. If it holds, a *stability-due-to-quantum-effects* occurs, cf. [9, 163, 169]. One meets here a non-trivial problem how to analyze the behavior of $m\varDelta_m^2$ for general Schrödinger operators like (3.212). A particular case of v being a polynomial of degree $\mathrm{r} \geq 2$ can be successfully handled by analytical perturbation methods. In this case the rigidity $m\Delta_m^2$ is a continuous function of the particle mass m; it gets small in the quasiclassical limit $m \to \infty$, see [169]. On the other hand, as was shown in [5, 169], one has

$$m\Delta_m^2 = \mathcal{O}(m^{-(r-1)/(r+1)}), \quad m \to +0. \tag{3.215}$$

Hence, (3.214) certainly holds in the *small mass* limit (cf. [6, 8]).

(ii) To compare the latter statement with Theorem 3.45 let us assume that $\mathbb{L} := \mathbb{Z}^d$ with $d \geq 3$, $J_{\ell\ell'} = J$ iff $|\ell - \ell'| = 1$, and all V_ℓ coincide with the polynomial given by (3.199). Then the gap parameter (3.213) obeys the estimate $\Delta_m < 1/2mt_*$, cf. [169], where t_* is the same as in (3.208), (3.209). In this case the condition (3.214) can be rewritten as

$$J < 1/8dmt_*^2. \tag{3.216}$$

It is known that $\theta_d > 1/d$ and $\theta_d \sim (d - 1/2)^{-1}$ *as* $d \to \infty$, see Theorem 5.1 in [100]. Hence, the estimates (3.208) and (3.216), which give sufficient conditions for the phase transition to occur or to be suppressed, become *asymptotically sharp* in large dimensions.

Thus, Theorem 3.46 provides a mathematical justification for the well-known *physical phenomenon* (see e.g., [48, 277]) that structural phase transition for a given mass m can be suppressed not only by thermal fluctuations (i.e., high temperatures $\beta^{-1} > \beta_{\mathrm{cr}}^{-1}$), but for the light particles (with $m < m_{\mathrm{cr}}$) also by the quantum fluctuations (i.e., tunneling in a double-well potential) simultaneously at all temperatures $\beta > 0$.

Consider again a translation invariant version of our model, i.e., $\mathbb{L} := \mathbb{Z}^d$. Set

$$\mathcal{F}_{\mathrm{Laguerre}} = \left\{ \varphi : \mathbb{R} \to \mathbb{R} \;\middle|\; \varphi(t) = \varphi_0 \exp(\gamma_0 t) t^n \prod_{j=1}^{\infty} (1 + \gamma_j t) \right\}, \tag{3.217}$$

where $\varphi_0 > 0$, $n \in \mathbb{N}_0$, and $\gamma_j \geq 0$ are such that $\sum_{j=1}^{\infty} \gamma_j < \infty$. Each $\varphi \in \mathcal{F}_{\mathrm{Laguerre}}$ can be extended to an entire function $\varphi : \mathbb{C} \to \mathbb{C}$ which has no zeros outside of $(-\infty, 0]$. These are *Laguerre entire functions*, for more details see [152, 168, 171].

Theorem 3.48 *Let the model we consider be translation invariant and the anharmonic potential be of the form*

$$V(x) = v(x^2) - hx, \quad h \in \mathbb{R}, \tag{3.218}$$

where $v(0) = 0$. *Furthermore, let there exist* $b_0 > 0$ *such that for all* $b \geq b_0$, *the derivative* v' *obeys the condition* $b + v' \in \mathcal{F}_{\mathrm{Laguerre}}$. *Then the set* $\mathcal{G}^{\mathrm{t}}$ *is a singleton if* $h \neq 0$.

For classical lattice models, the uniqueness at nonzero h was proven in [42, 181, 184]. This was done under the condition that the potential (3.218) possesses the *Lee-Yang property* which we establish below in Definition 3.75. The novelty of Theorem 3.48 is that it describes a quantum model and gives an explicit sufficient condition for V to possess such a property, cf. Remark 3.79. It turns out that this theorem is valid for the classical systems under the less restrictive condition that $b + v' \in \mathcal{F}_{\rm Laguerre}$ just *for some* value $b > 0$, which covers all the cases considered in [42, 181, 184]. In the quantum case a typical example satisfying Theorem 3.48 are the $(\phi^4)_2$-Euclidean fields, for which a similar uniqueness statement was independently proven in [124].

3.3.2 Stochastic order and extreme elements

First we make sure that the cone $K_+(\Omega^{\rm t})$ defined by (3.184) may be used to establish an order on $\mathcal{G}^{\rm t}$. This means that both the estimates $\mu(f) \leq \tilde{\mu}(f)$ and $\tilde{\mu}(f) \leq \mu(f)$ being valid for all $f \in K_+(\Omega^{\rm t})$ should imply $\mu = \tilde{\mu}$. Recall that *measure defining* classes of functions are usually established by means of monotone class theorems, see e.g. [47], pages 36–39. In our situation, a sufficient condition for a family of bounded continuous functions to be a measure defining class may be formulated as follows: is should (a) contain constant functions; (b) be closed under multiplication; (c) separate points of $\Omega^{\rm t}$. The class (3.184) does not meet (b); hence, to prove the stated we have to use additional arguments.

Proof of Lemma 3.38. Let us show that the cone $K_+(\Omega^{\rm t})$ contains a measure defining class for $\mathcal{G}^{\rm t}$. A continuous function $f : \Omega^{\rm t} \to \mathbb{R}$ is called a *cylinder function* if it can be represented as

$$f(\omega) = \phi(\omega_{\ell_1}(\tau_1), \ldots, \omega_{\ell_n}(\tau_n)), \tag{3.219}$$

with certain $\phi \in C(\mathbb{R}^n)$, $\ell_j \in \mathbb{L}$, $\tau_j \in S_\beta$, and $1 \leq j \leq n \in \mathbb{N}$. By $K_+^{\rm cyl}(\Omega^{\rm t})$ we denote the subset of $K_+(\Omega^{\rm t})$ consisting of cylinder functions. Suppose that (3.185) holds for all $f \in K_+^{\rm cyl}(\Omega^{\rm t})$. By (3.97) and a standard approximation argument, this implies the identity

$$\int_{\Omega^{\rm t}} \omega_\ell(\tau)\mu({\rm d}\omega) = \int_{\Omega^{\rm t}} \omega_\ell(\tau)\tilde{\mu}({\rm d}\omega), \qquad \text{for all} \ \ \ell, \tau. \tag{3.220}$$

For fixed $\ell_1, \ldots, \ell_n$ and $\tau_1, \ldots, \tau_n$, let P_n and $\tilde{P}_n$ be the corresponding projections of the measures μ and $\tilde{\mu}$ on $\mathbb{R}^n$. That is, each of P_n and $\tilde{P}_n$ obeys

$$\int_{\Omega^{\rm t}} f(\omega)\mu({\rm d}\omega) = \int_{\mathbb{R}^n} \phi(x_1, \ldots, x_n)P({\rm d}x)$$

for f and ϕ as in (3.219). Then by (3.185), it follows that $P_n \preceq \tilde{P}_n$, i.e.,

$$\int_{\mathbb{R}^n} \phi(x_1, \ldots, x_n)P_n({\rm d}x) \leq \int_{\mathbb{R}^n} \phi(x_1, \ldots, x_n)\tilde{P}_n({\rm d}x), \tag{3.221}$$

for all increasing ϕ. Let P be a probability measure on $\mathbb{R}^{2n}$ such that its marginal distributions coincide with P_n and $\tilde{P}_n$,

$$P_n({\rm d}x) = \int_{\mathbb{R}^n} P({\rm d}x, {\rm d}\tilde{x}), \qquad \tilde{P}_n({\rm d}\tilde{x}) = \int_{\mathbb{R}^n} P({\rm d}x, {\rm d}\tilde{x}).$$

In other words, $P \in \Pi(P_n, \tilde{P}_n)$ is a *coupling* of P_n and $\tilde{P}_n$. Of course, the above equations do not determine P uniquely. By the *Kantorovich-Rubinstein duality* relation (see Remark 2.32 (i)), the Wasserstein distance between the measures P_n and $\tilde{P}_n$ can be calculated as

$$\mathbf{W}(P_n, \tilde{P}_n) = \inf_{P \in \Pi(P_n, \tilde{P}_n)} \int_{\mathbb{R}^{2n}} |x - \tilde{x}| \hat{P}(\mathrm{d}x, \mathrm{d}\tilde{x}). \tag{3.222}$$

Consider

$$M = \{(x, \tilde{x}) \in \mathbb{R}^{2n} \mid x_i \leq \tilde{x}_i, \quad \text{for all } i = 1, \dots, n\},$$

which is a closed set in $\mathbb{R}^{2n}$. By *Strassen's theorem* (see page 129 of [192]), for $P_n \preceq \tilde{P}$ there exists a coupling $\widehat{P} \in \Pi(P_n, \tilde{P}_n)$ such that $\widehat{P}(M) = 1$. Thereby,

$$\begin{aligned}\mathbf{W}(P_n, \tilde{P}_n) &\leq \int_M |x - \tilde{x}| \widehat{P}(\mathrm{d}x, \mathrm{d}\tilde{x}) \leq \sum_{i=1}^n \int_{\mathbb{R}^{2n}} (\tilde{x}_i - x_i) \widehat{P}(\mathrm{d}x, \mathrm{d}\tilde{x}) \\ &= \sum_{i=1}^n \int_{\mathbb{R}^n} x_i \left[\tilde{P}_n(\mathrm{d}x) - P_n(\mathrm{d}x) \right] = 0,\end{aligned}$$

where the latter equality is implied by (3.220). Since the subset of $C_{\mathrm{b}}(\Omega^{\mathrm{t}})$ consisting of all cylinder functions (3.219) is a defining class for $\mathcal{P}(\Omega^{\mathrm{t}})$, this yields $\mu = \tilde{\mu}$. ∎

One observes that for (3.221) to hold, it was enough to have $\mu \preceq \tilde{\mu}$, cf., (3.184). Thus, we have the following important fact arising from the proof of the above lemma.

Corollary 3.49 *If for any pair of $\mu, \tilde{\mu} \in \mathcal{G}^{\mathrm{t}}$ such that $\mu \preceq \tilde{\mu}$, all their first moments coincide, i.e. (3.220) holds, then $\mu = \tilde{\mu}$.*

Remark 3.50 *For every ℓ, $t_\ell(\omega) \leq t_\ell(\tilde{\omega})$ if $\omega \leq \tilde{\omega}$. This means that the transformation θ_ℓ defined in (3.90) is order preserving.*

Proof of Theorem 3.41. In establishing the existence of the elements $\mu_\pm$ the main point was to prove Lemma 3.38. Thereby, the existence of $\mu_\pm$ can be shown by literal repetition of the arguments used in [42] for proving Theorem IV.3. They are unique by definition. Indeed, for two maximal elements, say μ_+ and $\tilde{\mu}_+$, one would have $\mu_+ \preceq \tilde{\mu}_+$ and $\tilde{\mu}_+ \preceq \mu_+$ at the same time. Thus, $\mu_+ = \tilde{\mu}_+$. The proof of the extremeness (respectively, the symmetry properties) of $\mu_\pm$ can be done by following the proof of Proposition V.1 (respectively, Proposition V.3) in [42]. ∎

Remark 3.51 (i) Let us give an *explicit construction* of the extreme measures $\mu_\pm$. For $\ell_0 \in \mathbb{L}$ and $b > 0$, let $\hat{\xi} = (\hat{\xi}_\ell)_{\ell \in \mathbb{L}}$ be the following constant (with respect to $\tau \in S_\beta$) configuration

$$\hat{\xi}_\ell(\tau) := [b \log(1 + |\ell - \ell_0|)]^{1/2}, \quad \ell \in \mathbb{L}. \tag{3.223}$$

Fix $\sigma \in (0, 1/2)$ and then respectively $b > d/\lambda_\sigma$ (see the proof of Theorem 2.19). In view of (3.58), $\hat{\xi}$ belongs to Ω^{t}. It also belongs to $\Xi(\ell_0, b, \sigma)$, and for every $\xi \in \Xi(b, \sigma) \subset \Omega^{\mathrm{t}}$ one finds $\Delta \Subset \mathbb{L}$ such that $\xi_\ell(\tau) \leq \hat{\xi}_\ell(\tau)$ for all τ and $\ell \in \Delta^c$. Therefore, for any cofinal sequence $\mathcal{L}$ one finds $\Delta \in \mathcal{L}$ such that $\pi_\Lambda(\cdot|\xi) \preceq \pi_\Lambda(\cdot|\hat{\xi})$ for all $\Lambda \in \mathcal{L}$

containing this Δ, see (3.253). As was established in the proof of Theorem 3.18, the sequence $\{\pi_\Lambda(\cdot|\xi)\}_{\Lambda\in\mathcal{L}}$ is relatively compact in any $\mathcal{W}_\alpha$, $\alpha\in\mathcal{I}$, which by Lemmas 3.29, 3.30 yields its $\mathcal{W}^{\rm t}$-relative compactness. Let $\hat{\mu}$ be any of the accumulation points of $\{\pi_\Lambda(\cdot|\hat{\xi})\}_{\Lambda\in\mathcal{L}}$. By Lemma 3.16 this $\hat{\mu}$ belongs to $\mathcal{G}^{\rm t}$ and by Lemma 3.33 it dominates every element of ex$(\mathcal{G}^{\rm t})$. Hence, $\hat{\mu}=\mu_+$ since the maximal element is unique; the same is true for the remaining accumulation points of $\{\pi_\Lambda(\cdot|\xi)\}_{\Lambda\in\mathcal{L}}$. Thus, for every cofinal sequence $\mathcal{L}$ and for any choice of ℓ_0 in (3.223), we have

$$\lim_{\mathcal{L}}\pi_\Lambda(\cdot|\pm\hat{\xi})=\mu_\pm. \tag{3.224}$$

(ii) As the configuration (3.223) is constant with respect to $\tau\in S_\beta$, the kernel $\pi_\Lambda(\cdot|\hat{\xi})$ may be considered as the one $\hat{\pi}_\Lambda(\cdot|0)$ corresponding to the Hamiltonian with the *external field* $\hat{\xi}$, that is

$$H_{\xi,\Lambda}(q_\Lambda):=H_\Lambda(q_\Lambda)-\sum_{\ell\in\Lambda}(q_\ell,\hat{\xi}_\ell). \tag{3.225}$$

We summarize the above discussion in the following statement, which will be *fundamental* for all further considerations.

Proposition 3.52 (Uniqueness criterion) *For the scalar ferromagnetic model (3.183), the next properties are equivalent:*

(a) $\mathcal{G}^{\rm t}$ is a singleton;

(b) For all $\ell\in\mathbb{L}$ it holds

$$\int_\Omega\omega_\ell(0)\mu_+(\mathrm{d}\omega)=\int_\Omega\omega_{\ell_0}(0)\mu_-(\mathrm{d}\omega). \tag{3.226}$$

If the model is symmetric, then (3.226) turns into $\mathbf{E}_{\mu_\pm}\omega_\ell(0)=0$;

(c) For all $\ell\in\mathbb{L}$, $\tau\in S_\beta$, and for any pair of boundary conditions $\xi,\tilde{\xi}\in\Xi(b,\sigma)$ with some $\sigma\in(0,1/2)$ and $b>d/\lambda_\sigma$ (see the proof of Theorem 2.19), it holds

$$\lim_{\mathcal{L}}\left[\int_\Omega\omega_\ell(\tau)\pi_\Lambda(\mathrm{d}\omega|\xi)-\int_\Omega\omega_\ell(\tau)\pi_\Lambda(\mathrm{d}\omega|\tilde{\xi})\right]=0, \tag{3.227}$$

along every cofinal sequence $\mathcal{L}$.

Proof. (a) $\Leftrightarrow$ (b): By Theorem 3.41 one has $|\mathcal{G}^{\rm t}|=1$ if and only if $\mu_+=\mu_-$, which by Corollary 3.49 is equivalent to

$$\mathbf{E}_{\mu_+}\omega_\ell(\tau)=\mathbf{E}_{\mu_-}\omega_\ell(\tau),\quad \text{for all } \ell,\ \tau.$$

By the τ-invariance property (3.19), $\mathbf{E}_{\mu\pm}\omega_\ell(\tau)=\mathbf{E}_{\mu_\pm}\omega_\ell(0)$.

(b) $\Leftrightarrow$ (c): By the observation made in Remark 3.51 (i), there exists $\Delta\in\mathcal{L}$ such that

$$\pi_\Lambda(\mathrm{d}\omega|-\hat{\xi})\preceq\pi_\Lambda(\mathrm{d}\omega|\xi),\quad \pi_\Lambda(\mathrm{d}\omega|\tilde{\xi})\preceq\pi_\Lambda(\mathrm{d}\omega|\hat{\xi}),$$

for all $\Lambda \in \mathcal{L}$, $\Delta \subset \Lambda$. Then, by the stochastic domination

$$\left| \int_{\Omega} \omega_\ell(\tau) \left[\pi_\Lambda(\mathrm{d}\omega|\xi) - \pi_\Lambda(\mathrm{d}\omega|\tilde{\xi}) \right] \right| \leq \int_{\Omega} \omega_\ell(\tau) \left[\pi_\Lambda(\mathrm{d}\omega|\hat{\xi}) - \pi_\Lambda(\mathrm{d}\omega| - \hat{\xi}) \right],$$

which by (3.224) implies the desired convergence (3.227). Choosing here the boundary conditions $\pm\hat{\xi}$, we immediately get the inverse claim (3.226). ■

Actually, the above criterion is a path space version of the known theorem of J. Lebowitz and A. Martin-Löf for ferromagnetic systems, which was originally proved for the Ising model in [181] and then extended to unbounded classical spins in [42, 184]. It says that the zero spontaneous magnetization $\mu_\pm(x_\ell) = 0$ in plus/minus boundary conditions implies uniqueness of the tempered Gibbs states. The ferromagneticity is thus a very helpful assumption – if it does not hold one should check instead of (3.227) that

$$\lim_{\mathcal{L}} \int_{\Omega} f(\omega_1(\tau_1), \ldots, \omega_{\ell_n}(\tau_n)) \left[\pi_\Lambda(\mathrm{d}\omega|\xi) - \pi_\Lambda(\mathrm{d}\omega|\tilde{\xi}) \right] = 0,$$

with all possible choices of $f \in C_{\mathrm{b}}(\mathbb{R}^N)$, $\ell_j \in \mathbb{L}$, $\tau_j \in S_\beta$, and $1 \leq j \leq n \in \mathbb{N}$.

3.3.3 Existence of the pressure

Here we consider a *translation invariant* version of our model with $\mathbb{L} := \mathbb{Z}^d$ and $V_\ell := V$, $J_{\ell\ell'} := J_{\ell-\ell'}$. Given $r > 0$ and $\Lambda \Subset \mathbb{L}$, let $\partial_r^+ \Lambda$ be the subset of all $\ell \in \Lambda^c$ such that $\mathrm{dist}(\ell, \Lambda) \leq r$. According to Definition 3.42, any van Hove sequence $\mathcal{L}$ certainly satisfies

$$\lim_{\mathcal{L}} |\partial_r^+ \Lambda| / |\Lambda| = 0, \quad \text{for each } \ r \geq 1. \tag{3.228}$$

An *important property* of the van Hove sequences $\mathcal{L}$ to be used below is that $||\mathbf{J}||_0 = \sum_{\ell \in \mathbb{L}} J_\ell < \infty$ implies the average convergence

$$\lim_{\mathcal{L}} \frac{1}{|\Lambda|} \sum_{\ell \in \Lambda, \ell' \in \Lambda^c} J_{\ell-\ell'} = 0, \tag{3.229}$$

see [184, 252]. Note that the existence of van Hove sequences means the *amenability* of the graph $(\mathbb{L}, \mathbb{E})$ with $\mathbb{E}$ being the set of all pairs ℓ, ℓ' such that $|\ell - \ell'| = 1$, cf. the definition (4.9). For nonamenable graphs, phase transitions with $h \neq 0$ are possible; hence statements like Theorem 3.48 do not hold, see [154, 198].

We conventionally begin with the study of the pressure functional, with *empty boundary* condition $\xi = \varnothing$, related to the Hamiltonian (3.189) introduced in Remark 3.40 (ii). In this case, the corresponding finite volume partition functions can be written as

$$\tilde{Z}_\Lambda := \tilde{Z}_\Lambda(\varnothing) := \int_{\Omega_\Lambda} \exp\left\{ -\sum_{\ell \in \Lambda} \int_0^\beta \tilde{V}(\omega_\ell) - \frac{1}{4} \sum_{\ell, \ell' \in \Lambda} J_{\ell\ell'} |\omega_\ell - \omega_{\ell'}|^2_{L^2_\beta} \right\} \chi_\Lambda(\mathrm{d}\omega_\Lambda), \tag{3.230}$$

where we set

$$\tilde{V}(q) := V(q) - \frac{||\mathbf{J}||_0}{2} q^2 - hq. \tag{3.231}$$

The free energy per site is defined by

$$\tilde{p}_\Lambda := \tilde{p}_\Lambda(\varnothing) := \frac{1}{|\Lambda|} \log \tilde{Z}_\Lambda(\varnothing). \tag{3.232}$$

Lemma 3.53 *The thermodynamic limit for the* ***free energy***

$$p := \lim_{\mathcal{L}} \tilde{p}_\Lambda(\varnothing) \tag{3.233}$$

exists for every van Hove sequence $\mathcal{L}$. It is independent of the particular choice of $\mathcal{L}$.

Proof. Without loss of generality we may assume that

$$\tilde{Z}_0 := \int_{C_\beta} \exp\left\{ - \int_0^\beta \tilde{V}(\omega_\ell(\tau)) \mathrm{d}\tau \right\} \chi(\mathrm{d}\omega_\ell) = 1 \tag{3.234}$$

(otherwise one has to apply a normalization, which leads to the factor $\log \tilde{Z}_\ell$ at the front of all $p_\Lambda(\varnothing)$ and p). Since the pair interaction in (3.189) is nonnegative, a useful observation made from (3.230) and (3.234) is that

$$\tilde{Z}_{\Lambda_1} \leq 1, \quad \tilde{Z}_{\Lambda_1 \sqcup \Lambda_2} \leq \tilde{Z}_{\Lambda_1} \cdot \tilde{Z}_{\Lambda_2}, \tag{3.235}$$

for all disjoint sets $\Lambda_1, \Lambda_2 \Subset \mathbb{L}$. On the other hand, by Assumptions $(\mathbf{V})$, $(\mathbf{J}_0)$,

$$\tilde{Z}_\Lambda \geq \int_{\Omega_\Lambda} \exp\left\{ - \sum_{\ell \in \Lambda} \int_0^\beta \left[\tilde{V}(\omega_\ell) + ||\mathbf{J}||_0 \omega_\ell^2 \right] \mathrm{d}\tau \right\} \chi_\Lambda(\mathrm{d}\omega_\Lambda) = \exp\left\{ -|\Lambda| C_{3.236} \right\}, \tag{3.236}$$

with a certain $C_{3.236} > 0$. Therefore, by (3.235) and (3.236), for all $\Lambda \Subset \mathbb{L}$

$$-C_{3.236} \leq \frac{1}{|\Lambda|} \log \tilde{Z}_\Lambda \leq 0, \quad |\tilde{p}_\Lambda| \leq C_{3.236}, \tag{3.237}$$

and hence the set $\{\tilde{p}_\Lambda\}_{\Lambda \Subset \mathbb{L}}$ has accumulation points in $\mathbb{R}$. For one of them, p, let $\{\Gamma_n\}_{n \in \mathbb{N}}$ be the sequence of parallelepipeds such that $p_{\Gamma_n} \to p$ as $n \to \infty$. Let also $\mathcal{L}$ be a van Hove sequence in the sense of Definition 3.42. Given $n_0 \in \mathbb{N}$ and $\Lambda \in \mathcal{L}$, let $\mathfrak{L}^-_{n_0}(\Lambda) \subset \mathfrak{G}(\Gamma_{n_0})$ (respectively, $\mathfrak{L}^+_{n_0}(\Lambda) \subset \mathfrak{G}(\Gamma_{n_0})$) consist of the translates of Γ_{n_0} which are contained in Λ (respectively, which have non-void intersections with Λ). Let also

$$\Lambda^\pm_{n_0} = \bigcup_{\Gamma \in \mathsf{L}^\pm_{n_0}} \Gamma, \quad |\Lambda^\pm_{n_0}| = N_\pm(\Lambda|\Gamma_{n_0}) \cdot |\Gamma_{n_0}|. \tag{3.238}$$

Note that $\tilde{Z}_\Gamma = \tilde{Z}_{\Gamma_{n_0}}$ for all $\Gamma \in \mathfrak{G}(\Gamma_{n_0})$, which follows from the translation invariance of the model. Now we standardly estimate $\tilde{p}_\Lambda - \tilde{p}_{\Gamma_{n_0}}$ by mimicking the proof of Theorem 2.7 in [184]. By (3.234), (3.235), and (3.238)

$$\begin{aligned}\tilde{p}_\Lambda &= \frac{1}{|\Lambda|}\log \tilde{Z}_\Lambda \leq \frac{1}{|\Lambda|}N_-(\Lambda|\Gamma_{n_0})\log \tilde{Z}_{\Gamma_{n_0}} + \frac{1}{|\Lambda|}\left[|\Lambda| - N_-(\Lambda|\Gamma_{n_0})\right]\log \tilde{Z}_0 \\ &\leq \frac{N_-(\Lambda|\Gamma_{n_0})}{N_-(\Lambda|\Gamma_{n_0})\cdot|\Gamma_{n_0}|}\log \tilde{Z}_{\Gamma_{n_0}} = \tilde{p}_{\Gamma_{n_0}}.\end{aligned}$$

On the other hand, by (3.235), (3.237), and (3.238)

$$\begin{aligned}\tilde{p}_\Lambda &= \frac{1}{|\Lambda|}\log \tilde{Z}_\Lambda \geq \frac{1}{|\Lambda|}N_+(\Lambda|\Gamma_{n_0})\log \tilde{Z}_{\Gamma_{n_0}} - \frac{1}{|\Lambda|}\left[N_+(\Lambda|\Gamma_{n_0}) - |\Lambda|\right]\log \tilde{Z}_0 \\ &\geq \frac{1}{|\Gamma_{n_0}|}\log \tilde{Z}_{\Gamma_{n_0}} - \left(\frac{N_+(\Lambda|\Gamma_{n_0})}{|\Lambda|} - 1\right)C_{3.236} \geq \tilde{p}_{\Gamma_{n_0}} - \epsilon,\end{aligned}$$

for any $\epsilon > 0$ and a large enough $n_0 \geq n_0(\epsilon) \in \mathbb{N}$. This yields the result

$$\lim_{\mathcal{L}} \tilde{p}_\Lambda = \lim_{n\to\infty} \tilde{p}_{\Gamma_n} =: p. \tag{3.239}$$

■

Proof of Theorem 3.43. The proof will be done if we show that for every $\mu \in \mathcal{G}^{\mathrm{t}}$ and any van Hove sequence $\mathcal{L}$

$$\lim_{\mathcal{L}} p^\mu_\Lambda = \lim_{\mathcal{L}} \frac{1}{|\Lambda|}\int_\Omega \log Z_\Lambda(\xi)\,\mu(\mathrm{d}\xi) = p, \tag{3.240}$$

with p being the same as in (3.233), (3.239). For each $\xi \in \Omega^{\mathrm{t}}$, the partition functions of both Hamiltonians (3.183) and (3.189) are related by

$$Z_\Lambda(\xi) = \tilde{Z}_\Lambda(\xi)\cdot\exp\left\{\frac{1}{2}\sum_{\ell\in\Lambda,\ \ell'\in\Lambda^c} J_{\ell\ell'}|\xi_{\ell'}|^2_{L^2_\beta}\right\}, \tag{3.241}$$

where we set, cf. (3.230),

$$\begin{aligned}\tilde{Z}_\Lambda(\xi) := \int_{\Omega_\Lambda}\exp\Bigg\{&-\sum_{\ell\in\Lambda}\int_0^\beta \tilde{V}(\omega_\ell)\mathrm{d}\tau \\ -\frac{1}{4}\sum_{\ell,\ell'\in\Lambda} J_{\ell\ell'}|\omega_\ell - \omega_{\ell'}|^2_{L^2_\beta} &- \frac{1}{2}\sum_{\ell\in\Lambda,\ \ell'\in\Lambda^c} J_{\ell\ell'}|\omega_\ell - \xi_{\ell'}|^2_{L^2_\beta}\Bigg\}\chi_\Lambda(\mathrm{d}\omega_\Lambda) \leq \tilde{Z}_\Lambda(\varnothing).\end{aligned} \tag{3.242}$$

Furthermore, by Jensen's inequality $\mathbf{E}_\mu(\exp f) \geq \exp(\mathbf{E}_\mu f)$ applied to $\mu := \tilde{\mu}_\Lambda$, one has

$$\tilde{Z}_\Lambda(\xi) \geq \tilde{Z}_\Lambda(\varnothing)\cdot\exp\left\{-\frac{1}{2}\int_{\Omega_\Lambda}\left[\sum_{\ell\in\Lambda,\ \ell'\in\Lambda^c} J_{\ell\ell'}|\omega_\ell - \xi_{\ell'}|^2_{L^2_\beta}\right]\tilde{\mu}_\Lambda(\mathrm{d}\omega_\Lambda)\right\}. \tag{3.243}$$

Combining (3.241)–(3.243), we obtain the following estimates

$$\tilde{p}_\Lambda - \frac{1}{2|\Lambda|} \sum_{\ell \in \Lambda,\ \ell' \in \Lambda^c} J_{\ell\ell'} \int_{\Omega_\Lambda} |\omega_\ell - \xi_{\ell'}|^2_{L^2_\beta} \tilde{\mu}_\Lambda(\mathrm{d}\omega_\Lambda) \leq \tilde{p}_\Lambda(\xi) \leq \tilde{p}_\Lambda, \tag{3.244}$$

$$p_\Lambda(\xi) = \tilde{p}_\Lambda(\xi) + \frac{1}{2|\Lambda|} \sum_{\ell \in \Lambda,\ \ell' \in \Lambda^c} J_{\ell\ell'} |\xi_{\ell'}|^2_{L^2_\beta}, \tag{3.245}$$

which relate the quantities

$$p_\Lambda(\xi) := (1/|\Lambda|) \log Z_\Lambda(\xi), \quad \tilde{p}_\Lambda(\xi) := (1/|\Lambda|) \log \tilde{Z}_\Lambda(\xi).$$

Next, we integrate in (3.244), (3.245) with respect to any $\mu \in \mathcal{G}^{\mathrm{t}}$ and arrive at

$$\tilde{p}_\Lambda - \frac{1}{|\Lambda|} \sum_{\ell \in \Lambda,\ \ell' \in \Lambda^c} J_{\ell\ell'} \left[\mu \left(|\xi_{\ell'}|^2_{L^2_\beta} \right) + \tilde{\mu}_\Lambda \left(|\omega_\ell|^2_{L^2_\beta} \right) \right] \ \leq\ \tilde{p}^\mu_\Lambda \ \leq\ \tilde{p}_\Lambda,$$

$$p^\mu_\Lambda = \tilde{p}^\mu_\Lambda + \frac{1}{2|\Lambda|} \sum_{\ell \in \Lambda,\ \ell' \in \Lambda^c} J_{\ell\ell'} \mu \left(|\xi_{\ell'}|^2_{L^2_\beta} \right). \tag{3.246}$$

By means of Theorem 3.19 and Lemma 3.29 one can bound both $\mu \left(|\xi_{\ell'}|^2_{L^2_\beta} \right)$ and $\tilde{\mu}_\Lambda \left(|\omega_\ell|^2_{L^2_\beta} \right)$ by positive constants independent of ℓ, ℓ'. Thereby, the property stated follows from (3.229) and Lemma 3.53. ■

Remark 3.54 As is clear from (3.244)–(3.246), it holds

$$\lim_{\mathcal{L}} p_\Lambda(\xi) = \lim_{\mathcal{L}} \tilde{p}_\Lambda = p, \tag{3.247}$$

for every van Hove sequence L and any boundary condition $\xi \in \Omega^{\mathrm{t}}$ obeying

$$\lim_{\mathcal{L}} \frac{1}{|\Lambda|} \sum_{\ell \in \Lambda,\ \ell' \in \Lambda^c} J_{\ell\ell'} |\xi_{\ell'}|^2_{L^2_\beta} = 0. \tag{3.248}$$

This is always the case if $\xi = 0$ or $\sup_\ell |\xi_{\ell'}|^2_{L^2_\beta} < \infty$. To have (3.247) for each ξ from $\Xi(b, \sigma)$, which by Proposition 3.21 is a universal support set for all $\mu \in \mathcal{G}^{\mathrm{t}}$, one needs a *stronger* than (3.229) regularity property

$$\lim_{\mathcal{L}} \frac{1}{|\Lambda|} \sum_{\ell \in \Lambda,\ \ell' \in \Lambda^c} J_{\ell\ell'} \log(1 + |\ell'|) = 0. \tag{3.249}$$

Proof of Proposition 3.44. First we note that, for each $\Lambda \Subset \mathbb{L}$ and $\xi \in \Omega^{\mathrm{t}}$, the local pressure $p_\Lambda(h, \xi)$ (and hence also the limiting one $p(h)$) is a *convex function* of $h \in \mathbb{R}$. This is a general fact, which is equivalent to the following property of the partition function

$$Z_\Lambda(\xi, h_1 + h_2) \leq [Z_\Lambda(\xi, h_1)]^{1/2} [Z_\Lambda(\xi, h_2)]^{1/2}$$

and is immediately seen from the Cauchy inequality. Furthermore, by direct calculations based on (3.191) and (3.192)

$$\frac{\partial}{\partial h} p_\Lambda(h,\xi) = \frac{1}{|\Lambda|}\sum_{\ell\in\Lambda}\int_0^\beta \pi_\Lambda[\omega_\ell(\tau)|\xi]\mathrm{d}\tau. \tag{3.250}$$

Then, for every $\mu \in \mathcal{G}^{\mathrm{t}}$ one has

$$\begin{aligned} \frac{\partial}{\partial h} p_\Lambda^\mu(h) &= \int_\Omega \frac{\partial}{\partial h}\left(p_\Lambda^\mu(h,\xi)\right)\mu(\mathrm{d}\xi) \\ &= \frac{1}{|\Lambda|}\sum_{\ell\in\Lambda}\int_0^\beta\int_\Omega \pi_\Lambda\left[\omega_\ell(\tau)|\xi\right]\mu(\mathrm{d}\xi)\mathrm{d}\tau \\ &= \frac{1}{|\Lambda|}\sum_{\ell\in\Lambda}\int_0^\beta \mu\left[\omega_\ell(\tau)\right]\mathrm{d}\tau. \end{aligned} \tag{3.251}$$

Note that in (3.250), (3.251) we have used Lebesgue's dominated convergence theorem and the à-priori bound (3.97) to interchange the differentiation in h and the integration over $\mu(\mathrm{d}\xi)$ and $\mathrm{d}\tau$. By Theorem 3.43 and Corollary 3.54 the sequences of differentiable convex functions $\{p_\Lambda(h,0)\}_{\Lambda\in\mathcal{L}}$, $\{p_\Lambda^{\mu\pm}(h)\}_{\Lambda\in\mathcal{L}}$ converge along any van Hove sequence $\mathcal{L}$ to one and the same limit $p(h)$, which holds for all $h\in\mathbb{R}$. Then if $p(h)$ is differentiable at a given h_0, the well-known result on the convex functions (see pages 34-35 of [258]) says that also each of the derivatives $\partial p^{\mu_\pm}(h_0)/\partial h$ should converge to $\partial p(h_0)/\partial h$. Both extreme measures $\mu_\pm$ are translation and shift invariant, which implies by (3.251) that

$$\frac{\partial}{\partial h} p_\Lambda^{\mu_+}(h_0) = \beta\mu_+(\omega_\ell(0)), \quad \frac{\partial}{\partial h} p_\Lambda^{\mu_-}(h_0) = \beta\mu_-(\omega_\ell(0)).$$

Hence, for all ℓ it holds $\mu_+(\omega_\ell(0)) = \mu_-(\omega_\ell(0))$, which by Proposition 3.52 (b) completes the proof. ■

Corollary 3.55 *The pressure $h \mapsto p(h)$ is a convex function on $\mathbb{R}$. If the potential V is even, i.e., $V(x) = v(x^2)$, then $h \mapsto p(h)$ is an even function growing on $\mathbb{R}_+$.*

Proof. The convexity of $p(h)$ has been already established by proving Proposition 3.44. If $V(x) = V(-x)$ for all $x \in \mathbb{R}$, then from (3.230) and (3.231) it is obvious that $p_\Lambda(h,0) = p_\Lambda(-h,0)$ for all $h\in\mathbb{R}$. By Lemma 3.53 and Theorem 3.43, the same property, i.e., $p(h) = p(-h)$, holds for the limiting pressure $p(h)$. ■

Remark 3.56 (i) From Proposition 3.44 and Corollary 3.55 it follows that the pressure $p(h)$ is differentiable and thus the corresponding set $\mathcal{G}^{\mathrm{t}}$ is a singleton, *except at most countable* number of values of $h\in\mathbb{R}$.

(ii) In proving Theorem 3.43 and Proposition 3.44 we do not require that the interaction possesses $\mathbb{Z}_2$-symmetry. In fact, the previous proofs work for any system with the Hamiltonian written in the form (3.189), where $J_{\ell-\ell'}(q_\ell - q_{\ell'})^2$ can be substituted by *general* pair potentials $W_{\ell-\ell'}(q_\ell, q_{\ell'}) \geq 0$.

(iii) As we have seen above, the knowledge of the à-priori bounds on $\mu\in\mathcal{G}^{\mathrm{t}}$ allows us to show existence of the limiting pressure in a rather elementary way, e.g. without applying *Ruelle's* technique of *superstability estimates* [184, 252, 254].

3.3.4 Correlation inequalities

Our main Theorems 3.45 and 3.46 will be proved *by comparing* the model considered with a certain (so-called *reference*) model, for which the property desired is being established directly. The comparison is based on *correlation inequalities* originated from classical spin systems, which we present in this subsection. For measures on path or loop spaces, such inequalities usually are derived in the framework of the lattice approximation technique, analogous to that of Euclidean quantum fields [255, 257]. For a systematic account on this matter the reader may refer to the review article [7]. Below we propose a simple approximation procedure (different from that employed in [7]), which allows us to get the required inequalities in a *much shorter* and *universal* way.

We begin with the *FKG* (*Fortuin-Kasteleyn-Ginibre)* inequalities, see [112]. Recall that the families of functions $K_+(\Omega)$ and $K_+^{\rm cyl}(\Omega)$ were introduced respectively in (3.184) and in the proof of Lemma 3.38.

Proposition 3.57 *For all* $\Lambda \Subset \mathbb{L}$, $\xi \in \Omega^{\rm t}$ *and any* $f, g \in K_+(\Omega)$, *it follows that*

$$\pi_\Lambda(f \cdot g|\xi) \ \geq \ \pi_\Lambda(f|\xi) \cdot \pi_\Lambda(g|\xi). \tag{3.252}$$

This inequality holds also for any continuous increasing functions, for which the corresponding integrals exist. This yields, in particular, that for all such functions

$$\xi \leq \tilde{\xi} \ \Rightarrow \ \pi_\Lambda(f|\xi) \preceq \pi_\Lambda(f|\tilde{\xi}). \tag{3.253}$$

The polynomial moments and covariances of *general* ferromagnets are nonnegative due to the *Griffiths* inequalities (see pages 74–76 in [129]), which in our case can be formulated as follows:

Proposition 3.58 *For all* $\xi \in \Omega^{\rm t}$, $\ell_1, \ldots, \ell_{N+M} \in \Lambda \Subset \mathbb{L}$, $\tau_1, \ldots, \tau_{N+M} \in S_\beta$, *and* $N, M \in \mathbb{N}$, *it holds*

$$\int_\Omega \left(\prod_{j=1}^N \omega_{\ell_j}(\tau_j) \right) \pi_\Lambda \left({\rm d}\omega|\xi\right) \geq 0; \tag{3.254}$$

$$\begin{aligned} & \int_\Omega \left(\prod_{j=1}^N \omega_{\ell_j}(\tau_j) \right) \cdot \left(\prod_{j=N+1}^{N+M} \omega_{\ell_j}(\tau_j) \right) \pi_\Lambda \left({\rm d}\omega|\xi\right) \\ & \geq \int_\Omega \left(\prod_{j=1}^N \omega_{\ell_j}(\tau_j) \right) \pi_\Lambda \left({\rm d}\omega|\xi\right) \cdot \int_\Omega \left(\prod_{j=N+1}^{N+M} \omega_{\ell_j}(\tau_j) \right) \pi_\Lambda \left({\rm d}\omega|\xi\right). \end{aligned} \tag{3.255}$$

For *even* ferromagnets, the above proposition is extended by the *GKS* inequalities (due to *J. Ginibre* and *Griffiths-Kelly-Sherman*; see pages 119–124 in [257]).

Proposition 3.59 *Let the anharmonic potentials have the form*

$$V_\ell(x) = v_\ell(x^2) - h_\ell x, \quad h_\ell \geq 0 \ \ \textit{for all} \ \ \ell \in \mathbb{L}, \tag{3.256}$$

with v_ℓ being continuous. Let also the continuous functions $f_1, \ldots, f_{N+M} : \mathbb{R} \to \mathbb{R}$ be polynomially bounded and such that every f_j is increasing nonnegative on $\mathbb{R}_+$ and either even or odd on the whole $\mathbb{R}$. Then the following inequalities hold for all $\ell_1, \ldots, \ell_{N+M} \in \Lambda \Subset \mathbb{L}$, $\tau_1, \ldots, \tau_{N+M} \in S_\beta$, and $N, M \in \mathbb{N}$,

$$\int_\Omega \left(\prod_{j=1}^N f_j(\omega_{\ell_j}(\tau_j)) \right) \pi_\Lambda\left(\mathrm{d}\omega|0\right) \geq 0; \tag{3.257}$$

$$\begin{aligned} &\int_\Omega \left(\prod_{j=1}^N f_j(\omega_{\ell_j}(\tau_j)) \right) \cdot \left(\prod_{j=N+1}^{N+M} f_j(\omega_{\ell_j}(\tau_j)) \right) \pi_\Lambda\left(\mathrm{d}\omega|0\right) \\ \geq &\int_\Omega \left(\prod_{j=1}^N f_j(\omega_{\ell_j}(\tau_j)) \right) \pi_\Lambda\left(\mathrm{d}\omega|0\right) \cdot \int_\Omega \left(\prod_{j=N+1}^{N+M} f_j(\omega_{\ell_j}(\tau_j)) \right) \pi_\Lambda\left(\mathrm{d}\omega|0\right). \end{aligned} \tag{3.258}$$

Given $\xi \in \Omega^{\mathrm{t}}$, $\ell, \ell' \in \Lambda \Subset \mathbb{L}$, and $\tau, \tau' \in S_\beta$, the *pair correlation* function is defined by

$$\begin{aligned} K^\Lambda_{\ell\ell'}(\tau, \tau'|\xi) \quad : \quad = &\int_\Omega \omega_\ell(\tau)\omega_{\ell'}(\tau')\pi_\Lambda(\mathrm{d}\omega|\xi) \\ - &\int_\Omega \omega_\ell(\tau)\pi_\Lambda(\mathrm{d}\omega|\xi) \cdot \int_\Omega \omega_{\ell'}(\tau')\pi_\Lambda(\mathrm{d}\omega|\xi). \end{aligned} \tag{3.259}$$

Note that by (3.253)

$$K^\Lambda_{\ell\ell'}(\tau, \tau'|\xi) \geq 0. \tag{3.260}$$

The next result is the path space version of the estimate (12.129), page 254 of [107].

Proposition 3.60 *Let all V_ℓ belong to the* BFS *class, i.e., they can be written in the form (3.256) with $h_\ell = 0$ and* convex *functions v_ℓ Then for all $\ell, \ell' \in \Lambda \Subset \mathbb{L}$, $\tau, \tau' \in S_\beta$, and for any $0 \leq \xi \in \Omega^{\mathrm{t}}$ it holds*

$$K^\Lambda_{\ell\ell'}(\tau, \tau'|\xi) \leq K^\Lambda_{\ell\ell'}(\tau, \tau'|0). \tag{3.261}$$

Let us consider

$$\begin{aligned} U^\Lambda_{\ell_1\ell_2\ell_3\ell_4}(\tau_1, \tau_2, \tau_3, \tau_4) := &\int_\Omega \omega_{\ell_1}(\tau_1)\omega_{\ell_2}(\tau_2)\omega_{\ell_3}(\tau_3)\omega_{\ell_4}(\tau_4)\pi_\Lambda(\mathrm{d}\omega|0) \\ &- K^\Lambda_{\ell_1\ell_2}(\tau_1, \tau_2|0)K^\Lambda_{\ell_3\ell_4}(\tau_3, \tau_4|0) \\ &- K^\Lambda_{\ell_1\ell_3}(\tau_1, \tau_3|0)K^\Lambda_{\ell_2\ell_4}(\tau_2, \tau_4|0) \\ &- K^\Lambda_{\ell_1\ell_4}(\tau_1, \tau_4|0)K^\Lambda_{\ell_2\ell_3}(\tau_2, \tau_3|0), \end{aligned} \tag{3.262}$$

which is the *Ursell function* for the measure $\pi_\Lambda(\cdot|0)$. The next statement gives the *Gaussian domination* and *Lebowitz inequalities* (in the classical case, see e.g. [69, 104, 266]).

Proposition 3.61 *Let V_ℓ be of the* BFS *type like as in Proposition 3.60. Then for all $\ell_1, \ldots, \ell_{2N} \in \Lambda \Subset \mathbb{L}$, $\tau_1, \ldots, \tau_{2N} \in S_\beta$, and $N \in \mathbb{N}$,*

$$\int_\Omega \omega_{\ell_1}(\tau_1)\omega_{\ell_2}(\tau_2)\cdots\omega_{\ell_{2n}}(\tau_{2N})\pi_\Lambda(\mathrm{d}\omega|0) \tag{3.263}$$

$$\leq \sum_\sigma \prod_{j=1}^N \int_\Omega \omega_{\ell_{\sigma(2j-1)}}(\tau_{\sigma(2j-1)})\omega_{\ell_{\sigma(2j)}}(\tau_{\sigma(2j)})\pi_\Lambda(\mathrm{d}\omega|0),$$

where the sum runs through the set of all partitions of $\{1, \ldots, 2N\}$ onto unordered pairs. In particular,

$$U^\Lambda_{\ell_1\ell_2\ell_3\ell_4}(\tau_1, \tau_2, \tau_3, \tau_4) \leq 0. \tag{3.264}$$

Proof of Propositions 3.57–3.61. Since the potentials V_ℓ and loops ω_ℓ are continuous, the integrals like $\int_0^\beta V_\ell(\omega_\ell(\tau))\mathrm{d}\tau$ are correctly defined through their partial Riemann sums. Respectively, we can approximate each of the measures $\mu_{\Lambda,\xi}(\mathrm{d}\omega_\Lambda) := \pi_\Lambda(\mathrm{d}\omega|\xi) \circ \mathbb{P}_\Lambda^{-1}$ by

$$\mu^{(n)}_{\Lambda,\xi}(\mathrm{d}\omega_\Lambda) := \frac{1}{Z^{(n)}_{\Lambda,\xi}} \exp\left\{-H_{\Lambda,n}(\omega_\Lambda|\xi)\right\} \chi_\Lambda(\mathrm{d}\omega_\Lambda), \quad n \in \mathbb{N},$$

where

$$\begin{aligned} H_{\Lambda,n}(\omega_\Lambda|\xi) \ : \ &= \frac{\beta}{n}\sum_{k=0}^{n-1}\sum_{\ell\in\Lambda} V_\ell\left(\omega_\ell\left(\beta\frac{k}{n}\right)\right) \\ &-\frac{\beta}{2n}\sum_{k=0}^{n-1}\sum_{\ell,\ell'\in\Lambda} J_{\ell\ell'}\omega_\ell\left(\beta\frac{k}{n}\right)\omega_{\ell'}\left(\beta\frac{k}{n}\right) \\ &-\frac{\beta}{n}\sum_{k=0}^{n-1}\sum_{\ell\in\Lambda,\ \ell'\in\Lambda^c} J_{\ell\ell'}\omega_\ell\left(\beta\frac{k}{n}\right)\xi_{\ell'}\left(\beta\frac{k}{n}\right) \end{aligned} \tag{3.265}$$

and χ_Λ is the free Gaussian process introduced in (3.32). For any function $f \in C_\mathrm{b}(\mathbb{R}^N)$ and for all $\ell_j \in \Lambda$ and $\tau_j \in S_\beta$ with $1 \leq j \leq N \in \mathbb{N}$, we then have

$$\int_{\Omega_\Lambda} f\left(\omega_{\ell_1}(\tau_1), ..., \omega_{\ell_N}(\tau_N)\right)\mu^{(n)}_{\Lambda,\xi}(\mathrm{d}\omega_\Lambda)$$

$$= \lim_{n\to\infty}\int_\Omega f\left(\omega_{\ell_1}\left(\beta\frac{k_{1,n}}{n}\right), ..., \omega_{\ell_N}\left(\beta\frac{k_{N,n}}{n}\right)\right)\mu^{(n)}_{\Lambda,\xi}(\mathrm{d}\omega_\Lambda), \tag{3.266}$$

whereby $k_{j,n} \in \{0, 1, ..., n-1\}$ are chosen in such a way that

$$\beta\frac{k_{j,n}}{n} \to \tau_j, \quad \text{as } n \to \infty. \tag{3.267}$$

The convergence in (3.266) follows from Lebesgue's theorem, since by Assumptions (**V**) and ($\mathbf{J}_0$) the partial Hamiltonians $H_{\Lambda,n}(\omega_\Lambda|\xi)$ are below bounded uniformly in ω_Λ and n. By a standard cut-off argument and Fatou's lemma, the relation (3.266) extends at least to all polynomially bounded f. We next look at the joint law of $\omega_\ell\left(\beta\frac{k}{n}\right)$, $\ell \in \Lambda$, $0 \le k \le n-1$, with respect to $\mu^n_{\Lambda,\xi}$. Taking into account (3.30)–(3.32), we observe that it has the form

$$\mu^{(n)}_{\Lambda,y}(\mathrm{d}x_{\Lambda,n}) = \frac{1}{Z^{(n)}_{\Lambda,y}} \exp\Bigg\{ c_n \sum_{k=0}^{n-1}\sum_{\ell\in\Lambda} x_{\ell,k}\, x_{\ell,k+1}$$
$$+\frac{\beta}{2n}\sum_{k=0}^{n-1}\sum_{\ell,\ell'\in\Lambda} J_{\ell\ell'} x_{\ell,k}\, x_{\ell',k} + \frac{\beta}{n}\sum_{k=0}^{n-1}\sum_{\ell\in\Lambda,\ \ell'\in\Lambda^c} J_{\ell\ell'} x_{\ell,k} y_{\ell',k}$$
$$+\sum_{k=0}^{n-1}\sum_{\ell\in\Lambda}\left[-\frac{\beta}{n}V_\ell(x_{\ell,k}) + h_\ell\frac{\beta}{n}x_{\ell,k} - b_n x^2_{\ell,k}\right]\Bigg\}\,\mathrm{d}x_{\Lambda,n}, \tag{3.268}$$

where we set $y_{\ell',k} := \xi_{\ell'}(\beta k/n)$ and introduce the positive parameters

$$b_n := \sqrt{am}\coth\left(\frac{\beta}{n}\sqrt{\frac{a}{m}}\right), \quad c_n := \sqrt{am}\left[\sinh\left(\frac{\beta}{n}\sqrt{\frac{a}{m}}\right)\right]^{-1}. \tag{3.269}$$

This is a probability distribution on $x_{\Lambda,n} = (x_{\ell,k}) \in \mathbb{R}^{n|\Lambda|}$ with $x_{\ell,n} := x_{\ell,0}$, which corresponds to a *classical ferromagnet* (which, moreover, is *even* or of the *BSF* type if all V_ℓ are such). So, it fulfills all the correlation inequalities listed above. Hence, the corresponding inequalities for $\mu_{\Lambda,\xi}$ will follow from their discretized versions for $\mu^{(n)}_{\Lambda,y}$ by taking the limit $n\to\infty$. For instance, starting from the classical *GKS* inequality

$$\int_{\mathbb{R}^{n|\Lambda|}}\left(\prod_{j=1}^N f_j(x_{\ell_j,k_j})\right)\mu^{(n)}_{\Lambda,y}(\mathrm{d}x_{\Lambda,n})$$
$$= \int_{\Omega_\Lambda}\left(\prod_{j=1}^N f_j\left(\omega_{\ell_{1_j}}\left(\beta\frac{k_{j,n}}{n}\right)\right)\right)\mu_{\Lambda,\xi}(\mathrm{d}\omega_\Lambda) \ge 0,$$

by (3.266) and (3.267) we immediately get its quantum version (3.257). Concerning the *FKG* inequalities we note that it suffices to check them on the cone $K^{\mathrm{cyl}}_+(\Omega^{\mathrm{t}})$ consisting of cylinder functions (3.219), see the proof of Lemma 3.38. ■

3.3.5 Reference models and a new comparison criterion

We shall prove Theorems 3.45 and 3.46 by comparing our initial model (3.183) with *two reference models* defined as follows. Let J and V be the same as in (3.197) and (3.199) respectively. For $\Lambda \Subset \mathbb{L} := \mathbb{Z}^d$, we set (in accordance with (3.3))

$$H^{\mathrm{low}}_\Lambda := \sum_{\ell\in\Lambda}\left[H^{\mathrm{har}}_\ell + V(q_\ell)\right] - \frac{1}{2}\sum_{\ell,\ell'\in\Lambda} J\epsilon_{\ell\ell'}q_\ell q_{\ell'}, \quad q_\ell\in\mathbb{R}, \tag{3.270}$$

where $H_\ell^{\rm har}$ is given by (3.29) and $\epsilon_{\ell\ell'} = 1$ if $|\ell - \ell'| = 1$ and $\epsilon_{\ell\ell'} = 0$ otherwise. The second reference model is defined on an arbitrary $\mathbb{L}$ satisfying (2.2). For $\Lambda \Subset \mathbb{L}$, we then set

$$H_\Lambda^{\rm up} := \sum_{\ell\in\Lambda} \left[H_\ell^{\rm har} + v(q_\ell^2)\right] - \frac{1}{2}\sum_{\ell,\ell'\in\Lambda} J_{\ell\ell'} q_\ell q_{\ell'} = \sum_{\ell\in\Lambda} \tilde{H}_\ell - \frac{1}{2}\sum_{\ell,\ell'\in\Lambda} J_{\ell\ell'} q_\ell q_{\ell'}, \qquad (3.271)$$

where $\tilde{H}_\ell$ is given by (3.212) and the interaction intensities $J_{\ell\ell'}$ are the same as in (3.3). Since both these models are *particular cases* of the basic model (3.183) we consider, their sets of Euclidean Gibbs measures have the properties established by Theorems 3.18–3.22. By $\mu_\pm^{\rm low}$, $\mu_\pm^{\rm up}$ we denote the corresponding *extreme elements*.

Remark 3.62 The anharmonic potentials of both reference models have the form (3.256) with the zero external field $h_\ell = 0$ and the functions v_ℓ being *convex*. Hence, they meet the conditions of all the statements of Subsection 3.3.4. By construction, the *low*-reference model is translation invariant. The *up*-reference model is translation invariant if $\mathbb{L}$ is a lattice and $J_{\ell\ell'}$ are translation invariant.

In the statements below the comparison with the *low*-reference model relates to the case of $\mathbb{L} := \mathbb{Z}^d$.

Lemma 3.63 *For every ℓ, it follows that*

$$\mu_+^{\rm low}(\omega_\ell(0)) \le \mu_+(\omega_\ell(0)) \le \mu_+^{\rm up}(\omega_\ell(0)). \qquad (3.272)$$

Proof. By (3.224) we have that for any cofinal $\mathcal{L}$,

$$\int_\Omega \omega_\ell(\tau)\mu_\pm({\rm d}\omega) = \lim_{\mathcal{L}} \int_\Omega \omega_\ell(\tau)\pi_\Lambda({\rm d}\omega|\pm\hat{\xi}), \quad \text{for all } \tau. \qquad (3.273)$$

Thus, the proof will be done if we show that for all $\ell \in \Lambda \Subset \mathbb{L}$,

$$\pi_\Lambda^{\rm low}(\omega_\ell(0)|\hat{\xi}) \le \pi_\Lambda(\omega_\ell(0)|\hat{\xi}) \le \pi_\Lambda^{\rm up}(\omega_\ell(0)|\hat{\xi}). \qquad (3.274)$$

First we prove the left-hand inequality in (3.274). Let us introduce the following family of measures parametrized by $t, s \in [0,1]$

$$\begin{aligned} \mu_\Lambda^{(t,s)}({\rm d}\omega_\Lambda) \quad : \quad &= \frac{1}{Y(t,s)} \exp\left(\frac{1}{2}\sum_{\ell,\ell'\in\Lambda} J\epsilon_{\ell\ell'}(\omega_\ell,\omega_{\ell'})_{L^2_\beta} + \sum_{\ell\in\Lambda}(\omega_\ell,\eta_\ell^{\ell_0,s})_{L^2_\beta}\right. \\ &- \sum_{\ell\in\Lambda}\int_0^\beta V(\omega_\ell(\tau)){\rm d}\tau + \frac{s}{2}\sum_{\ell,\ell'\in\Lambda}\left[J_{\ell\ell'} - J\epsilon_{\ell\ell'}\right](\omega_\ell,\omega_{\ell'})_{L^2_\beta} \\ &\left. - t\sum_{\ell\in\Lambda}\int_0^\beta \left[V_\ell(\omega_\ell(\tau)) - V(\omega_\ell(\tau))\right]{\rm d}\tau\right)\chi_\Lambda({\rm d}\omega_\Lambda), \end{aligned} \qquad (3.275)$$

where, cf. (3.223),

$$\eta_\ell^{\ell_0,s}(\tau) := \sum_{\ell'\in\Lambda^c} J\epsilon_{\ell\ell'}\hat{\xi}_{\ell'}(\tau) + s\sum_{\ell'\in\Lambda^c}\left[J_{\ell\ell'} - J\epsilon_{\ell\ell'}\right]\hat{\xi}_{\ell'}(\tau) \geq \sum_{\ell'\in\Lambda^c} J\epsilon_{\ell\ell'}\hat{\xi}_{\ell'}(\tau) > 0, \tag{3.276}$$

which in fact is independent of τ. The partition function is given by

$$\begin{aligned} Y(t,s) = \int_{\Omega_\Lambda} \exp\Bigg(&\frac{1}{2}\sum_{\ell,\ell'\in\Lambda} J\epsilon_{\ell\ell'}(\omega_\ell,\omega_{\ell'})_{L^2_\beta} + \sum_{\ell\in\Lambda}(\omega_\ell,\eta_\ell^{\ell_0,s})_{L^2_\beta} \\ &-\sum_{\ell\in\Lambda}\int_0^\beta V(\omega_\ell(\tau))\mathrm{d}\tau + \frac{s}{2}\sum_{\ell,\ell'\in\Lambda}\left[J_{\ell\ell'} - J\epsilon_{\ell\ell'}\right](\omega_\ell,\omega_{\ell'})_{L^2_\beta} \\ &- t\sum_{\ell\in\Lambda}\int_0^\beta \left[V_\ell(\omega_\ell(\tau)) - V(\omega_\ell(\tau))\right]\mathrm{d}\tau\Bigg)\chi_\Lambda(\mathrm{d}\omega_\Lambda). \end{aligned}$$

Since the site-dependent *"external field"* (3.276) is positive, the moments of the measure (3.275) obey the *GKS* inequalities. Therefore, for any $\ell \in \Lambda$, the function

$$\phi(t,s) = \mu_\Lambda^{(t,s)}(\omega_\ell(0)), \quad t,s \in [0,1], \tag{3.277}$$

is continuous and increasing in both variables. Indeed, taking into account (3.197) and (2.196), we get

$$\begin{aligned} \frac{\partial}{\partial s}\phi(t,s) &= \sum_{\ell'\in\Lambda}\left[J_{\ell\ell'} - J\epsilon_{\ell\ell'}\right]\hat{\xi}_{\ell'}(0) \\ &\times \int_0^\beta \left\{\mu_\Lambda^{(t,s)}\left[\omega_\ell(0)\omega_{\ell'}(\tau)\right] - \mu_\Lambda^{(t,s)}\left[\omega_\ell(0)\right]\cdot\mu_\Lambda^{(t,s)}\left[\omega_{\ell'}(\tau)\right]\right\}\mathrm{d}\tau \\ &+ \frac{1}{2}\sum_{\ell_1,\ell_2\in\Lambda}\left[J_{\ell_1\ell_2} - J\epsilon_{\ell_1\ell_2}\right]\left\{\mu_\Lambda^{(t,s)}\left[\omega_\ell(0)(\omega_{\ell_1},\omega_{\ell_2})_{L^2_\beta}\right]\right. \\ &- \left.\mu_\Lambda^{(t,s)}\left[\omega_\ell(0)\right]\cdot\mu_\Lambda^{(t,s)}\left[(\omega_{\ell_1},\omega_{\ell_2})_{L^2_\beta}\right]\right\} \geq 0, \\ \frac{\partial}{\partial t}\phi(t,s) &= \sum_{\ell'\in\Lambda}\int_0^\beta\left\{\mu_\Lambda^{(t,s)}\left(\omega_\ell(0)\cdot\left[V(\omega_{\ell'}(\tau)) - V_{\ell'}(\omega_{\ell'}(\tau))\right]\right)\right. \\ &- \left.\mu_\Lambda^{(t,s)}\left[\omega_\ell(0)\right]\cdot\mu_\Lambda^{(t,s)}\left[V(\omega_{\ell'}(\tau)) - V_{\ell'}(\omega_{\ell'}(\tau))\right]\right\}\mathrm{d}\tau \geq 0. \end{aligned}$$

But by (3.275) and (3.277)

$$\phi(0,0) = \pi_\Lambda^{\mathrm{low}}(\omega_\ell(0)), \quad \phi(1,1) = \pi_\Lambda(\omega_\ell(0)),$$

which yields the left-hand inequality in (3.274). To prove the right-hand one we have to employ the family of measures (3.275) with $s = 1$, $t \in [0,1]$, and $v(x_\ell^2)$ instead of $V(x_\ell)$. Thereafter we repeat the above steps by taking into account (3.177). ■

In the next statement, whose proof follows immediately from (3.272) and Lemma 3.52, we *summarize* the properties of the reference models.

Proposition 3.64 (Comparison Criterion) *The model considered undergoes a phase transition if the* low*-reference model does so. The uniqueness of tempered Euclidean Gibbs measures of the* up*-reference model implies that* $|\mathcal{G}^{\rm t}| = 1$.

We emphasize that the above criterion is *principally new* both for the classical and for the quantum spin systems. It offers a clear possibility to *extend essentially* the classes of models which allow for the study of their critical behavior. It does not matter which methods and results are used to establish the *phase transition* in the *low*-reference model (3.270) with the one-particle potential $V^{\rm low}(q) := V(q)$ and respectively the *uniqueness* in the *up*-reference model (3.271) with the potential $V^{\rm up}(q) := v(q^2)$. The above criterion immediately implies the same properties for the whole family of potentials $V_\ell(x)$ related by

$$V(q) - V(\tilde{q}) \leq V_\ell(q) - V_\ell(\tilde{q}) \leq v(q^2) - v(\tilde{q}^2), \quad \text{for } q^2 \leq \tilde{q}^2. \tag{3.278}$$

3.3.6 Estimates for pair correlation functions

In this subsection we study in more detail the *pair correlation* functions, cf. (3.259),

$$K^\Lambda_{\ell\ell'}(\tau,\tau'|\xi) := \mathbf{Cov}_{\pi_\Lambda(\mathrm{d}\omega|\xi)}\left[(\omega_\ell(\tau);\omega_{\ell'}(\tau')\right]. \tag{3.279}$$

For $\Delta \subset \Lambda$, $\ell,\ell' \in \Lambda$, $\tau,\tau' \in [0,\beta]$, and $t \in [0,1]$, we set

$$Q^\Lambda_{\ell\ell'}(\tau,\tau'|\Delta,t) := \int_{\Omega_\Lambda} \omega_\ell(\tau)\omega_{\ell'}(\tau')\mu^{(t)}_{\Lambda,\Delta}(\mathrm{d}\omega_\Lambda), \tag{3.280}$$

where this time we have denoted

$$\mu^{(t)}_{\Lambda,\Delta}(\mathrm{d}\omega_\Lambda) := \frac{1}{Y_{\Lambda,\Delta}(t)} \exp\Bigg\{ \frac{1}{2}\sum_{\ell_1,\ell_2\in\Lambda\setminus\Delta} J_{\ell_1\ell_2}(\omega_{\ell_1},\omega_{\ell_2})_{L^2_\beta} \tag{3.281}$$
$$+t\left(\sum_{\ell_1\in\Delta}\sum_{\ell_2\in\Lambda\setminus\Delta} J_{\ell_1\ell_2}(\omega_{\ell_1},\omega_{\ell_2})_{L^2_\beta} + \frac{1}{2}\sum_{\ell_1,\ell_2\in\Delta} J_{\ell_1\ell_2}(\omega_{\ell_1},\omega_{\ell_2})_{L^2_\beta}\right)$$
$$-\sum_{\ell\in\Lambda}\int_0^\beta V_\ell(\omega_\ell(\tau))\mathrm{d}\tau\Bigg\}\chi_\Lambda(\mathrm{d}\omega_\Lambda),$$

and

$$Y_{\Lambda,\Delta}(t) := \int_{\Omega_\Lambda} \exp\Bigg\{ \frac{1}{2}\sum_{\ell_1,\ell_2\in\Lambda\setminus\Delta} J_{\ell_1\ell_2}(\omega_{\ell_1},\omega_{\ell_2})_{L^2_\beta} \tag{3.282}$$
$$+t\left(\sum_{\ell_1\in\Delta}\sum_{\ell_2\in\Lambda\setminus\Delta} J_{\ell_1\ell_2}(\omega_{\ell_1},\omega_{\ell_2})_{L^2_\beta} + \frac{1}{2}\sum_{\ell_1,\ell_2\in\Delta} J_{\ell_1\ell_2}(\omega_{\ell_1},\omega_{\ell_2})_{L^2_\beta}\right)$$
$$-\sum_{\ell\in\Lambda}\int_0^\beta V_\ell(\omega_\ell(\tau))\mathrm{d}\tau\Bigg\}\chi_\Lambda(\mathrm{d}\omega_\Lambda).$$

By literal repetition of the arguments used for proving Lemma 3.63 one establishes the following

Proposition 3.65 *The above $Q^{\Lambda}_{\ell\ell'}(\tau,\tau'|\Delta,t)$ is an increasing continuous function of $t\in[0,1]$.*

Corollary 3.66 *Let the conditions of Proposition 3.59 be satisfied. Then for any pair $\Lambda\subset\Lambda'\Subset\mathbb{L}$ the functions (3.253) obey the estimate*

$$K^{\Lambda}_{\ell\ell'}(\tau,\tau'|0)\leq K^{\Lambda'}_{\ell\ell'}(\tau,\tau'|0), \tag{3.283}$$

which holds for all $\ell,\ell'\in\Lambda$ and $\tau,\tau'\in[0,\beta]$.

Now we derive bounds for the correlation functions of the reference models for a *one-point* $\Lambda=\{\ell\}$. Denote

$$K^{\rm up}_{\ell}(\tau,\tau'):=\pi^{\rm up}_{\ell}(\omega_\ell(\tau)\omega_\ell(\tau')|0),\quad K^{\rm low}_{\ell}(\tau,\tau'):=\pi^{\rm low}_{\ell}(\omega_\ell(\tau)\omega_\ell(\tau')|0). \tag{3.284}$$

We recall that the parameter $\varDelta_m$ was defined by (3.213).

Lemma 3.67 *For every β, it holds for the thermal average*

$$K^{\rm up}_{\ell}:=\int_0^\beta K^{\rm up}_{\ell}(\tau,\tau'){\rm d}\tau\leq 1/m\varDelta_m^2. \tag{3.285}$$

Proof. In view of (3.19) the above integral is independent of τ. Furthermore, by (3.18) and (3.20), it can be written as the *Duhamel two-point function* (see [100, 102]) corresponding to the multiplication operator q_ℓ in $\mathcal{H}_\ell:=L^2(\mathbb{R},{\rm d}q_\ell)$,

$$K^{\rm up}_{\ell}=\frac{1}{\tilde{Z}_\ell}\int_0^\beta {\rm trace}\left\{q_\ell e^{-\tau\tilde{H}_\ell}q_\ell e^{-(\beta-\tau)\tilde{H}_\ell}\right\}{\rm d}\tau,\quad \tilde{Z}_\ell={\rm trace}[e^{-\beta\tilde{H}_\ell}]. \tag{3.286}$$

The Hamiltonian $\tilde{H}_\ell$ was defined in (3.212) as

$$\tilde{H}_\ell=H^{\rm har}_\ell+v(q_\ell^2)=-\frac{1}{2m}\left(\frac{\partial}{\partial q_\ell}\right)^2+\frac{a}{2}q_\ell^2+v(q_\ell^2), \tag{3.287}$$

and its spectrum $\{E_n\}_{n\in\mathbb{N}}$ determines by (3.213) the parameter $\varDelta_m$. Integrating in (3.286) we get

$$\begin{aligned} K^{\rm up}_\ell &= \frac{1}{\tilde{Z}_\ell}\sum_{n,n'\in\mathbb{N}_0,\ n\neq n'}\left|(\psi_n,q_\ell\psi_{n'})_{L^2(\mathbb{R})}\right|^2\frac{(E_n-E_{n'})(e^{-\beta E_{n'}}-e^{-\beta E_n})}{(E_n-E_{n'})^2}\\ &\leq \frac{1}{\tilde{Z}_\ell}\cdot\frac{1}{\varDelta_m^2}\sum_{n,n'\in\mathbb{N}_0}\left|(\psi_n,q_\ell\psi_{n'})_{L^2(\mathbb{R})}\right|^2(E_n-E_{n'})(e^{-\beta E_{n'}}-e^{-\beta E_n})\\ &= \frac{1}{\varDelta_m^2}\cdot\frac{1}{\tilde{Z}_\ell}{\rm trace}\left\{\left[q_\ell,\left[\tilde{H}_\ell,q_\ell\right]\right]e^{-\beta\tilde{H}_\ell}\right\}=\frac{1}{m\varDelta_m^2}, \end{aligned} \tag{3.288}$$

where ψ_n, $n \in \mathbb{N}_0 := \mathbb{N} \cup \{0\}$, are the eigenfunctions of $\tilde{H}_\ell$ and $[\cdot,\cdot]$ stands for the commutator in $\mathcal{H}_\ell$. ■

For the functions $K_\ell^{\rm low}$, a representation like (3.286) is obtained by means of the following Hamiltonian

$$\hat{H}_\ell = H_\ell^{\rm har} + V(q_\ell) = -\frac{1}{2m}\left(\frac{\partial}{\partial q_\ell}\right)^2 + \frac{a}{2}q_\ell^2 + V(q_\ell), \tag{3.289}$$

and the associated quantum Gibbs state $\hat{\varrho}_\ell$, where m and a are the same as in (3.212) but V is given by (3.199). Thereby,

$$K_\ell^{\rm low}(0,0) = {\rm trace}[q_\ell^2 \exp(-\beta \hat{H}_\ell)]/{\rm trace}[\exp(-\beta \hat{H}_\ell)] := \hat{\varrho}_\ell(q_\ell^2). \tag{3.290}$$

Lemma 3.68 *Let t_* be the solution of (3.201), then $K_\ell^{\rm low}(0,0) \geq t_*$.*

Proof. We use the *Bogoliubov inequality* (see e.g. [63, 251, 258])

$$\beta\rho_\Lambda\{A^2\} \cdot \rho_\Lambda\{[B,[H_\Lambda,B]]\} \geq |\rho_\Lambda\{[B,A]\}|^2, \tag{3.291}$$

which holds for any admissible pair of self-adjoint operators A, B acting in the physical Hilbert space $\mathcal{H}_\Lambda := L^2(\mathbb{R}^{|\Lambda|} \to \mathbb{C})$. Here $[\cdot,\cdot]$ stands for commutator, H_Λ is a local Hamiltonian in $\mathcal{H}_\Lambda$, and the corresponding Gibbs state ρ_Λ is defined by (3.13). In our context this inequality will be applied to $\Lambda := \{\ell\}$, $H_\Lambda := \hat{H}_\ell$ and A, B both equal to the momentum operator $p_\ell = -\sqrt{-1}(\partial/\partial q_\ell)$. After calculations (3.291) becomes simply

$$\hat{\varrho}_\ell\left\{[p_\ell,[\hat{H}_\ell,p_\ell]]\right\} = \hat{\varrho}_\ell\left\{V_\ell''(q_\ell)\right\} + a \geq 0, \tag{3.292}$$

which further by (3.21), (3.199), and (3.200) yields

$$\begin{aligned} & a + 2b^{(1)} + \sum_{\rm s=2}^{\rm p} 2{\rm s}(2{\rm s}-1)b^{({\rm s})}\hat{\varrho}\left[q_\ell^{2({\rm s}-1)}\right] \\ &\quad = a + 2b^{(1)} + \sum_{\rm s=2}^{\rm p} 2{\rm s}(2{\rm s}-1)b^{({\rm s})}\pi_\ell^{\rm low}\left(\omega_\ell^{2({\rm s}-1)}(0)\middle|\, 0\right) \geq 0. \end{aligned}$$

Now we use the Gaussian domination inequality (3.263) and obtain $K_\ell^{\rm low}(0,0) = \pi_\ell^{\rm low}\left(\omega_\ell^2(0)|\,0\right) \geq t_*$. ■

In conclusion let us point out some links with the *analytical approach* to the Euclidean path measures which was systematically developed in the joint papers [10]–[13] and will be briefly described in Subsection 4.5.3. Let $\hat{\rho}_\Lambda$ be the quantum Gibbs state related to the local Hamiltonian $H_\Lambda^{\rm low}$, cf. (3.270). We remark that the following estimate generalizing (3.292)

$$\hat{\rho}_\Lambda\left\{V''(\omega_\ell)\right\} = \int_\Omega V''(\omega_\ell(\tau))\pi_\Lambda({\rm d}\omega|0) \geq -a, \qquad \forall \ell \in \Lambda \Subset \mathbb{L}, \tag{3.293}$$

can be easily derived from the *integration by parts* formula for the measure $\pi_\Lambda^{\mathrm{low}}(\mathrm{d}\omega|0)$, cf. Proposition 4.58. Let us chose the constant direction $\xi \in \Omega$, such that $\xi_{\ell'}(\tau) \equiv 1$ if $\ell' = \ell$ and $\xi_{\ell'}(\tau) \equiv 0$ otherwise. Then the measure $\pi_\Lambda^{\mathrm{low}}(\mathrm{d}\omega|0)$ is differentiable along this ξ with the partial logarithmic derivative, cf. (4.301),

$$b_\xi(\omega) := \frac{\mathrm{d}}{\mathrm{d}\theta}\left[\frac{\mathrm{d}\pi_\Lambda^{\mathrm{low}}(\omega - \theta\xi|0)}{\mathrm{d}\pi_\Lambda^{\mathrm{low}}(\omega|0)}\right]_{\theta=0} = -\int_0^\beta \partial_\ell H_\Lambda^{\mathrm{low}}(\omega_\Lambda(\tau))\mathrm{d}\tau, \tag{3.294}$$

where

$$\partial_\ell H_\Lambda^{\mathrm{low}}(q_\Lambda) := V'(q_\ell) + aq_\ell - J\sum_{\ell' \in \Lambda} \epsilon_{\ell\ell'} q_{\ell'}.$$

Then, by integrating by parts (for all rigorous details see e.g. [13]) we get

$$\beta\int_\Omega (V'' + a)(\omega_\ell(\tau))\pi_\Lambda^{\mathrm{low}}(\mathrm{d}\omega|0) = \int_0^\beta \int_\Omega (V'' + a)(\omega_\ell(\tau))\pi_\Lambda^{\mathrm{low}}(\mathrm{d}\omega|0)\mathrm{d}\tau$$
$$= \frac{\mathrm{d}}{\mathrm{d}\theta}\left[\int_\Omega \partial_\ell H_\Lambda^{\mathrm{low}}((\omega + \theta\xi)(\tau))\pi_\Lambda^{\mathrm{low}}(\mathrm{d}\omega|0)\right]_{t=0} = \int_\Omega \left[b_\xi(\omega)\right]^2 \pi_\Lambda^{\mathrm{low}}(\mathrm{d}\omega|0) \geq 0,$$

which yields the required relation (3.293).

3.3.7 Periodic states and phase transitions

In our language, the model described by the Hamiltonian (3.1), (3.2) undergoes a *phase transition* if $|\mathcal{G}^{\mathrm{t}}| > 1$ at certain values of the interaction parameters and the temperature β^{-1}. The strategy of proving non-uniqueness of $\mu \in \mathcal{G}^{\mathrm{t}}$ for the scalar ferromagnetic model (3.183) can be divided into *two steps*. First, we apply Proposition 3.64 and compare the initial model with a certain (*low*-) reference model, cf. (3.270). The latter model is translation invariant and possesses some specific *symmetry properties*, which makes its investigation much easier. Below we shall perform the remaining second step and establish the phase transitions in the reference model, which will complete the proof of Theorem 3.45.

(i) Periodic Euclidean Gibbs states

To this end we shall crucially employ the *translation invariance* and *reflection positivity* of the *low*-reference model. With this connection we construct its *periodic* Euclidean Gibbs states $\mu^{\mathrm{per}} \in \mathcal{G}^{\mathrm{t}}$, which are always translation invariant (cf. Subsection 4.3 in [122] for a general framework). An idea beyond is to show that there exist some μ^{per}, which is however *non-ergodic* with respect to the group $\mathbb{L}_0$ of translations of the lattice. Hence, it could not be a pure phase, that implies a non-uniqueness (see Theorem 14.15 in [122] and Subsection 2.3.5 (ii)).

Consider any cubic box in $\mathbb{L} := \mathbb{Z}^d$ of the form

$$\Lambda = (-L/2, L/2]^d \bigcap \mathbb{L}, \quad L \in \mathbb{N}, \tag{3.295}$$

and let $\mathfrak{T}(\Lambda) \cong \mathbb{L} / L\mathbb{L}$ be the *torus* obtained by identifying its opposite walls. The distance on $\mathfrak{T}(\Lambda)$ is given by

$$|\ell - \ell'|^2_{\mathsf{T}(\Lambda)} := \sum_{j=1}^{d} \left[\min\left\{|\ell_j - \ell'_j|\,;\, L - |\ell_j - \ell'_j|\right\}\right]^2.$$

Next, we can define the Λ-periodic modification of the interaction (cf. (3.270))

$$I^{\rm per}_\Lambda(\omega_\Lambda) := -\frac{J}{2} \sum_{\ell,\ell' \in \Lambda} \epsilon^\Lambda_{\ell\ell'} (\omega_\ell, \omega_{\ell'})_{L^2_\beta} + \sum_{\ell\in\Lambda} \int_0^\beta V\left(\omega_\ell(\tau)\right) {\rm d}\tau, \tag{3.296}$$

where $\epsilon^\Lambda_{\ell\ell'} = 1$ if $|\ell - \ell'|_{\mathsf{T}(\Lambda)} = 1$ and $\epsilon^\Lambda_{\ell\ell'} = 0$ otherwise. Clearly, $I^{\rm per}_\Lambda$ is invariant with respect to the translations of the torus $\mathfrak{T}(\Lambda)$. The energy functional $I^{\rm per}_\Lambda$ corresponds to the following Hamiltonian

$$H^{\rm per}_\Lambda := \sum_{\ell\in\Lambda} \left[H^{\rm har}_\ell + V(q_\ell)\right] - \frac{J}{2} \sum_{\ell,\ell'\in\Lambda} \epsilon^\Lambda_{\ell\ell'} q_\ell q_{\ell'}, \tag{3.297}$$

in the same sense as I_Λ given by (3.51) corresponds to H_Λ given by (3.3). Now we introduce the associated Λ-periodic Euclidean kernels (cf. (3.78))

$$\pi^{\rm per}_\Lambda(B) := \frac{1}{Z^{\rm per}_\Lambda} \int_{\Omega_\Lambda} \exp\left\{-I^{\rm per}_\Lambda(\omega_\Lambda)\right\} \mathbf{1}_B(\omega_\Lambda \times 0_{\Lambda^c}) \chi_\Lambda({\rm d}\omega_\Lambda), \quad B \in \mathcal{B}(\Omega), \tag{3.298}$$

where

$$Z^{\rm per}_\Lambda := \int_{\Omega_\Lambda} \exp\left\{-I^{\rm per}_\Lambda(\omega_\Lambda)\right\} \chi_\Lambda({\rm d}\omega_\Lambda).$$

Thereby, for every box Λ, the above $\pi^{\rm per}_\Lambda$ is a probability measure on $\Omega^{\rm t}$. By $\mathcal{L}_{\rm per}$ we denote the sequence of boxes (3.295) indexed by $L \in \mathbb{N}$. Since the pair interaction $J\epsilon^\Lambda_{\ell\ell'}$ is of nearest neighbor type, there is a consistency relation

$$\int_\Omega \int_\Omega f(\omega_\Lambda) \pi_\Lambda({\rm d}\omega_\Lambda|\xi) \pi^{\rm per}_{\Lambda'}({\rm d}\xi) = \int_\Omega f(\omega) \pi^{\rm per}_{\Lambda'}({\rm d}\omega), \tag{3.299}$$

valid for any local function $f \in C_{\rm b}(\Omega_\Lambda)$ and all $\Lambda, \Lambda' \in \mathcal{L}_{\rm per}$ such that $\Lambda^+ := \{\ell' \in \mathbb{L} \mid {\rm dist}(\ell', \Lambda) \leq 1\} \subseteq \Lambda'$. For a given $\alpha \in \mathcal{I}$, let us choose some $\kappa > 0$ such that the estimate (3.122) holds.

Lemma 3.69 *For every box Λ, $\alpha \in \mathcal{I}$, and $\sigma \in (0, 1/2)$, the measure $\pi^{\rm per}_\Lambda$ obeys the estimate*

$$\int_\Omega \|\omega\|^2_{\alpha,\sigma} \pi^{\rm per}_\Lambda({\rm d}\omega) \leq C_{3.300}. \tag{3.300}$$

Thereby, the sequence $\{\pi^{\rm per}_\Lambda \mid \Lambda \in \mathcal{L}_{\rm per}\}$ is $\mathcal{W}^{\rm t}$-relatively compact.

Proof. For a fixed $\ell \in \Lambda$ such that the Euclidean distance $\mathrm{dist}(\ell, \Lambda^c) > 1$, we set $\Delta_\ell = \mathbb{L} \setminus \{\ell\}$. Then let $\mu_{\Lambda,\ell}$ be the projection of $\pi_\Lambda^{\mathrm{per}}$ onto $\mathcal{B}(\Omega_{\Delta_\ell})$. For $\xi \in \Omega$, let $\nu_\ell(\cdot|\xi)$ be the following probability measure on the single spin space $\Omega_\ell := C_\beta$

$$\nu_\ell(\mathrm{d}\omega_\ell|\xi) := \frac{1}{N_\ell(\xi)} \exp\left\{ J \sum_{\ell'} \epsilon_{\ell\ell'} (\omega_\ell, \xi_{\ell'})_{L^2_\beta} - \int_0^\beta V(\omega_\ell(\tau))\mathrm{d}\tau \right\} \chi(\mathrm{d}\omega_\ell). \tag{3.301}$$

Then, disintegrating $\pi_\Lambda^{\mathrm{per}}$ through the *DLR* equation (3.80), we get

$$\pi_\Lambda^{\mathrm{per}}(\mathrm{d}\omega) = \nu_\ell(\mathrm{d}\omega_\ell|\omega_{\Delta_\ell})\mu_{\Lambda,\ell}(\mathrm{d}\omega_{\Delta_\ell}). \tag{3.302}$$

Like in Lemma 3.26 and Corollary 3.27 one proves that the measure $\nu_\ell(\cdot|\xi)$ obeys

$$\int_{C_\beta} \exp\left\{ \lambda_\sigma |\omega_\ell|^2_{C^\sigma_\beta} + \kappa |\omega_\ell|^2_{L^2_\beta} \right\} \nu_\ell(\mathrm{d}\omega_\ell|\omega_{\Delta_\ell}) \leq \exp\left\{ \Upsilon + J \sum_{\ell'} \epsilon_{\ell\ell'} |\omega_{\ell'}|^2_{L^2_\beta} \right\},$$

where $\lambda_\sigma$and κ are as in (3.109), (3.115). Now we integrate both sides of this inequality with respect to $\mu_{\Lambda,\ell}$ and get similarly to (3.121), (3.122) that

$$n_\ell^{\mathrm{per}}(\Lambda) := \log\left\{ \int_\Omega \exp[\lambda_\sigma |\omega_\ell|^2_{C^\sigma_\beta} + \kappa |\omega_\ell|^2_{L^2_\beta}] \pi_\Lambda^{\mathrm{per}}(\mathrm{d}\omega) \right\} \leq \Upsilon_{3.118}. \tag{3.303}$$

By the periodicity property of $\pi_\Lambda^{\mathrm{per}}(\mathrm{d}\omega)$, the bound (3.303) is actually valid for all $\ell \in \Lambda$. Then the estimate (3.300) is obtained in the same way as (3.125) was proven. The tightness of $\{\pi_\Lambda^{\mathrm{per}} \,|\, \Lambda \in \mathcal{L}_{\mathrm{per}}\}$ in Ω_α follows from (3.300) and the compactness of the embeddings $\Omega_{\alpha,\sigma} \hookrightarrow \Omega_{\alpha'}$, $\alpha < \alpha'$. The $\mathcal{W}^{\mathrm{t}}$-compactness of this family is a consequence of Lemma 3.30. ■

Lemma 3.70 *Every $\mathcal{W}^{\mathrm{t}}$-accumulation point μ^{per} of the sequence $\{\pi_\Lambda^{\mathrm{per}} \,|\, \Lambda \in \mathcal{L}_{\mathrm{per}}\}$ is a Euclidean Gibbs measure of the* low-*reference model.*

Proof. Let $\mathcal{L} \subset \mathcal{L}_{\mathrm{per}}$ be the subsequence along which $\{\pi_\Lambda^{\mathrm{per}}\}_{\Lambda \in \mathcal{L}}$ converges to $\mu^{\mathrm{per}} \in \mathcal{P}(\Omega^{\mathrm{t}})$ in the topology $\mathcal{W}^{\mathrm{t}}$. Employing the Feller property (Lemma 3.12) we can pass to the limit along this $\mathcal{L}$ in the both sides of (3.299), where we consider all possible choices of the function $f \in C_{\mathrm{b}}(\Omega_\Lambda)$. Since such local functions constitute a measure determining class, we conclude that the limit point μ^{per} satisfies the *DLR* equation (3.91), see Remark 2.8. Note that the desired $\mathbb{L}_0$-invariance of μ^{per} follows from the invariance of any $\pi_\Lambda^{\mathrm{per}}$ with respect to all translations of the torus $\mathfrak{T}(\Lambda)$. ■

(ii) Infrared estimates and the proof of Theorem 3.45

For translation invariant lattice models, phase transitions are established by means of the *infrared estimates*, see [38, 37, 100, 140, 162, 228]. Here we use a version of the technique developed in those papers and the corresponding correlation inequalities which allow us to compare the model considered with its translation invariant version (3.270).

In view of Propositions 3.52 it is needed to show that

$$\mu_+^{\rm low}(\omega_\ell(0)) > 0, \tag{3.304}$$

provided the conditions of Theorem 3.45 are satisfied. Given a box Λ, we introduce the parameter

$$P_\Lambda(\beta) = \int_\Omega \left(\frac{1}{\beta|\Lambda|} \sum_{\ell\in\Lambda} \int_0^\beta \omega_\ell(\tau) {\rm d}\tau \right)^2 \pi_\Lambda^{\rm per}({\rm d}\omega). \tag{3.305}$$

Let $\ell \in \Lambda$ be such that ${\rm dist}(\ell, \Lambda^c) > 1$. Then by Corollary 3.66 and Lemma 3.68 we get

$$\int_\Omega [\omega_\ell(0)]^2 \pi_\Lambda^{\rm per}({\rm d}\omega) \geq K_\ell^{\rm low}(0,0) \geq t_*. \tag{3.306}$$

Thus, by the *Bruch-Falk inequality* (see Theorem VI.7.5, page 392 of [258] or Theorem 3.1 in [102]) it holds

$$\int_\Omega \left(\frac{1}{\beta} \int_0^\beta \omega_\ell(\tau) {\rm d}\tau \right)^2 \pi_\Lambda^{\rm per}({\rm d}\omega) \geq t_* f(\beta/4mt_*). \tag{3.307}$$

Recall that t_* is the unique solution of the equation (3.206), whereas the function f was defined in (3.209). By periodicity argument, both (3.306) and (3.307) indeed are valid for all $\ell \in \Lambda$ and $\Lambda \in \mathcal{L}_{\rm per}$. The infrared estimates, based on the *reflection positivity* of the *low*-reference model, then standardly lead to the following bound, see [100, 102, 107, 258],

$$P_\Lambda(\beta) \geq t_* f(\beta/4mt_*) - \frac{1}{2\beta J|\Lambda|} \sum_{p\in\Lambda_*\setminus\{0\}} \frac{1}{E(p)}, \tag{3.308}$$

where $E(p)$ is given by (3.205) and the sum runs over the dual lattice

$$\Lambda_* := \left\{ p = (p^j)_{j=1}^d \;\middle|\; p^j := \pi s_j/L, \quad -L \leq s_j \leq L, \;\; 1 \leq j \leq d \right\}.$$

Note that for $d \geq 3$

$$|\Lambda|^{-1} \sum_{p\in\Lambda_*\setminus\{0\}} \frac{1}{E(p)} \to \theta_d, \quad \text{as } \; |\Lambda| \to \infty. \tag{3.309}$$

Analyzing (3.308) by means of (3.207)–(3.209) and (3.309), we conclude that

$$P_\Lambda(\beta) \geq t_* f(\beta/4mt_*) - \theta_d/2\beta J > 0, \tag{3.310}$$

for all $\beta > \beta_*$ and large enough boxes Λ. This yields the positivity of the *long-range parameter*

$$P(\beta) := \limsup_{\mathcal{L}_{\rm per}} P_\Lambda(\beta) > 0 \tag{3.311}$$

and indicates the non-ergodicity of a certain limit point $\mu^{\rm per} := \lim_{\mathcal{L}_{\rm per}} \pi_\Lambda^{\rm per} \in \mathcal{G}^{\rm t}$. Finally, by means of the *Griffiths theorem* (see [102], Theorem 1.1 and the Corollaries) one can prove that

$$\mu^{\rm per}(\omega_\ell(0)) \geq \sqrt{P(\beta)}. \tag{3.312}$$

Therefore, the estimate (3.304) holds if the right-hand side of (3.312) is positive, which can be ensured by taking $\beta > \beta_*$. ■

Remark 3.71 As was mentioned in Introduction, there are two general methods for proving phase transitions (i.e., non-uniqueness of $\mu \in \mathcal{G}_t$) at low temperatures β^{-1}, namely: (i) the *reflection positivity* (*for* $d \geq 3$); and (ii) the *Peierls-type argument* (*for* $d \geq 2$) as a part of the *Pirogov-Sinai contour method.* The first method we also used above, enables us to show the positivity of a long-range order parameter $P(\beta)$, for big enough $m > m_*$ and $\beta > \beta_*(m_*)$, via the *infrared (Gaussian)* upper bounds on two-point correlation functions $\pi_\Lambda^{\rm per}\left(\beta^{-2}\int_0^\beta \omega_\ell(\tau){\rm d}\tau \cdot \int_0^\beta \omega_{\ell'}(\tau'){\rm d}\tau'\right)$. Note that a majority of the papers here dealt with the $P(\varphi)$-models, see e.g. [100, 102, 118, 129, 228]. More general classes of potentials were treated only in the *"semi-classical"* asymptotical regime $\beta \to \infty$ (see [38, 37]) or at the zero temperature $\beta = \infty$ (see [140]). The second method, which is a quantum modification of the Peierls argument, was first implemented in [129, 130] to the $(\varphi^4)_2$-*model* of Euclidean field theory and then in [14, 100, 116, 265] to its lattice approximation. One defines a *"collective spin variable"* $\sigma_\ell := \text{sign} \int_0^\beta \omega_\ell(\tau)d\tau$ taking values ± 1 and a long-range parameter $\Pi(\beta) := \limsup_{\mathcal{L}_{\rm per}} \pi_\Lambda^{\rm per}(\sigma_\ell \sigma_{\ell'})$. Then, the occurrence of phase transition would follow from the estimate $\Pi(\beta) > 1/2$ valid for large enough values of m and β. We emphasize that our comparison criterion, cf. Proposition 3.64, immediately allows to *extend the previously known results* by taking as a *low-reference* model any concrete model of even ferromagnets investigated in the above papers.

3.3.8 Uniqueness due to quantum effects

In this subsection we establish the strongest uniqueness result for the ferromagnetic system (3.183), Theorem 3.46, which reveals the influence of *quantum effects* and displays the mass in the uniqueness condition. The proof will combine the classical ideas of [42, 184, 266] based on the use of the *FKG*, *GKS*, and *Lebowitz correlation inequalities* with the *spectral analysis* of the single-particle oscillators (3.287) specific for the quantum case.

First we make *precise* the parameter ι participating in the condition (3.60) of Assumption $(\mathbf{J}_\alpha)$. Our aim is to have the relation

$$||\mathbf{J}||_0 < ||\mathbf{J}||_\alpha < m\varDelta_m^2, \tag{3.313}$$

where $\varDelta_m > 0$ was defined by (3.213). So, in what follows we set $\iota := m\varDelta_m^2 - ||\mathbf{J}||_0 > 0$ and fix the corresponding $\alpha \in \mathcal{I}$. Recall that in Example 3.7 we has analyzed how to check (3.313) in some typical situations.

Now let us turn to the proof of Theorem 3.46. By Proposition 3.64 it is enough to show the uniqueness for the *up*-reference model, which in turn by Proposition 3.52 is equivalent to

$$\mu_+^{\rm up}(\omega_\ell(0)) = 0, \quad \text{for all } \ \beta > 0 \ \text{ and } \ \ell \in \mathbb{L}. \tag{3.314}$$

Given $\Lambda \Subset \mathbb{L}$, we introduce a symmetric matrix $\mathbf{K}^\Lambda := (K_{\ell\ell'}^\Lambda \geq 0)_{\ell,\ell'\in\mathbb{L}}$ with the entries

$$K_{\ell\ell'}^\Lambda := \int_0^\beta K_{\ell\ell'}^{\rm up,\Lambda}(\tau,\tau'){\rm d}\tau' = \int_0^\beta \pi_\Lambda^{\rm up}\left[\omega_\ell(\tau)\omega_{\ell'}(\tau')\,|0\right]{\rm d}\tau'. \tag{3.315}$$

By (3.19) the above integral is independent of τ. Furthermore, we set

$$K_{\ell\ell'}^{\rm up} := \lim_{\Lambda\nearrow\mathbb{L}} K_{\ell\ell'}^\Lambda \leq +\infty, \tag{3.316}$$

since by Corollary 3.66 the correlations $K_{\ell\ell'}^\Lambda(\tau,\tau')$ are growing as $\Lambda \nearrow \mathbb{L}$

Lemma 3.72 *If (3.214) is satisfied, then there exists $\alpha \in \mathcal{I}$ such that the matrix $\mathbf{K} := (K_{\ell\ell'}^{\rm up})_{\ell,\ell'\in\mathbb{L}}$ defines a bounded operator in the Banach space $l^\infty(w_\alpha)$.*

Proof. The proof will be based on a generalization of the method used in [8] for proving Lemma 4.7. For $t \in [0,1]$, let the family of measures $\mu_\Lambda^{(t)} \in \mathcal{P}(\Omega_\Lambda)$ be defined by

$$\mu_\Lambda^{(t)}({\rm d}\omega_\Lambda) := \frac{1}{Y_\Lambda(t)} \exp\left\{ \frac{t}{2} \sum_{\ell_1,\ell_2\in\Lambda} J_{\ell_1\ell_2}(\omega_{\ell_1},\omega_{\ell_2})_{L^2_\beta} \right. \tag{3.317}$$

$$\left. - \sum_{\ell\in\Lambda} \int_0^\beta v([\omega_\ell(\tau)]^2){\rm d}\tau \right\} \chi_\Lambda({\rm d}\omega_\Lambda),$$

$$Y_\Lambda(t) := \int_{\Omega_\Lambda} \exp\left\{ \frac{t}{2} \sum_{\ell_1,\ell_2\in\Lambda} J_{\ell_1\ell_2}(\omega_{\ell_1},\omega_{\ell_2})_{L^2_\beta} \right. \tag{3.318}$$

$$\left. - \sum_{\ell\in\Lambda} \int_0^\beta v([\omega_\ell(\tau)]^2){\rm d}\tau \right\} \chi_\Lambda({\rm d}\omega_\Lambda),$$

where v is the same as in (3.210), (3.212). Then by (3.271)

$$\mu_\Lambda^{(0)} = \prod_{\ell\in\Lambda} \pi_\ell^{\rm up}(\cdot|0), \quad \mu_\Lambda^{(1)} = \pi_\Lambda^{\rm up}(\cdot|0), \quad \text{for any } \ \Lambda \Subset \mathbb{L}. \tag{3.319}$$

The corresponding Duhamel two-point functions are given by

$$K_{\ell\ell'}^\Lambda(t) := \int_0^\beta \mu_\Lambda^{(t)}\left[\omega_\ell(\tau)\omega_{\ell'}(\tau')\right]{\rm d}\tau', \quad t \in [0,1], \quad \ell,\, \ell' \in \mathbb{L}. \tag{3.320}$$

One can show that for every fixed ℓ, ℓ', the above $K_{\ell\ell'}^\Lambda(t)$ is differentiable on the interval $t \in (0,1)$ and continuous at its endpoints, where (see (3.285))

$$K_{\ell\ell'}^\Lambda(0) = \delta_{\ell\ell'} K_{\ell'}^{\rm up} \leq \delta_{\ell\ell'}/m\varDelta_m^2, \quad K_{\ell\ell'}^\Lambda(1) = K_{\ell\ell'}^\Lambda, \tag{3.321}$$

and $\delta_{\ell\ell'}$ is the Kronecker delta. Computing the derivative in (3.320) we get

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t}K^{\Lambda}_{\ell\ell'}(t) &= \frac{1}{2}\sum_{\ell_1,\ell_2} J_{\ell_1\ell_2}\int_0^{\beta}\int_0^{\beta} U^{\Lambda}_{\ell\ell'\ell_1\ell_2}(t;\tau,\tau',\tau_1,\tau_1)\mathrm{d}\tau'\mathrm{d}\tau_1 \\ &+ \sum_{\ell_1,\ell_2} K^{\Lambda}_{\ell\ell_1}(t)J_{\ell_1\ell_2}K^{\Lambda}_{\ell_2\ell'}(t). \end{aligned} \tag{3.322}$$

Here $U^{\Lambda}_{\ell\ell'\ell_1\ell_2}(t;\tau,\tau',\tau_1,\tau_1)$ is the *Ursell function* of the measure $\mu^{(t)}_{\Lambda}$, which obeys the estimate (3.264) since the function v is convex. Except for the trivial case $J_{\ell\ell'} \equiv 0$, the first term in (3.322) is strictly negative, which implies

$$\frac{\mathrm{d}}{\mathrm{d}t}K^{\Lambda}_{\ell\ell'}(t) \leq \sum_{\ell_1,\ell_2} K^{\Lambda}_{\ell\ell_1}(t)J_{\ell_1\ell_2}K^{\Lambda}_{\ell_2\ell'}(t), \quad t \in (0,1), \quad \ell,\, \ell' \in \mathbb{L}. \tag{3.323}$$

Let us consider the following Cauchy problem

$$\frac{\mathrm{d}}{\mathrm{d}t}L_{\ell\ell'}(t) = \sum_{\ell_1,\ell_2} L^{\Lambda}_{\ell\ell_1}(t)J_{\ell_1\ell_2}L^{\Lambda}_{\ell_2\ell'}(t), \qquad t \in (0,1], \quad \ell,\, \ell' \in \mathbb{L}, \tag{3.324}$$

subject to the initial condition

$$L_{\ell\ell'}(0) = \lambda\delta_{\ell\ell'}, \quad L_{\ell\ell'}(0) \geq \delta_{\ell\ell'}/m\varDelta_m^2 = K^{\Lambda}_{\ell\ell'}(0) \tag{3.325}$$

where $\lambda \in (1/m\varDelta_m^2, 1/||\mathbf{J}||_{\alpha})$ and $\alpha \in \mathcal{I}$ is chosen from the relation (3.313). For such α, one can uniquely solve the problem (3.324), (3.325) in the space $l^{\infty}(w_{\alpha})$ (see Remark 2.1) and obtain

$$\mathbf{L}(t) = \lambda\left(\mathbf{I} - t\lambda\mathbf{J}\right)^{-1}, \quad ||\mathbf{L}(t)||_{l^{\infty}(w_{\alpha})} \leq \lambda\left(1 - t\lambda||\mathbf{J}||_{\alpha}\right)^{-1}, \tag{3.326}$$

where $\mathbf{I}$ is the identity operator. Note that the operator $\mathbf{L}(t) = (L_{\ell\ell'}(t))_{\mathbb{L}\times\mathbb{L}}$ is represented through the Neumann series $\sum_{n=0}^{\infty} t^n\lambda^{n+1}\mathbf{J}^{n+1}$and thus its matrix elements $L_{\ell\ell'}(t)$ are nonnegative. It remains to compare (3.323) and (3.324) taking into account (3.321) and (3.325). Using Theorem V, page 65 of [282], we conclude that

$$K^{\mathrm{up}}_{\ell\ell'} := \sup_{\Lambda \Subset \mathbb{L}} K^{\Lambda}_{\ell\ell'} \leq L_{\ell\ell'}(1) := \lambda\left[(\mathbf{I} - \lambda\mathbf{J})^{-1}\right]_{\ell\ell'}, \quad \ell,\, \ell' \in \mathbb{L}, \tag{3.327}$$

which in view of (3.326) yields the proof. ■

Proof of Theorem 3.46: For $\ell \in \Lambda \Subset \mathbb{L}$ and $t \in [0,1]$, we set

$$M^{\Lambda}_{\ell}(t) = \int_{\Omega} \omega_{\ell}(0)\pi^{\mathrm{up}}_{\Lambda}(\mathrm{d}\omega|t\xi), \tag{3.328}$$

where the constant boundary condition $\xi := \hat{\xi}$ is the same as in (3.223). The function $M^{\Lambda}_{\ell}(t)$ is obviously differentiable on the interval $t \in (0,1)$ and continuous at its endpoints. Thus, by the mean-value theorem

$$0 \leq M^{\Lambda}_{\ell}(1) \leq \sup_{t\in[0,1]} \frac{\mathrm{d}M^{\Lambda}_{\ell}}{\mathrm{d}t}(t). \tag{3.329}$$

The derivative is

$$\frac{\mathrm{d}M_\ell^\Lambda}{\mathrm{d}t}(t) = \sum_{\ell_1\in\Lambda,\ \ell_2\in\Lambda^c} J_{\ell\ell_1}\xi_{\ell_2}\int_0^\beta K^\Lambda_{\ell_1\ell_2}(0,\tau|t\xi)\mathrm{d}\tau, \quad t\in(0,1), \tag{3.330}$$

where, see the notation (3.259),

$$K^\Lambda_{\ell_1\ell_2}(0,\tau|t\xi) := \mathbf{Cov}_{\pi^{\mathrm{up}}_\Lambda(\mathrm{d}\omega|t\xi)}\left[\omega_{\ell_1}(0);\omega_{\ell_2}(\tau)\right]$$

and the *"external field"* $\xi_{\ell'} := [b\log(1+|\ell'-\ell_0|)]^{1/2}$ is positive at each site $\ell'\in\mathbb{L}$. Thus, we may use the correlation bound (3.261) and obtain

$$\frac{\mathrm{d}M_\ell^\Lambda}{\mathrm{d}t}(t) \leq \sum_{\ell_1\in\Lambda,\ \ell_2\in\Lambda^c} J_{\ell\ell_1}K^\Lambda_{\ell_1\ell_2}\xi_{\ell_2} := (\mathbf{K}^\Lambda\mathbf{J}\xi_{\Lambda^c})_\ell, \quad \forall t\in(0,1). \tag{3.331}$$

Herefrom, employing Lemma 3.72 and the estimate (3.326) in particular, we conclude that

$$\frac{\mathrm{d}M_\ell^\Lambda}{\mathrm{d}t}(t) \leq ||\mathbf{K}||_\alpha||\mathbf{J}||_\alpha||\xi_{\Lambda^c}||_{l^1(w_\alpha)}, \tag{3.332}$$

where by the condition (3.58) in Assumption $(\mathbf{J}_\alpha)$ one has $\xi\in l^1(w_\alpha)$ with any $\alpha\in\mathcal{I}$. Thus, the right-hand side of (3.332) tends to zero as $\Lambda\nearrow\mathbb{L}$, which by (3.273), (3.328), and (3.329) finally yields (3.314). ■

Lemma 3.72 and the correlation bound (3.261) immediately imply the *uniform decay* of the corresponding Duhamel functions.

Corollary 3.73 *Under assumptions of Theorem 3.46, it holds for any* $\lambda\in(1/m\varDelta_m^2, 1/||\mathbf{J}||_\alpha)$ *and all* $\ell,\ell'\in\mathbb{L}$, $\tau\in S_\beta$,

$$\sup_{0\leq\xi\in\Omega^{\mathrm{t}}}\sup_{\Lambda\Subset\mathbb{L}}\int_0^\beta \pi^{\mathrm{up}}_\Lambda\left[\omega_\ell(\tau)\omega_\ell(\tau')\,|\xi\right]\mathrm{d}\tau' \leq w_\alpha(\ell,\ell')\frac{\lambda||\mathbf{J}||_\alpha}{1-\lambda||\mathbf{J}||_\alpha}. \tag{3.333}$$

Remark 3.74 (i) Actually, the key estimate (3.327) can be looked upon as a *correlation inequality* relating the pair correlations $K^\Lambda_{\ell\ell'}$ of the measures $\pi^{\mathrm{up}}_\Lambda(\cdot|0)$ with the quantities $L_{\ell\ell'}(1)$, which in turn can be identified with the correlation functions of a certain *Gaussian model.* For the *classical* even ferromagnets, this elegant comparison argument was first suggested by A. Sokal in [266]. In the latter models it is well known (see e.g. [42, 184]) that the exponential decay of the pair correlations implies zero spontaneous magnetization (i.e., $\mathbf{E}_{\mu_\pm}x_\ell = \lim_{\Lambda\nearrow\mathbb{L}}\int_\Omega x_\ell\pi_\Lambda(\mathrm{d}\omega|\pm y) = 0$), which in turn by the theorem of J. Lebowitz and A. Martin-Löf (its quantum analog is Proposition 3.52) yields the uniqueness of $\mu\in\mathcal{G}^{\mathrm{t}}_{\mathrm{cl}}$.

(ii) It is instructive to compare Theorem 3.46 with the corresponding results relying on *Dobrushin's uniqueness criterion*, see Subsection 2.3.4. To this end, let us consider a classical ferromagnet on $\mathbb{L} := \mathbb{Z}^d$ with the nearest-neighbor interaction $W_{\ell\ell'}(x_\ell,x_{\ell'}) := J|x_\ell-x_{\ell'}|^2/2\geq 0$ and the self-interaction $V(x_\ell) := v(x_\ell^2)$ of the *BSF* type (see (3.210)).

Analyzing (3.321), (3.325), and (3.327), we conclude that the statement of Lemma 3.72 will be implied by the following *mean-field condition* (cf. Equations (20), (21) in [266])

$$\sup_{\ell} \int_{\Omega} x_\ell^2 \mathrm{d}\pi_\ell(x|0) < (2\beta d J)^{-1}. \tag{3.334}$$

The moments in (3.334) can be estimated by means of the one-point Poincaré inequality (2.191) which leads to the uniqueness condition (2.195), previously obtained by Dobrushin's criterion and discussed in Remark 2.35. Alternatively, to estimate (3.334) one may use the integration by parts method (cf. Subsection 2.4.1) yielding the moment bound (2.244).

(iii) In [105, 214] dealing with the quantum $P(\varphi)$-models, the convergence of *cluster expansions* for small masses m (independently of the boundary condition) has been proved *uniformly* for all values of the temperature including the ground state case $\beta = \infty$. However, as is typical for unbounded spins, such convergence of cluster expansions does not yet imply the *DLR* uniqueness.

3.3.9 Uniqueness at nonzero external field

In statistical mechanics phase transitions are traditionally associated with non-analyticity of thermodynamic characteristics considered as functions of the external field $h \in \mathbb{R}$. In special cases one can oversee at which values of h this non-analiticity can occur. The *Lee-Yang theorem* states that the only such value is $h = 0$; hence no phase transitions can occur at nonzero h. In the theory of classical lattice models these arguments were first employed e.g. in [181, 183, 184, 253]. We refer also to Sections 4.5, 4.6 in [129] and Sections IX.3 – IX.5 in [255], where further applications to quantum field theory are discussed.

In the case of lattice models with the single spin space $\mathbb{R}$ the validity of the Lee-Yang theorem depends on the properties of the anharmonic potentials. For the polynomials $V(x) = x^4 + ax^2$, $a \in \mathbb{R}$, the Lee-Yang theorem holds, see e.g. Theorem IX.15 on page 342 in [255]. But no other examples of this kind were known, see the discussion on page 71 in [129]. Below we give a sufficient condition for the potentials V to have the corresponding property and discuss some examples. Here we use the family $\mathcal{F}_{\text{Laguerre}}$ defined by (3.217). We also prove a number of lemmas, which allow us to apply the arguments based on the Lee-Yang theorem to our quantum model and hence to prove Theorem 3.48.

Recall that the elements of $\mathcal{F}_{\text{Laguerre}}$ can be continued to entire functions $\varphi : \mathbb{C} \to \mathbb{C}$, which have no zeros outside of $(-\infty, 0]$.

Definition 3.75 *A probability measure ν on the real line is said to have the* ***Lee-Yang property*** *if there exists $\varphi \in \mathcal{F}_{\text{Laguerre}}$ such that*

$$\int_{\mathbb{R}} \exp(hx)\, \nu(\mathrm{d}x) = \varphi(h^2), \quad \forall h \in \mathbb{R}.$$

In [168] the following fact was proven.

Proposition 3.76 *Let the function $u : \mathbb{R} \to \mathbb{R}$ be such that for a certain $b \geq 0$, its derivative obeys the condition $b + u' \in \mathcal{F}_{\mathrm{Laguerre}}$. Then the probability measure*

$$\nu(\mathrm{d}x) := C \exp[-u(x^2)]\mathrm{d}x, \tag{3.335}$$

has the Lee-Yang property.

Set

$$f(h^2) := \int_{\mathbb{R}^n} \exp\left[h\sum_{i=1}^{n} x_i + \sum_{i,j=1}^{n} C_{ij}x_ix_j\right]\prod_{i=1}^{n}\nu(\mathrm{d}x_i), \quad h \in \mathbb{R}. \tag{3.336}$$

By Theorem 3.2 of [190], we have the following

Proposition 3.77 *If in (3.336) $C_{ij} \geq 0$ for all $1 \leq i, j \leq n$ and the measure ν is as in Proposition 3.76, then the function f, if exists, belongs to $\mathcal{F}_{\mathrm{Laguerre}}$. It certainly exists if u' is not constant.*

Now let the one-particle potential be of the form $V(x) := v(x^2) - hx$, cf. (3.218). Recall that $p_\Lambda(h)$ stands for the pressure (3.191) with $\xi = 0$, which by Corollary 3.55 is an even function of h. Define

$$\varphi_\Lambda(h^2) := p_\Lambda(h), \quad h \in \mathbb{R}. \tag{3.337}$$

Lemma 3.78 *If V obeys the conditions of Theorem 3.48, the function* $\exp(|\Lambda|\varphi_\Lambda)$ *belongs to $\mathcal{F}_{\mathrm{Laguerre}}$.*

Proof. With the help of the lattice discretization technique described in Subsection 3.3.4, the function

$$\begin{aligned}\exp\left(|\Lambda|\varphi_\Lambda\right)(h^2) \;&=\; \int_{\Omega_\Lambda} \exp\left\{h\sum_{\ell\in\Lambda}\int_0^\beta \omega_\ell \mathrm{d}\tau + \frac{1}{2}\sum_{\ell,\ell'\in\Lambda} J_{\ell\ell'}(\omega_\ell, \omega_{\ell'})_{L^2_\beta}\right.\\ &\left. -\sum_{\ell\in\Lambda}\int_0^\beta V(\omega_\ell)\mathrm{d}\tau\right\}\chi_\Lambda(\mathrm{d}\omega_\Lambda)\end{aligned}$$

may be approximated by $f^{(N)}(h^2)$, $N \in \mathbb{N}$, having the form (3.336) with non-negative coefficients $C_{ij}^{(N)}$. In doing so, see (3.268), the reference measures $\nu^{(N)}$ can be written in the form (3.335) with functions $u^{(N)}(t) = v^{(N)}(t) + a^{(N)}t/2$ as required in Proposition 3.76. The coefficients $a^{(N)} > 0$ can be exactly calculated from (3.269) and are *growing* to infinity as $N \to \infty$. For every $h \in \mathbb{R}$ we have $f_N(h^2) \to \exp(|\Lambda|\varphi_\Lambda(h^2))$ as $N \to \infty$. The entire functions f_N are ridge, with the ridge being $[0, \infty)$. For sequences of such functions, their pointwise convergence on the ridge implies via the *Vitali theorem* (see e.g. Proposition VIII.19 in [255]) the uniform convergence on compact subsets of $\mathbb{C}$, which yields the property stated (for more details, see [170, 171]). ■

Proof of Theorem 3.48. By Lemma 3.78, for every $\Lambda \Subset \mathbb{L}$, the pressure $p_\Lambda(h)$ can be extended to a function of $h \in \mathbb{C}$, holomorphic in the right and left open half-planes. By standard arguments (see e.g. Lemma 39, page 34 of [170], and Lemma 3.53) it follows that the limit of such extensions $p(h)$ is holomorphic in certain subsets of those half-planes containing the real line, except possibly for the point $h = 0$. Therefore, $p(h)$ is differentiable at each $h \neq 0$, which by Corollary 3.44 yields the result. ■

Remark 3.79 (i) Theorem 3.48 together with Proposition 3.76 give a *sufficient condition*, the lack of which was mentioned on page 71 of [129]. Turning to the classical analog (with $m \to +\infty$ and $a = 0$) of the model described by Theorem 3.48 and going through the above proofs, we conclude that $|\mathcal{G}^{\rm t}_{\rm cl}| = 1$ at all $h \neq 0$ provided

$$b + v' \in \mathcal{F}_{\rm Laguerre}, \quad for\ some\ b > 0. \tag{3.338}$$

In particular, the function $v(t) := t^3 + b_2 t^2 + b_1 t$ obeys (3.338) if and only if $b_2 \geq 0$ and $b_1 \leq b_2^2/3$. Thus we certainly have $|\mathcal{G}^{\rm t}_{\rm cl}| = 1$ if $b_2 > 0$ and $b_1 \leq 0$. On the other hand, the example of a polynomial given in [129], for which the corresponding classical model undergoes phase transitions at nonzero h, in our notations is $v(t) = t^3 - 2t^2 + (\alpha + 1)t$ with $\alpha > 0$. It certainly does not meet the assumption (3.338).

(ii) In the quantum case the class of potentials V satisfying conditions of Theorem 3.48 is more restrictive and has been *completely* described by C. M. Newman. By Theorem 2 in [219], $b_0 + v' \in \mathcal{F}_{\rm Laguerre}$ for all $b \geq b_0 > 0$ if and only if

$$v(t) := \left(b_2 t^2 + b_1 t\right) - \Psi(t), \quad t \geq 0, \tag{3.339}$$

$$\text{with} \quad \Psi(t) := \log\left\{ t^{-n} \prod_j \left[(1 + t/a_j^2)^{-1} \exp\left(t/a_j^2\right)\right]\right\}, \tag{3.340}$$

where $n \in \mathbb{N} \cup \{0\}$, $a_j \neq 0$, $\sum_j a_j^{-4} < \infty$, and $b_2 > 0$, $b_1 \in \mathbb{R}$ (or else $b_2 = 0$ and $b_1 + \sum_j a_j^{-2} > 0$). The product in (3.340) may be taken over an empty, finite, or infinite subset of $j \in \mathbb{N}$. This means that all admitted potentials $V(x) := v(x^2)$ can be represented as the following "*perturbation*" of the φ^4-polynomials

$$V(x) := \left(b_2 x^4 + b_1 x^2\right) - \Psi(x^2). \tag{3.341}$$

(iii) After this section was written, Yu. Kozitsky communicated me that there exists an alternative approach to the uniqueness problem at nonzero external fields which was discovered for the Ising model by C. J. Preston in [232]. This aproach seems to be much simpler and is based on the *GHS* inequalities valid for the even ferromagnets with the self-potentials V_ℓ belonging to the *EMN* class, cf. (3.199). Its extension to the quantum case would employ the properties of the pressure functional (3.240) established above together with the uniform moment estimates (3.97), (3.119). A report on this issue, which to be published as an addendum to our joint paper [174], is in preparation.

Chapter 4

Stochastic Dynamics on Graphs

4.1 Interacting spin systems on graphs

A graph is the most general mathematical description of a set of elements connected by some kind of pairwise relation. The fast growing area of research in physics is concerned with applications of graph theory to modelling of different complex systems and inhomogeneous structures (e.g. communication networks, statistical models of algorithms, polymers, biomoleculas, disordered materials, etc.). Of prime interest here is the question of how the *geometry* of an underlying graph can influence the physical properties observed in the models. For background material on infinite graphs, especially in the context of their applications in statistical mechanics, see e.g. [2, 61, 64, 122, 147, 154, 198, 284].

The subject of our last Chapter 4 can be generally characterized as *"statistical-mechanics type Markov processes on graphs"*. We return to the classical spin systems, but now the particles will be attached to the vertices $v \in \mathbb{V}$ of an infinite *graph* $\mathbb{G}(\mathbb{V}, \mathbb{E})$, instead of a lattice $\mathbb{L}$ as in Chapters 2 and 3. The fact of interaction between the particles marked by $v, v' \in \mathbb{V}$ means that the corresponding vertices are joined by the edge $e = [v, v'] \in \mathbb{E}$. Given interaction potentials V_v, $W_{vv'}$, we then can define the local specification $\Pi := \{\pi_\Lambda\}_{\Lambda \Subset \mathbb{V}}$ and the associated Gibbs measures $\mu \in \mathcal{G}$ as *Markov fields* on $\mathbb{V}$. However, the major difference will concern the aims of research. While the preceding chapters were focused on static properties of Gibbs measures, here the emphasis is shifted towards dynamical questions. The key object will be the *Glauber dynamics*, as a model of stochastic evolution (actual or in computer simulations) of the underlying physical system towards its thermal equilibrium. Our main result will state the *pointwise exponential relaxation* of the Glauber dynamics to the unique invariant (i.e., Gibbs) measure, provided the strength of interaction is small enough.

In this introductory section we shall develop a standard *DLR* framework for the Gibbs measures on graphs. In Subsection 4.1.1 we introduce a reasonable class of infinite graphs, which can be used in statistical mechanics as indexing sets for the interacting particle systems. Note that a principal restriction imposed on the graph $\mathbb{G}(\mathbb{V}, \mathbb{E})$ is that it has *uniformly bounded degree*. Hypotheses on the interaction potentials are listed in Subsection 4.1.2, they ought to guarantee a dynamical stability our system and hence are much stronger that those in the preceding chapters. In Subsection 4.1.3 we define the set of tempered Gibbs measures $\mu \in \mathcal{G}^{\mathrm{t}}$ and adapt to them the basic results of Chapter 2 establishing the existence and à-priori estimates.

4.1.1 Geometry of the graph

Here we turn back to the setup already mentioned in Subsections 2.2.5, 3.2.6. The spin systems of our present interest are living on some *infinite graph* $\mathbb{G}(\mathbb{V}, \mathbb{E})$, which consists of a countable set of *vertices* (or *nodes*) $v \in \mathbb{V}$ and a set of unordered *edges* (or *bonds*) $e = [v, v'] \in \mathbb{E}$. This graph is *simple*, that means without loops, isolated vertices, and multiple edges. It is endowed with the *combinatorical distance* $\rho(v, v')$, which is the length of the shortest path γ connecting the vertices $v, v' \in \mathbb{V}$. We write $v \sim v'$ for the adjacent vertices (or *nearest neighbors*) with the unit distance $\rho(v, v') = 1$. For each vertex v, we define its *vicinity* $\partial v := \{v' \in \mathbb{V} \mid \rho(v, v') = 1\}$ and the *degree* $m_v := |\partial v|$. In the subsequent, we shall restrict ourselves to the graphs having the *uniformly bounded degree* (or *valence*)

$$m_{\mathbb{G}} := \sup_{v \in \mathbb{V}} m_v < \infty. \tag{4.1}$$

Obviously, for every such graph one finds a nonnegative $\delta_0 \leq \log m_{\mathbb{G}}$ such that, for all $\delta > \delta_0$ and each initial point $o \in \mathbb{V}$,

$$\sum_{v} \exp\{-\delta\rho(\nu, o)\} < \infty. \tag{4.2}$$

Furthermore, we shall require the following *uniform* version of (4.2):

Assumption $(\mathbf{G}_\delta)$ *There exists* $\delta_{\mathbb{G}} \geq 0$ *such that for all* $\delta > \delta_{\mathbb{G}}$

$$\Xi_\delta := \sup_{o \in \mathbb{V}} \sum_{v} \exp\{-\delta\rho(\nu, o)\} < \infty. \tag{4.3}$$

For certain purposes (starting from Subsection 4.2.2) we shall need to strengthen this hypothesis:

Assumption $(\mathbf{G}_0)$ *Condition (4.3) holds with* $\delta_{\mathbb{G}} = 0$.

A large class of graphs, including the lattice $\mathbb{Z}^d$, which meets the latter assumption is suggested by the following:

Lemma 4.1 *Assumption* $(\mathbf{G}_0)$ *is fulfilled by all graphs* $\mathbb{G}$ *possessing the so-called* ***doubling property****, which means the following bound*

$$\limsup_{r \geq 1} |B_{2r}(o)|/|B_r(o)| =: C_{\mathbb{G}} < \infty \tag{4.4}$$

on the number of vertices in balls

$$B_r(o) := \{v \in \mathbb{V} \mid \rho(o, v) \leq r\}, \quad r \in \mathbb{N}, \ o \in \mathbb{V}. \tag{4.5}$$

Proof. Let us denote $S_r(o) := \{v \in \mathbb{V} \mid \rho(o,v) = r\} \subseteq B_r(o)$. Then, (4.3) follows by a simple calculation

$$\begin{aligned}\sum_v \exp\{-\delta\rho(\nu,o)\} &= 1 + \sum_{N\in\mathbb{N}} \exp\{-\delta N\} \cdot |S_N(o)| \\ &\leq 1 + (m_{\mathbb{G}})^r \sum_{N\in\mathbb{N}} \exp\{-\delta N\} \cdot N^{\log_r C_{\mathbb{G}}} < \infty, \quad \forall \delta > 0.\end{aligned}$$

■

As well known, the class (4.4) contains all graphs $\mathbb{G}(\mathbb{V},\mathbb{E})$ of *polynomial growth*, which are characterized by

$$\textbf{Assumption } (\mathbf{G}_d) \qquad \sup_{r\in\mathbb{N},\ o\in\mathbb{V}} \left\{\frac{1}{r^d}|B_r(o)|\right\} =: d_{\mathbb{G}} < \infty. \tag{4.6}$$

The number $d_{\mathbb{G}} \in (0,\infty)$ is called the *upper dimension* of the graph.

For the reasons explained in [64], the amenable graphs satisfying (4.1), (4.6), and (4.9) are often called *"physical"* graphs.

The geometric properties (4.5), (4.6) will be relevant later in connection with the ergodicity result of Theorem 4.9.

In general, we shall try to keep the system of notation adopted in the preceding chapters. As before, $|\Lambda|$ stands for the cardinality and Λ^c for the complement of a set $\Lambda \subseteq \mathbb{V}$; for shorthand we write $\Lambda \Subset \mathbb{V}$ if $1 \leq |\Lambda| < \infty$. The distance between two sets $\Lambda, \Lambda' \subseteq \mathbb{V}$ is defined by $\mathrm{dist}(\Lambda,\Lambda') := \inf_{v\in\Lambda,\ v'\in\Lambda'} \rho(v,v')$. There is an elementary relation valid for all $v, v' \in \mathbb{V}$

$$\rho(v,v') - \mathrm{diam}(v' \cup \Lambda) \leq \mathrm{dist}(v,\Lambda) \leq \rho(v,v') + \mathrm{dist}(v',\Lambda), \tag{4.7}$$

where $\mathrm{diam}(\Lambda) := \sup_{v,v'\in\Lambda} \rho(v,v')$ and $\mathrm{dist}(v,\Lambda) := \mathrm{dist}(\{v\},\Lambda)$. By

$$\begin{aligned}\partial^-\Lambda := \{v' \in \Lambda \mid \mathrm{dist}(v';\Lambda^c) = 1\}, \quad \partial^+\Lambda := \{v' \in \Lambda^c \mid \mathrm{dist}(v';\Lambda) = 1\}, \\ \Lambda^- := \Lambda\backslash\partial^-\Lambda = \{v' \in \Lambda \mid \partial v \subset \Lambda\}, \quad \Lambda^+ := \Lambda \cup \partial^+\Lambda,\end{aligned} \tag{4.8}$$

we denote respectively the vertex *boundaries* (or *surfaces*) of the set $\Lambda \subseteq \mathbb{V}$ and its *interior* and *closure*. The graph $\mathbb{G}(\mathbb{V},\mathbb{E})$ is called *amenable* (respectively *nonamenable*) if the vertex-isoperimetric constant

$$\iota_{\mathbb{G}} := \inf_{\Lambda\Subset\mathbb{V}} \left\{\frac{|\partial^+\Lambda|}{|\Lambda|}\right\} = 0 \quad (\text{respectively} \quad \iota_{\mathbb{G}} > 0). \tag{4.9}$$

4.1.2 Assumptions on the interaction potentials

The *configuration space* $\Omega := [\mathbb{R}^\nu]^{\mathbb{V}}$ now consists of all sequences $x = (x_v)_{v\in\mathbb{V}}$, their components $x_v := (x_v^i)_{i=1}^\nu \in \mathbb{R}^\nu$ are called *spins*. The potential energy of the configuration $x \in \Omega$ is given by a *formal Hamiltonian*

$$H(x) = \sum_v V_v(x_v) + \frac{1}{2}\sum_{v\sim v'} W_{vv'}(x_v, x_{v'}), \tag{4.10}$$

where the sums are running over all $v \in \mathbb{V}$ and ordered pairs $(v, v') \in \mathbb{V}^2$ with $\rho(v, v') = 1$. The interaction potentials are *twice continuously differentiable* functions

$$V_v \in C^2(\mathbb{R}^\nu \to \mathbb{R}), \quad W_{vv'} \in C^2(\mathbb{R}^{2\nu} \to \mathbb{R})$$

satisfying the following list of hypotheses:

Assumption $(\mathbf{W}^*)$ *There exist* $C_W, D_W, J \geq 0$ *such that for all* $v \sim v'$ *and* $x_v, x_{v'} \in \mathbb{R}^\nu$

$$|W_{vv'}(x_v, x_{v'})| \leq \frac{1}{2} J(C_W + |x_v|^2 + |x_{v'}|^2), \tag{4.11}$$

$$\left| \frac{\partial}{\partial x_v} W_{vv'}(x_v, x_{v'}) \right| \leq J(D_W + |x_v|^2 + |x_{v'}|^2)^{1/2}, \tag{4.12}$$

$$\left| \frac{\partial^2}{\partial x_v^2} W_{vv'}(x_v, x_{v'}) \right|_{\mathcal{L}(\mathbb{R}^\nu)}, \quad \left| \frac{\partial^2}{\partial x_v \partial x_{v'}} W_{vv'}(x_v, x_{v'}) \right|_{\mathcal{L}(\mathbb{R}^\nu)} \leq J, \tag{4.13}$$

where in (4.12) and (4.13) we consider respectively the Euclidean norm in $\mathbb{R}^\nu$ *and the operator norm in* $\mathcal{L}(\mathbb{R}^\nu)$.

Assumption $(\mathbf{V}^*)$ *(i) There exist constants* $P \geq 2$, $A_V > \frac{3}{2} m_{\mathbb{G}} J$, $B_V \in \mathbb{R}$, *and* $C_V > 0$, *such that uniformly for all* $v \in \mathbb{V}$ *and* $x_v \in \mathbb{R}^\nu$

$$A_V |x_v|^2 + B_V \leq V_v(x_v) \leq C_V \left(1 + |x_v|^P\right), \tag{4.14}$$

$$|V_v'(x_v)| \leq C_V \left(1 + |x_v|^{P-1}\right). \tag{4.15}$$

(ii) For any $\vartheta > 0$ *there exist* $K_\vartheta, L_\vartheta \geq 0$, *such that*

$$\vartheta \Delta V_v(x_v) \leq |V_v'(x_v)|^2 + K_\vartheta |x_v|^2 + L_\vartheta. \tag{4.16}$$

(ii) Furthermore, each of V_v *can be written in the form*

$$V_v = U_v + Q_v, \quad V_v,\ Q_v \in C^2(\mathbb{R}^\nu \to \mathbb{R}), \tag{4.17}$$

where, respectively, U_v *is strictly convex and* Q_v *is globally bounded together with its derivatives. This decomposition is uniform in the following sense: there exist* $a_U > 3 m_{\mathbb{G}} J$ *and* $\delta_Q, \epsilon', \epsilon'' \geq 0$, *such that for all* $v \in \mathbb{V}$ *and* $x_v \in \mathbb{R}^\nu$

$$U_v''(x_v) \geq a_U \cdot \mathbf{Id}_\nu, \quad \mathbf{Osc}\ Q_\ell \leq \delta_Q, \tag{4.18}$$

$$|U_v'(0)| + |Q_v'(x_v)| \leq \epsilon', \quad |Q_v''(x_v)|_{\mathcal{L}(\mathbb{R}^\nu)} \leq \epsilon''. \tag{4.19}$$

Remark 4.2 Obviously, (4.17)–(4.19) imply the *coercivity* and *semi-monotonicity* properties (cf. Assumptions $(\mathbf{A}_6)$ and $(\mathbf{A}_7)$ respectively in Subsections 2.4.1 and 2.4.2)

$$(V_v'(x_v), x_v) \geq A_6 |x_v|^2 - B_6, \tag{4.20}$$

$$(V_v'(x_v) - V_v'(\tilde{x}_v), x_v - \tilde{x}_v) \geq \left\{ \begin{array}{l} A_7 |x_v - \tilde{x}_v|^2 - B_7 \\ A_8 |x_v - \tilde{x}_v|^2 \end{array} \right., \tag{4.21}$$

valid with any $A_6 = A_7 \in (3 m_{\mathbb{G}} J, a_U)$, $A_8 := a_U - \epsilon_Q'' \in \mathbb{R}$, and $B_6 = B_7 := \epsilon'(A_6 - a_U)^{-1} \geq 0$. Without loss of generality, we may assume that $A_V \geq a_U/2$.

The *stability properties* of the system (both in the *thermodynamical* and in the *stochastic* sense) will be described by a couple of parameters

$$\Delta_V := A_V - \frac{3}{2} m_{\mathbb{G}} J, \quad \Delta_U := \frac{1}{2}(a_U - 3m_{\mathbb{G}} J), \quad \Delta_V \geq \Delta_U > 0. \tag{4.22}$$

As usual, we suppose that the pair potentials $W_{vv'}$ vanish at the diagonal and are invariant with respect to permutations of the coordinates v, v' and variables $x_v, x_{v'}$.

4.1.3 DLR framework

(i) Tempered configurations and measures

Given $\Lambda \subseteq \mathbb{V}$, we set

$$\Omega_\Lambda := \{x_\Lambda = (x_v)_{v \in \Lambda} \mid x_v \in \mathbb{R}^\nu\}, \quad \Omega := \Omega_{\mathbb{V}}. \tag{4.23}$$

Each Ω_Λ is a Polish space endowed with the product topology and with the corresponding Borel σ-algebra $\mathcal{B}(\Omega_\Lambda)$. By $\mathcal{P}(\Omega)$ and $\mathcal{P}(\Omega_\Lambda)$ we denote the set of all probability measures on $(\Omega, \mathcal{B}(\Omega))$ and $(\Omega_\Lambda, \mathcal{B}(\Omega_\Lambda))$ respectively. The *projections* $\mathbb{P}_\Lambda \mu \in \mathcal{P}(\Omega_\Lambda)$ of a measure $\mu \in \mathcal{P}(\Omega)$ under the mapping $\Omega \ni x \mapsto \mathbb{P}_\Lambda x := x_\Lambda \in \Omega_\Lambda$ are defined by

$$(\mathbb{P}_\Lambda \mu)[B] := \mu[\mathbb{P}_\Lambda^{-1}(B)], \qquad B \in \mathcal{B}(\Omega_\Lambda). \tag{4.24}$$

Next, we introduce the scale of weighted Hilbert spaces

$$\Omega_\delta := \left\{ x \in \Omega \,\middle|\, ||x||_{o,\delta} := \left[\sum_v |x_v|^2 \exp\{-\delta \rho(\nu, o)\} \right]^{1/2} < \infty \right\}, \quad \delta > 0. \tag{4.25}$$

A special role will be played by the *tangent* Hilbert space (corresponding to $\delta = 0$) of square summable sequences over $\mathbb{V}$

$$\Omega_0 := l^2(\mathbb{V} \to \mathbb{R}^\nu) := \left\{ x \in \Omega \,\middle|\, ||x||_{l^2} := \left[\sum_v |x_v|^2 \right]^{1/2} < \infty \right\} \tag{4.26}$$

with the natural orthonormal basis

$$\left\{ h_{(v,i)} \mid v \in \mathbb{V},\, 1 \leq i \leq \nu \right\} \subset \Omega_{\mathrm{fin}}, \quad h_{(v,i)} := (\delta_{vv'} \delta_{ii'})_{v' \in \mathbb{V},\, 1 \leq i' \leq \nu}, \tag{4.27}$$

where $\delta_{vv'}$ and $\delta_{ii'}$ are Kronecker's delta. Obviously, the norms $||x||_{o,\delta}$ and $||x||_{o',\delta}$ are equivalent for different o, o'. Assumption $(\mathbf{G}_\delta)$ can be reformulated as

$$||\mathbf{1}||^2_{o,\delta} \leq \mathbf{\Xi}_\delta < \infty. \quad \text{for any } \delta > \delta_{\mathbb{G}}.$$

Another important observation is that the embeddings $\Omega_\delta \hookrightarrow \Omega_{\delta'}$ are *compact* whenever $\delta' > \delta$. Now, we define the subset of (*exponentially*) *tempered configurations*

$$\Omega^{\mathrm{t}} := \bigcap_{\delta > \delta_{\mathbb{G}}} \Omega_\delta \tag{4.28}$$

and, respectively, the subset of *tempered measures*

$$\mathcal{P}(\Omega^{\mathrm{t}}) := \bigcap_{\delta > \delta_{\mathbb{G}}} \mathcal{P}(\Omega_\delta) = \left\{ \mu \in \mathcal{P}(\Omega) \;\middle|\; \mu(\Omega^{\mathrm{t}}) = 1 \right\}. \tag{4.29}$$

The above Ω^{t} is a Polish space with the *projective limit topology* generated by the system of norms $||x||_{o,\delta}$, $\delta > \delta_{\mathbb{G}}$. The set $\mathcal{P}(\Omega)$ (as well as its subsets $\mathcal{P}(\Omega^{\mathrm{t}})$, $\mathcal{P}(\Omega_{o,\delta})$) will be endowed with the corresponding *weak* topology $\mathcal{W}$ (respectively $\mathcal{W}^{\mathrm{t}}$, $\mathcal{W}_\delta$), which is standardly defined by means of all bounded continuous functions. With these topologies the sets $\mathcal{P}(\Omega)$, $\mathcal{P}(\Omega^{\mathrm{t}})$, and $\mathcal{P}(\Omega_{o,\delta})$ become Polish spaces.

(ii) Local specification and Gibbs states

Fixed an inverse temperature $\beta > 0$, we define the *local specification* $\Pi := \{\pi_\Lambda\}_{\Lambda \Subset \mathbb{V}}$ as a family of probability kernels

$$\mathcal{B}(\Omega) \times \Omega \ni (B, y) \mapsto \pi_\Lambda(B|y) \in [0,1], \tag{4.30}$$
$$\pi_\Lambda(B|y) := Z_\Lambda^{-1}(y) \int_{\Omega_\Lambda} \exp\{-\beta H_\Lambda(x_\Lambda|y)\} \mathbf{1}_B(x_\Lambda \times y_{\Lambda^c}) \times_{v\in\Lambda} \mathrm{d}x_v,$$

where

$$H_\Lambda(x_\Lambda|y) := \sum_{v\in\Lambda} V_v(x_v) + \sum_{v\in\Lambda,\ v'\in\Lambda\cap\partial v} W_{vv'}(x_v, x_{v'}) + \sum_{v\in\Lambda,\ v'\in\Lambda^c\cap\partial v} W_{vv'}(x_v, y_{v'}) \tag{4.31}$$

is the interaction in volume $\Lambda \Subset \mathbb{V}$ under the boundary condition $y \in \Omega$. Their *finite volume* projections under the mappings $\mathbb{P}_\Lambda : \Omega \ni x \mapsto \mathbb{P}_\Lambda x := x_\Lambda \in \Omega_\Lambda$ are given by

$$\mu_{\Lambda,y}(\mathrm{d}x_\Lambda) := \mathbb{P}_\Lambda \pi_\Lambda(\mathrm{d}x|y) \in \mathcal{P}(\Omega_\Lambda). \tag{4.32}$$

From the above assumptions it follows the exponential integrability for any positive $\kappa < \Delta_V + m_{\mathbb{G}} J$,

$$\int_\Omega \exp\left\{\beta\kappa \sum_{v\in\Lambda} |x_v|^2\right\} \pi_\Lambda(\mathrm{d}x|y) < \infty. \tag{4.33}$$

A measure $\mu \in \mathcal{P}(\Omega)$ is called a *Gibbs measure* (or *state*) for the local specification (4.30) if it satisfies the *DLR* equilibrium equation $\pi_\Lambda \mu = \mu$, $\Lambda \Subset \mathbb{V}$, cf. Definition 2.4. From here on we shall be concerned with the subset of tempered Gibbs measures

$$\mathcal{G}^{\mathrm{t}} := \mathcal{G} \cap \mathcal{P}(\Omega^{\mathrm{t}}) = \{\mu \in \mathcal{G} \mid \forall \delta > \delta_{\mathbb{G}} :\ \mu(\Omega_\delta) = 1\}. \tag{4.34}$$

We summarize the preceding discussion (see Chapter 2) on the existence and à-priori estimates for the Gibbs measures in the following

Proposition 4.3 *The set of tempered Gibbs measures is not empty, i.e.,* $\mathcal{G}^{\mathrm{t}} \neq \varnothing$. *In particular, it contains each* $\mathcal{W}$*-accumulation point* $\mu \in \mathcal{P}(\Omega^{\mathrm{t}})$ *of the family* $\pi_\Lambda(\mathrm{d}\omega|y)$,

$\Lambda \Subset \mathbb{V}$, $y \in \Omega$. *For every* $\kappa < \Delta_V$ *there exists a positive constant* $C_{4.35} := C_{4.35}(\beta, k)$ *such that, uniformly for all* $\mu \in \mathcal{G}^{\mathrm{t}}$ *and boundary conditions* $y \in \Omega^{\mathrm{t}}$,

$$\begin{aligned} & \sup_v \int_\Omega \exp\left\{\beta\kappa |x_v|^2\right\} \mu(\mathrm{d}x) \qquad (4.35) \\ \leq \; & \sup_v \limsup_{\Lambda \nearrow \mathbb{V}} \int_\Omega \exp\left\{\beta\kappa |x_v|^2\right\} \pi_\Lambda(\mathrm{d}x|y) := \mathcal{C}_{4.35}. \end{aligned}$$

4.2 Ergodicity of the Glauber dynamics

This section, as well as the subsequent ones 4.3, 4.4, will be mainly devoted to the proof of the ergodicity Theorem 4.9. A unique solution to the infinite system of *SDE*'s describing the Glauber dynamics will be constructed in Subsection 4.2.2. After preliminary definitions, the *ergodicity result* itself will be precisely stated in Subsection 4.2.3. There we also shall write down its formal proof, which involves three technically different steps. The realization of *Step I* includes a proper approximation of the infinite volume solution $x(t, y) \in \Omega^{\mathrm{t}}$ by the solutions $x^\Lambda(t, y) \in \Omega^{\mathrm{t}}$ of the *cut-off* problems. Precise estimate on the L^p-convergence of such approximations, which are based on the so-called *finite propagation* property for locally interacting diffusions, will be established in Subsection 4.2.4. In Subsection 4.2.5 we consider the set $\mathcal{I}^{\mathrm{t}}$ of all tempered invariant measures and obtain *à-priori moment bounds* on its elements. Finally, in Subsection 4.2.6 we briefly discuss a possible generalization of the stochastic dynamics method to the Euclidean Gibbs states.

4.2.1 Outline of the main result

Analysis of the ergodic properties is the most difficult task in studying stochastic evolutions in infinite dimensions. According to a common knowledge, there are missing efficient criteria for the ergodicity in general situations, except a few nice cases when the stochastic system is dissipative or the corresponding transition semigroup is strong Feller. In this section we shall prove our central result, Theorem 4.9, about the *pointwise ergodicity* of the *nonequilbrium Glauber dynamics* associated with the interacting spin system (4.10). We show that, starting from any initial value $y \in \Omega^{\mathrm{t}}$, the dynamics will *converge exponentially* in the *Wasserstein metric* to the Gibbs measure $\mu \in \mathcal{G}^{\mathrm{t}}$ which thus has to be unique. The result is valid under the assumption of *weak dependence*, which typically holds when the strength of the interaction is small or the temperature is high enough. Furthermore, we give *computable bounds* on the critical values of these parameters and on the speed of the relaxation. This seems to be the first explicit statement about the ergodicity of the infinite system of interacting diffusions. To prove the result we shall combine different probabilistic and analytical tools such as the Lyapunov function method, log-Sobolev and Talagrand's inequalities, and Dobrushin's contraction technique. The supporting material on these topics are contained in the subsequent sections. However, it should be recognized that such approach is essentially limited to the systems of a *gradient type* and presumes the existence of the Gibbs (i.e., *symmetrizing*) distribution.

The history of the problem can be summarized as follows. The equivalence between the mixing properties of the Gibbs measures and the exponential convergence of the associated Glauber dynamics is well understood in compact spin spaces (cf. [202]–[204], [269, 270]). For lattice systems of unbounded spins, the investigation of this problem was initiated by B. Zegarlinski in [295]. Using semigroup methods, he proved that the *uniform log-Sobolev* inequality (**ULS**) implies the exponential convergence with the rate C_{ULS} of a certain (properly constructed) Glauber dynamics in the infinite volume. By means of the stochastic calculus this result was refined by N. Yoshida [290], who showed that (**ULS**) yields the uniqueness of $\mu \in \mathcal{G}^{\mathrm{t}}$ and the convergence of *all* transition probabilities $\mu_{t,y}$, $y \in \Omega^{\mathrm{t}}$, of the infinite volume process

$$\int f\mu_{t,y} \to \int f\mu, \quad \text{as} \;\; t \to \infty, \quad \text{for local smooth functions } f : \Omega \to \mathbb{R}.$$

At the same time, G. Royer [249] has observed that for the uniqueness of $\mu \in \mathcal{G}^{\mathrm{t}}$ it suffices to have a weaker form of (**ULS**) for $\mu_{\Lambda,y}$ with a fixed boundary condition, say $y = 0$. These results were established only for the *translation invariant* systems with *attractive harmonic* interactions (*"classical ferromagnets"*) and strongly used the à-priori estimates for the Gibbs states $\mu \in \mathcal{G}^{\mathrm{t}}$ known for such systems from the early paper of J. Bellissard and R. Høegh-Krohn [42] (see Subsection 2.2.1). It worth mentioning that the pointwise ergodicity of the infinite systems of *non-gradient* diffusions has been studied by the method of cluster expansions in the asymptotic regime $J \to 0$ in the series of papers of V. Malyshev and his collaborates, cf. [150, 151]. Another still open, challenging problem is to show a dynamical *separation of phases*, i.e., non-ergodicity of the stochastic dynamics, which has to occur in the unbounded spin systems outside the uniqueness regime for $\mu \in \mathcal{G}^{\mathrm{t}}$.

Thus, as compared with [249, 290, 295], the *progress* achieved in Theorem 4.9 is as follows:

– The result extends to the systems of vector spins with *general* non-translation interactions of at most quadratic growth, whose indexing set is a graph;

– The result is clearly stated as the *ergodic theorem*, that says that the dynamics possesses the unique *invariant measure* $\mu \in \mathcal{P}(\Omega^{\mathrm{t}})$ and there is a *local weak convergence* of the laws $\nu_t \to \mu$ for all initial distributions $\nu \in \mathcal{P}(\Omega^{\mathrm{t}})$;

– The *Wasserstein distances* are used to estimate the convergence of the transition probabilities $\mu_{t,y} \to \mu$. Then for each finite volume projection $\mathbb{P}_\Lambda$, the distance $\mathbf{W}(\mathbb{P}_\Lambda\mu_{t,y}, \mathbb{P}_\Lambda\mu)$ exponentially decays as $C(\Lambda, y)\exp\{-tC_{\mathrm{ULS}}\}$ with C_{ULS} being the *uniform log-Sobolev constant* for all local Gibbs measures $\mu_{\Lambda,y}$. The factor $C(\Lambda, y)$ is an integrable function of $y \in \Omega^{\mathrm{t}}$, whose growth is determined by the one-particle potentials;

– The *explicit bounds* on C_{ULS} in terms of the temperature and parameters of the interaction are given;

– The mechanism of such ergodicity is clarified by using *Talagrand's transportation inequality* for the Wasserstein distances;

– It is shown that the pointwise convergence of the dynamics implies its *exponential L^2-decay* with the same speed $C_{\rm ULS}$.

4.2.2 Unique solvability of the Cauchy problem

Here we introduce a notion of solution to the Glauber dynamics associated with the spin system (4.10).

Let us fix some probability space $(\Omega, \mathcal{F}, \mathbf{P})$ with a filtration $(\mathcal{F}_t)_{t\geq 0}$ and a family $w(t) = (w_v(t))_{v\in\mathbb{V}}$ of independent, $\mathbb{R}^\nu$-valued standard Brownian motions on it. We shall consider the following infinite system of locally interacting Itô's diffusions

$$\mathrm{d}x_v(t) = \frac{1}{2} b_v(x(t))\mathrm{d}t + \mathrm{d}w_v(t), \quad t > 0, \quad v \in \mathbb{V}. \tag{4.36}$$

The drift term

$$b = (b_v)_{v\in\mathbb{V}} \in C_{\mathrm{b,loc}}(\varOmega^{\mathrm{t}} \to \varOmega^{t}) \tag{4.37}$$

has a gradient form, whereby its components coincide with the *partial logarithmic derivatives* of the measure μ, cf. (2.205),

$$\varOmega \ni x \mapsto b_v(x) := -\beta \left[V_v'(x_v) + \sum_{v'\in\partial v} \partial_{x_v} W_{vv'}(x_v, x_{v'}) \right] \in \mathbb{R}^\nu. \tag{4.38}$$

A peculiarity, caused by the infinite number of components and possible nonlinear growth of V_v', is that the vector field (4.37) in general cannot be defined pointwise in any fixed weighted space $\varOmega_\delta$ of the scale (4.25). But, as is apparent from (4.12) and (4.15), it maps continuously each $\varOmega_\delta$ into a large space $\varOmega_{\delta'}$ with $\delta' \geq \delta P$ and satisfies the polynomial bound

$$||b(x)||_{o,\delta'} \leq C_{\delta,\delta'} \left(1 + ||x||_\delta^P\right), \quad x \in \varOmega_\delta. \tag{4.39}$$

Furthermore, employing the canonical basis (4.27), we have that

$$b := \sum_{v\in\mathbb{V},\ 1\leq i\leq \nu} b_v^i e_v^i \in C_{\mathrm{b,loc}}(\varOmega_\delta \to \varOmega_{\delta'}), \tag{4.40}$$

where the series in (4.40) converges uniformly on every ball in $\varOmega_\delta$.

Definition 4.4 *Given a (nonrandom) initial value $y \in \varOmega^{\mathrm{t}}$, by the strong solution of the corresponding* ***Cauchy problem*** *we mean a continuous process $x(t,y) = (x_v(t,y))_{v\in\mathbb{V}} \in \varOmega^{\mathrm{t}}$, $t \geq 0$, satisfying (almost surely)*

$$x_v(t,y) = y_v + \frac{1}{2}\int_0^t b_v(x(s,y))\mathrm{d}s + w_v(t), \quad t \geq 0, \quad v \in \mathbb{V}. \tag{4.41}$$

The applications of the stochastic dynamics method in classical statistical mechanics goes back to the paper of R. J. Glauber (1963) on the time-dependent statistics of the Ising model. The stochastic equation (4.36) is usually called the *Glauber* or *gradient dynamics* associated with the Hamiltonian (4.10). During the last three decades such *SDE*'s in infinite dimensions have been extensively studied in the literature, we just mention (among others) early contributions [98, 114, 145, 187, 188, 248, 261] and more recent ones [24, 56, 249, 291].

Concerning the list of hypotheses on the potentials V_v, $W_{vv'}$ made in Subsection 4.1.2, we stress that our goal here is not to release (as much as possible) regularity assumptions on the drifts $b_v(x)$ which are required at some intermediate steps, but to examine the qualitative behavior of the system as $t \to \infty$.

Agreement. *From here on we always suppose that the graph $\mathbb{G}$ obeys Assumption* $(\mathbf{G}_0)$.

This is a technical hypothesis, but it will essentially simplify the control of the stochastic dynamics. Below we collect the principal results known so far about the solutions of (4.36).

Proposition 4.5 *Under Assumptions* $(\mathbf{V}^*)$, $(\mathbf{W}^*)$, *for each* $y \in \Omega^{\mathrm{t}}$ *the Cauchy problem (4.41) has a unique solution* $x(t, y)$, $t \geq 0$, *in* Ω^{t}. *Furthermore, for all* $t > 0$, $p \geq 1$, *and* $y \in \Omega^{\mathrm{t}}$

$$\sup_v \sup_{0 \leq s \leq t} \mathbf{E}|x_v(s, y)|^{2p} < \infty. \tag{4.42}$$

A family of all such solutions constitutes a time homogeneous ***Markov process*** $x(t) = (x_v(t))_{v \in \mathbb{V}} \in \Omega^{\mathrm{t}}$, $t \geq 0$, *with the transition probabilities*

$$\mu_{t,y}(B) := P(x(t, y) \in B), \quad t \geq 0, \quad y \in \Omega^{\mathrm{t}}, \quad B \in \mathcal{B}(\Omega^{\mathrm{t}}), \tag{4.43}$$

obeying the Kolmogorov-Chapman equality

$$\mu_{t+\Delta t, y}(B) = \int_\Omega \mu_{\Delta t, x}(B) \mu_{t,,y}(\mathrm{d}x), \quad t, \Delta t \geq 0. \tag{4.44}$$

The associated ***transition semigroup*** $\mathbb{T}_t f$, $t \geq 0$, *which is defined by*

$$\mathbb{T}_t f(y) := \mathbf{E} f(x(t, y)) = \int_\Omega f(x) \mu_{t,y}(\mathrm{d}x), \quad f \in C_b(\Omega^{\mathrm{t}}),\ t \geq 0, \tag{4.45}$$

is Feller, i.e., preserves the Banach space $C_b(\Omega^{\mathrm{t}})$. *Its dual semigroup* $\mu \mathbb{T}_t$, $t \geq 0$, *acts in the Banach space* $\mathcal{P}(\Omega^{\mathrm{t}})$ *as*

$$\mu \mathbb{T}_t(B) := \int_\Omega \mu(\mathrm{d}y) \mu_{t,y}(B), \quad B \in \mathcal{B}(\Omega^{\mathrm{t}}),\ t \geq 0. \tag{4.46}$$

A measure $\mu \in \mathcal{P}(\Omega^{\mathrm{t}})$ *is said to be the* ***invariant*** *(or* ***stationary****) distribution for the Markov process* $x(t)$ *if*

$$\mu \mathbb{T}_t = \mu, \quad t \geq 0.$$

The set $\mathcal{I}^{\mathrm{t}}$ of tempered invariant distributions is nonempty, whereby all its elements satisfy the following à-priori estimate

$$\sup_{\mu\in\mathcal{I}^{\mathrm{t}}}\sup_{v}\int_{\Omega}|x_v|^{2p}\mu(\mathrm{d}x)\le C_{4.47}^{(p)}<\infty,\quad \textit{for any}\ \ p\ge 1. \tag{4.47}$$

Most of the above properties are standard and come out by a formal substitution of the indexing set $\mathbb{L}=\mathbb{Z}^d$ by a graph $\mathbb{G}$. A full account of them can be find in [24], but already formulated in a more cumbersome situation of the stochastic dynamics on loop spaces. Later on, in Subsection 4.2.4, we shall clarify some key issues concerning how to construct the solutions of (4.41) and derive the uniform bound (4.47).

Definition 4.6 *The Markov process $x(t)$ is called (locally weak)* ***ergodic****, if*

(i) There exists exactly one invariant distribution $\mu\in\mathcal{I}(\Omega^{\mathrm{t}})$;

(ii) For every $\nu\in\mathcal{P}(\Omega^{\mathrm{t}})$ and any $\Lambda\Subset\mathbb{V}$, it takes place the convergence in the weak topology on Ω_Λ

$$\mathbb{P}_\Lambda(\nu\mathbb{T}_t)\to\mathbb{P}_\Lambda\mu,\quad \textit{as}\ \ t\to\infty.$$

Remark 4.7 Recall that a measure $\mu\in\mathcal{P}(\Omega^{\mathrm{t}})$ is said to be the *reversible distribution* for the Markov process $x(t)$ if

$$\int_{\Omega}(\mathbb{T}_t f)g\,d\mu=\int_{\Omega}f(\mathbb{T}_t g)\,d\mu,\quad \forall\, f,g\in C_b(\Omega^{\mathrm{t}}),\quad t\ge 0. \tag{4.48}$$

The set of all tempered reversible distributions, which will be denoted by $\mathcal{R}^{\mathrm{t}}$, is à-priori contained in $\mathcal{I}^{\mathrm{t}}$. Then $\mathbb{T}_t$, $t\ge 0$, uniquely extends to a symmetric contraction semigroup of C_0*-type* (strongly continuous) on $L^2(\mu)$. Furthermore, this semigroup is *sub-Markovian* (positivity and identity preserving) and hence contractive in all $L^p(\mu)$, $p\in[1,+\infty]$. Let $(\mathbb{H},\ \mathcal{D}(\mathbb{H}))$ be its infinitesimal generator in $L^2(\mu)$; it is clear that $\mathbb{H}\ge 0$ and $\mathbf{1}\in\mathcal{D}(\mathbb{H})$ with $\mathbb{H}\mathbf{1}=0$. In other words, (4.48) means that $\mu\in\mathcal{R}^{\mathrm{t}}$ is *symmetrizing* for $\mathbb{H}$, i.e.,

$$(\mathbb{H}f,g)_{L^2(\mu)}=(f,\mathbb{H}g)_{L^2(\mu)},\quad \forall\, f,g\in\mathcal{D}(\mathbb{H}). \tag{4.49}$$

Hence it is also *infinitesimally invariant*, i.e.,

$$\int_{\Omega}\mathbb{H}f\,d\mu=0,\quad \forall\, f\in\mathcal{D}(\mathbb{H}). \tag{4.50}$$

The basic relation between the above classes of measures is expressed by

$$\mathcal{G}^{\mathrm{t}}=\mathcal{R}^{\mathrm{t}}\subseteq\mathcal{I}^{\mathrm{t}}. \tag{4.51}$$

In a rather general context, the equivalence between the Gibbsian property and the stochastic reversibility has been established e.g. in [76, 87, 98, 114, 119, 110, 145, 153, 165, 247, 248, 261]. Another conceptually important question – whether the sets of invariant and reversible distributions for classical spin systems coincide – still remains unresolved; in some particular situations (e.g. for $\mathbb{L} := \mathbb{Z}^d$ with $d = 1, 2$) the positive answer was given in [59, 60, 115, 145]. Existence and regularity properties of (infinitesimally) invariant measures, as an intermediate step to the ergodicity problem, have been studied intensively; for the present state of research and available techniques we refer to [54]–[56], [77, 85, 132, 189, 241, 242].

The next statement allows us to identify the *generator* of the transition semigroup.

Proposition 4.8 *Let Assumptions* $(\mathbf{V}^*)$, $(\mathbf{W}^*)$ *be satisfied, and consider any* $\mu \in \mathcal{G}^{\mathrm{t}} = \mathcal{R}^{\mathrm{t}}$. *Then the generator of the semigroup* $\mathbb{T}_t$, $t \geq 0$, *acting in* $L^2(\mu)$ *coincides with the* Dirichlet operator $\mathbb{H}_\mu$ *of the measure* μ, *which was introduced in Subsection 2.3.5 (ii). The later is given by the second order differential expression, cf. (2.206),*

$$\mathbb{H}_\mu f(x) = -\sum_{v \in \mathbb{V}} [\Delta_v f + (b_v, \partial_{x_v} f)] \tag{4.52}$$

on the set of smooth cylinder functions $f \in \mathcal{F}C_{\mathrm{b}}^\infty(\Omega)$, *which moreover is its domain of essential self-adjointness.*

The proof will be postponed to Subsection 4.2.4, where we shall look in more technical details at the properties of the process $x(t, y)$, $t \geq 0$.

4.2.3 Scheme of the ergodicity result

Here we present the *main result* of this chapter.

Theorem 4.9 *Suppose that the graph* $\mathbb{G}$ *obeys the polynomial growth, see Assumption* $(\mathbf{G}_d)$. *Let the interaction parameters satisfy the following relation*

$$Jm_{\mathbb{G}} < \frac{a_U}{1 + e^{\beta \delta_Q}}. \tag{4.53}$$

Then, the Markov process solving (4.41) is ergodic in the sense of Definition 4.6 and its transition probabilities converge exponentially quickly to the unique invariant measure $\mu \in \mathcal{I}^{\mathrm{t}} = \mathcal{G}^{\mathrm{t}}$. *More precisely, let us introduce the (uniform log-Sobolev) constant*

$$C_{\mathrm{ULS}} := \beta \left[(a_U - Jm_{\mathbb{G}}) e^{-2\beta\delta_Q} - Jm_{\mathbb{G}} \right] > 0. \tag{4.54}$$

Then, for any $C_{4.55} \in (0, C_{\mathrm{ULS}})$ *one finds a corresponding* $\delta_0 := \delta_0(C_{4.55}) > 0$, *such that for each* $\delta \in (0, \delta_0)$, $t \geq 1$, *and all* $y \in \Omega^{\mathrm{t}}$, $o \in \Lambda \Subset \mathbb{V}$, *the following estimate for finite volume projections holds in the Wasserstein metric on* $\mathcal{P}(\Omega_\Lambda)$:

$$\mathbf{W}_\Lambda(\mathbb{P}_\Lambda \mu_{t,y}, \mathbb{P}_\Lambda \mu) \leq \exp\{|\Lambda| K_{4.55} - t C_{4.55}\} \sum\nolimits_v \left(1 + |y_v|^{P/2}\right) e^{-\delta \rho(v,o)}, \tag{4.55}$$

where $K_{4.55} > 0$ *depends only on* $C_{4.55}$ *and* δ_0.

Necessary details about the Wasserstein distance $\mathbf{W}_\Lambda$ will be recalled in Subsection 4.4.1. Here we consider each Ω_Λ as a Banach space with the l^1-type norm $|x_\Lambda - \tilde{x}_\Lambda|_\Lambda := \sum_{v\in\Lambda} |x_v - \tilde{x}_v|$, but of course the similar result will be true for other choices of the norm including the Euclidean one borrowed from $\mathbb{R}^{\nu|\Lambda|}$. In infinite dimensional setting, a main reason for using the Wasserstein distance, which metricizes the topology of weak convergence, is that the transition probabilities $\mu_{t,y} \in \mathcal{P}(\Omega^{\mathrm{t}})$ are mutually singular and hence their strong convergence, in the total variation distance, does not hold.

Now we explain a general strategy how to prove the ergodicity result of Theorem 4.9. The proof will be done in *three steps* which are focused on the following problems:

I. Finite volume approximation of the stochastic dynamics;

II. Uniform ergodicity of the finite volume dynamics;

III. Exponential convergence of the local conditional distributions.

Each of the above problems will require its own *techniques*, which respectively can be characterized as follows:

I. *Stochastic Approach:* methods from the theory of finite and infinite dimensional SDE's (Subsections 4.2.4, 4.2.5);

II. *Analytic Approach:* entropy estimates, Poincaré and log-Sobolev inequalities for the corresponding Dirichlet operators (Subsections 4.3.1–4.5.4);

III. *DLR* or *Markovian Approach:* use of the associated Gibbs structure and Dobrushin's contraction estimates (Subsections 4.4.1, 4.4.2).

Of course, any single problem taken from this list has been studied by many authors, from different viewpoints, and for various aims. A *major new input* of the present work is to show how a proper combination of such distinct techniques leads to the ergodicity result of Theorem 4.9.

To realize this program we shall follow a standard way for constructing and studying solutions to the system (4.41) in the whole $\mathbb{V}$, which is based on its *finite volume approximation*. For any fixed initial data $y \in \Omega^{\mathrm{t}}$ and $\Lambda \Subset \mathbb{V}$, let us consider the Cauchy problem derived from (4.41) by a cut-off procedure with respect to y_v, $v \notin \Lambda$,

$$x_v^\Lambda(t,y) = \begin{cases} y_v + \frac{1}{2}\int_0^t b_v(x^\Lambda(s,y))\mathrm{d}s + w_v(t), & v \in \Lambda\ , \\ \qquad\qquad x_v(t,y) = y_v, & v \in \Lambda^c, \end{cases} \qquad t \geq 0. \tag{4.56}$$

Then, to each finite volume $\Lambda \Subset \mathbb{V}$, there corresponds a unique non-exploiding strong solution $x^\Lambda(t,y) \in \Omega^{\mathrm{t}}$ starting from $y \in \Omega^{\mathrm{t}}$. These solutions obey the polynomial integrability similar to (4.42) and with probability one approximate the solution to the infinite volume problem, i.e.,

$$\mathbf{P}\left\{\lim_{\Lambda\nearrow\mathbb{V}} \left[\sup_{0\leq s\leq t} ||x(s,y) - x^\Lambda(s,y)||_{o,\delta}\right] = 0\right\} = 1. \tag{4.57}$$

By $\mu_{t,y}^{\Lambda} \in \mathcal{P}(\Omega^{\mathrm{t}})$ we denote the law of the random variable $x^{\Lambda}(t,y) \in \Omega^{\mathrm{t}}$.

We stress that (4.56) is a particular case of SDE's (though with globally non-Lipschitz drifts), whose general theory is much elaborated in finite dimensions, see e.g. the standard sources [127, 139, 193, 243]. Our goal however is going beyond such theory, in so far as we need the *quantitative* and *dimension free* results describing the rate of convergence of $x^{\Lambda}(t,y) \to x(t,y)$ as $\Lambda \nearrow \mathbb{V}$ and $t \to \infty$. Loosely speaking, we should be able to commute the above limit procedures in volume and in time. In doing *Step I*, a key role will be played by the so-called *finite propagation property* to be discussed in Subsection 4.2.4, which gives precise estimates on the L^p-convergence in (4.57). By the classical *Khasminskii theorem* (cf. e.g. Theorem 4.4.1 in [139]; Proposition 4.1 in [97]) we already know that each diffusion process $(x_v^{\Lambda}(t,y))_{v\in\Lambda} \in \Omega_{\Lambda}$ is ergodic, whereby its unique invariant (moreover, reversible) distribution is the local Gibbs measure $\mu_{\Lambda,y}$ in volume Λ with the boundary condition y. This theorem however cannot help us to pass to the thermodynamic limit $\Lambda \nearrow \mathbb{V}$, since it contains no information how quickly the probability laws $\mathbb{P}_{\Lambda}\mu_{t,y}^{\Lambda}$ of $(x_v^{\Lambda}(t,y))_{v\in\Lambda}$ would converge to $\mu_{\Lambda,y}$ as $t \to \infty$. To control such convergence we shall employ *analytical tools* based on the entropy estimates and log-Sobolev inequalities, which will constitute *Step II* in the above scheme of proof. And finally, to check the convergence of invariant measures $\mu_{\Lambda,y}$ as $\Lambda \nearrow \mathbb{V}$, at *Step III* we shall refer to their interpretation as the conditional Gibbs distributions and apply the corresponding *DLR* techniques.

To be more specific, the above scheme gives rise to the following chain of estimates

$$
\begin{aligned}
& & & \mathbf{W}_{\Lambda}(\mathbb{P}_{\Lambda}\mu_{t,y}\,,\,\mathbb{P}_{\Lambda}\mu) && (4.58)\\
(\mathbf{I}) & & \leq\quad & \mathbf{W}_{\Lambda}(\mathbb{P}_{\Lambda}\mu_{t,y}\,,\,\mathbb{P}_{\Lambda}\mu_{t,y}^{\Lambda(t)}) && (4.59)\\
(\mathbf{II}) & & +\quad & \mathbf{W}_{\Lambda(t)}(\mu_{t,y}^{\Lambda(t)}\,,\,\mu_{\Lambda(t),y}) && (4.60)\\
(\mathbf{III}) & & +\quad & \mathbf{W}_{\Lambda}(\mathbb{P}_{\Lambda}\mu_{\Lambda(t),y}\,,\,\mathbb{P}_{\Lambda}\mu). && (4.61)
\end{aligned}
$$

From a technical viewpoint our strategy can be outlined as follows. For each fixed $\Lambda \Subset V$, we allow for the intermediate volumes $\Lambda(t)$ to grow quickly enough so that $\mathrm{dist}\,(\Lambda,[\Lambda(t)]^c) \geq 1 + Bt$ as the time $t \to \infty$. The *"speed"* parameter $B > 0$ will be explicitly given in Proposition 4.16. As will be shown below, in such regime all three terms in the right-hand side in (4.58) are exponentially convergent in time like $C(y)e^{-tC_{\mathrm{ULS}}}$, whereby the functions $C(y)$ behave them polynomially as $||y_v||_{o,\delta}^{P/2}$ and hence are integrable with respect to $\mu \in \mathcal{I}^{\mathrm{t}}$.

Proof of Theorem 4.9. The proof formally can be written down as follows.

The *first term*, (4.59), is estimated by Corollary 4.17 as

$$
\begin{aligned}
(\mathbf{I}) \quad \leq \quad & C_{4.98}^{(M,\delta_1)}|\Lambda|^{1/2}\exp\left\{\frac{1}{2}\delta_1\mathrm{diam}(\Lambda) - M\mathrm{dist}(\Lambda,[\Lambda(t)]^c)\right\} && (4.62)\\
& \times\sum_v (1+|y_v|)e^{-\frac{1}{2}\delta_1\rho(v,o)}.
\end{aligned}
$$

The *second term*, (4.60), is estimated by Corollary 4.34 as

$$\begin{aligned}(\mathbf{II}) \quad &\leq \quad C^{(\delta_2)}_{4.189}|\Lambda(t)|\exp\{\delta_2\mathrm{diam}[\Lambda(t)] - tC_{\mathrm{ULS}}\} \\ &\quad \times \sum_v (1 + |y_v|^{P/2})e^{-\delta_2\rho(v,o)}.\end{aligned} \tag{4.63}$$

The *third term*, (4.61), is estimated by Corollary 4.40 as

$$\begin{aligned}(\mathbf{III}) \quad &\leq \quad (1 + C_{4.210})\,|\Lambda|\exp\left\{\delta_3\mathrm{diam}(\Lambda) - \frac{\delta_3}{2}\mathrm{dist}\,(\Lambda, [\Lambda(t)]^c)\right\} \\ &\quad \times (1 - ||\mathbf{D}||_0 \exp\delta_3)^{-1}\sum_v (1 + |y_v|)\,e^{-\delta_3\rho(\nu,o)/2}.\end{aligned} \tag{4.64}$$

Here we may chose arbitrary $M, \delta_1, \delta_2 > 0$, $\delta_3 \in (0, \log ||\mathbf{D}||_0^{-1})$, $t \geq 1$, $y \in \Omega^{\mathrm{t}}$, and consider the bounded domains $\Lambda \subset \Lambda(t)$ containing an initial vertex $o \in \mathbb{V}$, such that

$$\mathrm{dist}(\Lambda, [\Lambda(t)]^c) > 1 + tB^{(M,\delta_1)}_{4.84}.$$

Now we set $M = \delta_1 = 1$, $\delta_3 := \frac{1}{2}\log ||\mathbf{D}||_0^{-1}$, and

$$\mathcal{R} := \max\left\{4C_{\mathrm{ULS}}(1 + \log ||\mathbf{D}||_0^{-1})^{-1},\ 2B^{(1,1)}_{4.84}\right\}. \tag{4.65}$$

Supposing that

$$\begin{gathered}\mathrm{dist}(\Lambda, [\Lambda(t)]^c) \geq t\mathcal{R}, \\ \sup_{t\geq 1}\left\{\frac{1}{t}\mathrm{diam}[\Lambda(t)]\right\} := \mathcal{S} < \infty, \quad \limsup_{t\geq 1}\left\{\frac{1}{t}\log|\Lambda(t)|\right\} = 0,\end{gathered} \tag{4.66}$$

we next put $\delta_2 := (C_{\mathrm{ULS}} - C_{4.55})/2\mathcal{S} > 0$. Summing (4.62), (4.63), and (4.64) together gives us the desired estimate (4.55) with $\delta_0 := \frac{1}{2}\min\{\delta_1, \delta_2, \delta_3\}$. So, to complete the proof it remains to construct a sequence $\Lambda(t) \nearrow \mathbb{V}$ obeying the properties (4.66). It is naturally to take

$$\Lambda(t) := \partial^+_{t\mathcal{R}}\Lambda := \{v' \in \mathbb{V} \mid \mathrm{dist}(v'; \Lambda) \leq t\mathcal{R}\} \tag{4.67}$$

with

$$\begin{gathered}\mathrm{diam}[\Lambda(t)] \leq \mathrm{diam}(\Lambda) + 2t\mathcal{R}, \\ |\Lambda(t)| \leq |B_{r(t)}(o)|, \quad \text{and} \quad r(t) := \mathrm{diam}(\Lambda) + t\mathcal{R}.\end{gathered} \tag{4.68}$$

Note that the factor $\exp\{const|\Lambda|\}$ in the estimate (4.55) allows us the uniform control with respect to all initial sets $\Lambda \Subset \mathbb{V}$. Thus, as (4.68) shows, the cofinal sequence (4.67) would fit (4.66) if

$$\sup_{r\in\mathbb{N},\ o\in\mathbb{V}}\left\{\frac{1}{r}\log|B_r(o)|\right\} < \infty.$$

But the last property surely holds for each graph $\mathbb{G}$ which obeys the polynomial growth condition (4.6). For the moment we leave open the question about the ergodicity in the sense of Definition 4.6 and shall clarify this in Proposition 4.12 below. ■

The previously known results about the exponential relaxation of the stochastic dynamics in ferromagnetic systems, see [290, 295], were stated in the *dual form* which describes the action of the semigroup $\mathbb{T}_t$, $t \geq 0$, on smooth local functions, without mentioning the Wasserstein distance. Furthermore, no explicit estimates on the rate of convergence in terms of the parameter interaction have yet been given. The following assertion presents the dual form of Theorem 4.9 and considerably improves the related result of N. Yoshida (see Theorems 2.1 and 2.2 in [290]).

Corollary 4.10 *In the dual form, the estimate (4.55) can be rewritten as*

$$|(\mathbb{T}_t f)(y) - < f >_\mu| = \left| \int_\Omega f(x_\Lambda) \left[\mu_{t,y}(\mathrm{d}x) - \mu(\mathrm{d}x) \right] \right| \tag{4.69}$$
$$\leq [f]_\Lambda e^{K_{K_{4.55}}|\Lambda| - tC_{4.55}} \sum\nolimits_v \left(1 + |y_v|^{P/2}\right) e^{-\delta\rho(v,o)}, \quad \forall f \in \mathrm{Lip}(\Omega_\Lambda),$$

where, see (2.169), (2.170),

$$\mathrm{Lip}(\Omega_\Lambda) := \left\{ f : \mathbb{R}^{\nu|\Lambda|} \to \mathbb{R} \,\middle|\, [f]_\Lambda := \sup_{x_\Lambda \neq \tilde{x}_\Lambda} \frac{|f(x_\Lambda) - f(\tilde{x}_\Lambda)|}{|x_\Lambda - \tilde{x}_\Lambda|_\Lambda} < \infty \right\}. \tag{4.70}$$

Proof. The left-hand sides in (4.55) and (4.69) coincide by the Kantorovich-Rubinstein relation (4.194). ■

The next result says that the *pointwise* exponential relaxation of the stochastic dynamics (4.55) ensures the *spectral gap* of size C_{ULS} for its generator $\mathbb{H}$ (see the definitions (2.212), (2.213)) or, what is equivalent, the L^2*-exponential decay* (4.71) of the semigroup $\mathbb{T}_t$, $t \geq 0$..

Corollary 4.11 *In situation of Theorem 4.9, for all $f \in L^2(\mu)$*

$$\mathbf{Var}_\mu^{1/2}(\mathbb{T}_t f) := \|\mathbb{T}_t f - \mathbf{E}_\mu f\|_{L^2_\mu} \leq e^{-tC_{\mathrm{ULS}}} ||f||_{L_2(\mu)}, \quad t \geq 0. \tag{4.71}$$

Proof. Integrating (4.69) with respect to $\mu \in \mathcal{I}^{\mathrm{t}}$ and crucially using the à-priori bound (4.47), we get that

$$\|\mathbb{T}_t f - \mathbf{E}_\mu f\|_{L^2_\mu} \leq 2[f]_\Lambda \cdot \Xi_\delta e^{K_{4.55}|\Lambda| - tC_{4.55}} \left(1 + C_{4.47}^{(P/2)}\right)^{1/2}.$$

Finally, we refer to the general fact (see e.g. page 374 in [156], Theorem 2.3 in [244], or Proposition 2.9 in [287]) saying that the (much weaker) bound

$$\mathbf{Var}_\mu(\mathbb{T}_t f) \leq \mathcal{K}(f) e^{-tC_{4.47}}, \quad \forall t \geq 0, \tag{4.72}$$

holding with its own constant $\mathcal{K}(f) > 0$ for each function f from a dense domain $\mathcal{D}$ in $L^2(\mu)$, indeed yields the spectral gap estimate (4.71) with $C_{4.47}$. Since these $C_{4.47}$ can be taken arbitrarily close to C_{ULS}, by the continuity argument we immediately obtain the result. ■

Proposition 4.12 *From (4.69) is follows that $\mu \in \mathcal{G}^{\mathrm{t}}$ is the unique tempered invariant measure.*

Proof. Suppose that there exists one more $\tilde{\mu} \in \mathcal{I}^{\mathrm{t}}$, which by Theorem 4.21 has to satisfy the a-priori bound (4.47). Then, by Corollary 4.10 we get that for all $\Lambda \subseteq \mathbb{V}$ and $f \in \mathrm{Lip}(\Omega_\Lambda)$

$$\int_\Omega f(x_\Lambda)\tilde{\mu}(\mathrm{d}x) = \lim_{t\to\infty} \int_\Omega \mathbb{T}_t f(x)\tilde{\mu}(\mathrm{d}x) = \int_\Omega f(x_\Lambda)\mu(\mathrm{d}x), \tag{4.73}$$

which implies $\mu = \tilde{\mu}$. The passage to the limit $t \to \infty$ is allowed by (4.47) and Lebesgue's dominated convergence theorem. In a similar way, but additionally employing the L^∞-contractivity of the semigroup $\mathbb{T}_t$, $t \geq 0$, one proves that for each $\nu \in \mathcal{P}(\Omega^{\mathrm{t}})$ and all bounded Lipschitz continuous functions f

$$\int_\Omega f(x_\Lambda)\mathbb{T}_t\nu(\mathrm{d}x) = \int_\Omega \mathbb{T}_t f(x)\nu(\mathrm{d}x) \underset{t\to\infty}{\to} \int_\Omega f(x_\Lambda)\mu(\mathrm{d}x) \tag{4.74}$$

(which means the convergence $T_t\nu \to \mu$ in the *Fortet-Mourier distance*). Since $C^1_{\mathrm{b}}(\Omega_\Lambda)$ (as a subset of $\mathrm{Lip}(\Omega_\Lambda) \cap L^\infty(\Omega_\Lambda)$) is densely embedded in $C_{\mathrm{b}}(\Omega_\Lambda)$, by a standard approximation argument the convergence in (4.74) extends further to all $f \in C_{\mathrm{b}}(\Omega_\Lambda)$, which implies the required ergodicity in the sense of Definition 4.6. ■

Remark 4.13 (i) Under additional integrability assumptions on the initial distributions $\nu \in \mathcal{P}(\Omega^{\mathrm{t}}$, it is also possible to control the convergence $T_t\nu \to \mu$ in the Wasserstein distances $\mathbf{W}_p$, $p \geq 1$.

(ii) The spectral gap estimate (4.71) by itself does not yet imply the pointwise ergodicity like in (4.69) *for all* $y \in \Omega^{\mathrm{t}}$. The sufficient conditions for (4.71) are discussed in Subsections 2.3.5 (ii) and 4.5.1.

(iii) Actually, Theorem 4.9 could be stated with the *best* exponent $C_{4.55} := C_{\mathrm{ULS}}$ in (4.55), if we replace (4.60) by a more accurate estimate for $\mathbf{W}_\Lambda(\mathbb{P}_\Lambda\mu^{\Lambda(t)}_{t,y}, \mathbb{P}_\Lambda\mu_{\Lambda(t),y})$. Note that the projections $\mathbb{P}_\Lambda\mu_{\Lambda(t),y}$ satisfy the log-Sobolev inequality with the same constant C_{ULS}. So, it only remains to get an upper bound for the entropy $\mathbf{H}(\mathbb{P}_\Lambda\mu^{\Delta}_{t,y}, \mathbb{P}_\Lambda\mu_{\Delta,y})$, which has to be similar to that in Theorem 4.33 and *independent* of $\Delta \supseteq \Lambda$. This is surely possible and will be discussed elsewhere.

4.2.4 Finite propagation property for stochastic dynamics

Here we perform *Step I* in proving Theorem 4.9. Along with rather standard Propositions 4.15, 4.18, and 4.20 describing the properties of the solutions in finite and infinite volumes, the most principal issue is Theorem 4.16 and its Corollary 4.17 about the so-called *finite propagation property* for locally interacting diffusions. This property can also be established in a rather abstract context, including the case of general (e.g. N-particle) interactions. As will be seen from the proof, a necessary condition for its validity is that the drift terms (which need not to be of a gradient form) are *local*, i.e., there exists some finite $\rho > 0$ such that each component $b_v(x)$ depends only on $x_{v'}$ with $\rho(v, v') \le \rho$.

The finite propagation property is well known for stochastic dynamics with compact spin spaces, cf. e.g. [146, 269, 270]. Its extension to the unbounded spin systems was first performed in Proposition 1.4 of [295], by constructing a finite volume approximation of the semigroup $\mathbb{T}_t$, $t \ge 0$. This was done by means of pure analytical methods based on the Duhamel formula. The analog of that result in our situation is the dual estimate (4.99) in Corollary 4.17. Thereafter, in Lemma 3.1 of [290], by applying probabilistic methods there was obtained the strongest result of such type, which states the L^{2p}-convergence of the random variables $x^\Lambda(t,y) \to x(t,y)$ as $\Lambda \nearrow \mathbb{V}$. Theorem 4.16 to be proved below improves those results in the following directions: (i) we are able to consider general drift terms, whereas [295, 290] dealt exceptionally with translation invariant harmonic interactions on a lattice; and (ii) the formulations and proofs in [290] are true only for $p = 1$, and there is a principal mistake in their argument for $p > 1$ which we shall correct.

Technically, the proofs below will rely on a simple observation made from (4.13), (4.20) and (4.20), that the drifts (4.38) satisfy the following *coercivity* and *semi-dissipativity properties*:

Lemma 4.14 *For all $v \in \Lambda \subseteq \mathbb{V}$ and $x_\Lambda, \tilde{x}_\Lambda \in \Omega_\Lambda$, $y_{\Lambda^c} \in \Omega_{\Lambda^c}$,*

$$\begin{aligned}
(i) \qquad & (b_v(x_\Lambda, y_{\Lambda^c}), x_v) - \beta \frac{J}{2}\left[D_W + \sum_{v' \in \Lambda \cap \partial v} |x_{v'}|^2 + \sum_{v' \in \Lambda^c \cap \partial v} |y_{v'}|^2\right] \\
\le \;& -\beta\,(A_6 - J m_{\mathbb{G}})\,|x_v|^2 - \beta B_6; && (4.75)\\
(ii) \qquad & \big(b_v(x_\Lambda, y_{\Lambda^c}) - b_v^\Lambda(\tilde{x}_\Lambda, y_{\Lambda^c}), x_v - \tilde{x}_v\big) \\
& -\beta\frac{J}{2}\left[\sum_{v' \in \Lambda \cap \partial v} |x_{v'} - \tilde{x}_{v'}|^2 + \sum_{\ell' \in \Lambda^c \cap \partial v} |y_{v'} - \tilde{y}_{v'}|^2\right] \\
\le \;& \begin{cases} -\beta\left(A_7 - \frac{3}{2} J m_{\mathbb{G}}\right)|x_v - \tilde{x}_v|^2 - \beta B_7, \\ \beta A_8 |x_v - \tilde{x}_v|^2\;. \end{cases} && (4.76)
\end{aligned}$$

Recall that here $A_8 \ge a_U - \varepsilon''$ and $A_6, A_7 \in (3Jm_{\mathbb{G}}, a_U)$ can be chosen arbitrarily.

We start with the à-priori moment estimates on the solutions of (4.41) and (4.56), assuming nothing more than (4.75) and (4.76). The proposition below shows the *exponentially weak dependence* of $x_v^\Lambda(t,y)$ on boundary values $y_{v'}$ as $\rho(v,v') \to \infty$.

Proposition 4.15 *Fixed $\delta > 0$, for any $p \ge 1$ there exist positive $C_{4.77}^{(p)} := C_{4.77}^{(p)}(\delta)$ and $C_{4.78} := C_{4.77}^{(1)}(\delta)$, such that for all $t \ge 0, y \in \Omega_\delta$, and $v \in \Lambda \Subset \mathbb{V}$,*

$$(i) \qquad \sup_{0 \le s \le t} \mathbf{E}|x_v^\Lambda(s,y)|^{2p} \le 2 e^{2tC_{4.77}^{(p)}} \sum_{v'} (1 + |y_{v'}|^{2p}) e^{-p\delta\rho(v,v')}; \qquad (4.77)$$

$$(ii) \quad \mathbf{E}\left(\sup_{0\le s\le t}|x_v^\Lambda(s,y)|^2\right) \le 3e^{3tC_{4.78}}\sum_{v'}(1+|y_{v'}|^2)e^{-\delta\rho(v,v')}. \tag{4.78}$$

All stated above is also valid for $\Lambda := \mathbb{V}$ *and* $x(t,y) := x^{\mathbb{V}}(t,y)$.

Proof. The proof is entirely standard (cf. e.g. [290] in the translation invariant case with $p=1$). So, our aim is to get the estimates which are uniform with respect to all shifted norms $||\cdot||_{o,\delta}$. By Itô's formula we have that for each $v\in\Lambda$

$$\begin{aligned}|x_v^\Lambda(t,y)|^{2p} \quad &\le \quad |y_v|^{2p} + p\int_0^t |x_v^\Lambda(s,y)|^{2p-2}\left[\left(x_v^\Lambda(s,y), b_v(x^\Lambda(s,y)\right) + \nu(2p-1)\right]\mathrm{d}s \\ &\quad +2p\int_0^t (x_v^\Lambda(s,y)|x_v^\Lambda(s,y)|^{2p-2}, \mathrm{d}w_v(s)), \quad t\ge 0;\end{aligned} \tag{4.79}$$

for $p=1$ there is just an equality in (4.79). Herefrom, by Hölder's and Young's inequalities and the coercivity property (4.75), we conclude that for all $v\in\mathbb{V}$

$$\begin{aligned}&E_v^{(p)}(t) =: \mathbf{E}|x_v^\Lambda(t,y)|^{2p} \\ &\le |y_v|^{2p} + C_{4.80}^{(p)}\int_0^t\left[1+m_{\mathbb{G}}E_v^{(p)}(s) + \sum\nolimits_{v'\in\partial v}E_{v'}^{(p)}(s)\right]\mathrm{d}s,, \quad t\ge 0,\end{aligned} \tag{4.80}$$

with a constant $C_{4.80}^{(p)} := p\left[\nu(2p-1)+\beta|B_6|+\beta J(D_W+1)/2\right]$. Applying to (4.80) the infinite dimensional version of Gronwall's inequality (see Lemma 4.22 below), we readily obtain that

$$\begin{aligned}&\sup_{0\le s\le t}E_v^{(p)}(s) \le \sum\nolimits_{v'} e^{-p\delta\rho(v,v')}\sup_{0\le s\le t}E_v^{(p)}(s) \\ \le \quad & e^{tC_{4.80}^{(p)}m_{\mathbb{G}}\left(1+e^{p\delta}\right)}\sum\nolimits_{v'} e^{-p\delta\rho(v,v')}\left[|y_{v'}|^{2p}+tC_{4.80}^{(p)}\right] \\ \le \quad & 2e^{2tC_{4.77}^{(p)}}\sum\nolimits_{v'} e^{-p\delta\rho(v,v')}\left(1+|y_{v'}|^{2p}\right) \le 2e^{tC_{4.77}^{(p)}}\left(\Xi_\delta + ||y||_{v,\delta}^2\right)^p < \infty,\end{aligned} \tag{4.81}$$

which proves the claim (i) with $C_{4.77}^{(p)}(\delta) := C_{4.80}^{(p)}m_{\mathbb{G}}\left(1+e^{p\delta}\right)$. To prove (ii) we use Doob's maximal inequality for martingales implying that

$$\begin{aligned}\mathbf{E}\left(\sup_{0\le s\le t}\left|\int_0^t (x_v^\Lambda(s,y),\mathrm{d}w_v(s))\right|\right) &\le \left[\mathbf{E}\left(\sup_{0\le s\le t}\left|\int_0^t (x_v^\Lambda(s,y),\mathrm{d}w_v(s))\right|^2\right)\right]^{1/2} \\ &\le 2\left[\int_0^t \mathbf{E}|x_v^\Lambda(s,y)|^2\mathrm{d}s\right]^{1/2}.\end{aligned} \tag{4.82}$$

Thereafter, repeating the preceding estimates for $p=1$, we find that the following quantities are finite

$$E_v(t) := \mathbf{E}\sup_{0\le s\le t}|x_v^\Lambda(t,y)|^2, \quad v\in\mathbb{V},$$

and satisfy the system of inequalities

$$E_v(t) \le 2+|y_v|^2 + C_{4.80}^{(1)}\int_0^t\left[1+2E_v(s)+\sum\nolimits_{v'\in\partial v}E_{v'}(s)\right]\mathrm{d}s, \quad t\ge 0.$$

Lemma 4.22 yields us finally that

$$\begin{aligned} E_v(t) &\leq \sum\nolimits_{v'} e^{-\delta\rho(v,v')} E_{v'}(t) \\ &\leq e^{tC^{(1)}_{4.80}\left(2+e^{\delta}m_{\mathbb{G}}\right)} \sum\nolimits_{v'} e^{-\delta\rho(v,v')} \left[2+|y_{v'}|^2 + tC^{(1)}_{4.80}\right] \\ &\leq 3e^{3tC^{(1)}_{4.77}} \sum\nolimits_{v'} e^{-\delta\rho(v,v')} \left(1+|y_{v'}|^2\right) = 3e^{3tC_{4.78}} \left[\mathbf{\Xi}_\delta + ||y||^2_{v,\delta}\right]. \end{aligned}$$

■

The next statement describes the so-called property of *finite speed of propagation* of interaction (cf. [295], page 408) for the system of locally interacting diffusions (4.41). Because such property is also of independent interest and can be used more widely, we shall formulate it in a much generality. Indeed, for proving the ergodicity result of Theorem 4.9 we need only its Corollary 4.17.

Theorem 4.16 *Fixed $\delta > 0$ and $M > 0$, for each $p \geq 1$ one finds positive constants $B^{(p)}_{4.83} = B^{(p)}_{4.83}(\delta, M)$ and $C^{(p)}_{4.83} = C^{(p)}_{4.83}(\delta, M)$ (respectively, $B_{4.84} = B_{4.84}(\delta, M)$ and $C_{4.84} := C_{4.84}(\delta, M)$), such that for all domains $\Lambda \subset \Delta \subseteq \mathbb{V}$, boundary conditions $y, \tilde{y} \in \Omega_\delta$, $y \equiv \tilde{y}$ on Λ, and inner points $v \in \Lambda^-$ with*

$$\mathrm{dist}(v, \Lambda^c) \geq 1 + tB^{(p)}_{4.83} \quad (\textit{respectively, } \mathrm{dist}(v, \Lambda^c) \geq 1 + tB_{4.84}),$$

the following estimates for the corresponding solutions hold:

$$\begin{aligned} (i) \qquad & \sup_{0\leq s\leq t} \mathbf{E}|x^\Lambda_v(t,y) - x^\Delta_v(t,\tilde{y})|^{2p} \\ &\leq C^{(p)}_{4.83} e^{-2M\mathrm{dist}(v,\Lambda^c)} \sum_{v'} (1+|y_{v'}|^{2p} + |\tilde{y}_{v'}|^{2p}) e^{-p\delta\rho(v,v')}; \end{aligned} \tag{4.83}$$

$$\begin{aligned} (ii) \qquad & \mathbf{E}\left(\sup_{0\leq s\leq t} |x^\Lambda_v(t,y) - x^\Delta_v(t,\tilde{y})|^2\right) \\ &\leq C_{4.84} e^{-2M\mathrm{dist}(v,\Lambda^c)} \sum_{v'} (1+|y_{v'}|^2 + |\tilde{y}_{v'}|^2) e^{-\delta\rho(v,v')}. \end{aligned} \tag{4.84}$$

In particular, all the above applies to $\Delta := \mathbb{V}$ and $x(t, \tilde{y}) := x^{\mathbb{V}}(t, \tilde{y})$.

Proof. The proof to be presented here (at least for $p = 1$) is based mainly on that of [290], Lemma 3.1 (b). Set $z(t) := x^\Lambda(t, y) - x^\Delta(t, \tilde{y})$, then obviously $z_v(0) = 0$ for $v \in \Lambda$. For each point $v \in \Lambda^-$, by integrating by parts we have

$$|z_v(t)|^{2p} = p \int_0^t |z_v(s)|^{2p-2} \left(z_v(s), b_v(x^\Lambda(s,y)) - b_v(x^\Delta(s,\tilde{y}))\right) \mathrm{d}s. \tag{4.85}$$

Employing here Young's inequality and the semi-dissipativity property (4.76), we further get that

$$\begin{aligned} |z_v(t)|^{2p} &\leq \beta p \left[A_8 \int_0^t |z_v(s)|^{2p} \mathrm{d}s + \frac{J}{2} \sum_{v'\in\partial v} \int_0^t |z_v(s)|^{2p-2} |z_{v'}(s)|^2 \mathrm{d}s\right] \\ &\leq \beta \left[p\left(A_8 + m_{\mathbb{G}} \frac{J}{2}\right) \int_0^t |z_v(s)|^{2p} \mathrm{d}s + \frac{J}{2} \sum_{v'\in\partial v} \int_0^t |z_{v'}(s)|^{2p} \mathrm{d}s\right]. \end{aligned} \tag{4.86}$$

For all other points $v \notin \Lambda^-$ we simply write

$$\mathbf{E}|z_v(t)|^{2p} \leq 2^{2p-1}\left[\mathbf{E}|x_v^{\Lambda}(t,y)|^{2p} + \mathbf{E}|x_v^{\Delta}(t,\tilde{y})|^{2p}\right]. \tag{4.87}$$

Combining (4.86) and (4.87) we arrive at

$$\begin{aligned} I_v(t) &:= \sup_{0\leq s\leq t} \mathbf{E}|z_v(s)|^{2p} \\ &\leq c_v(t) + \sum\nolimits_{v'} Q_{vv'} \int_0^t I_{v'}(s)\mathrm{d}s, \quad v \in \mathbb{V}, \end{aligned} \tag{4.88}$$

where

$$c_v(t) := \begin{cases} 0, & \text{if } v \in \Lambda^-, \\ 2^{2p+1} e^{2tC^{(p)}_{4.77}} \sum\limits_{v'} (1 + |y_{v'}|^{2p} + |\tilde{y}_{v'}|^{2p}) e^{-p\delta\rho(v,v')}, & \text{otherwise}, \end{cases} \tag{4.89}$$

and

$$Q_{vv'} := \begin{cases} p\left(A_8 + Jm_{\mathbb{G}}/2\right), & \text{if } \rho(v,v') \leq 1, \\ 0, & \text{otherwise}. \end{cases} \tag{4.90}$$

In turn, (4.88) can be looked upon as a vector inequality

$$I(t) \leq c(t) + \int_0^t \mathbf{Q} I(s)\mathrm{d}s, \quad 0 \leq t \leq T < \infty, \tag{4.91}$$

for $I(t) := (I_v(t))_{v\in\mathbb{V}}$ taking values in the Banach space (cf. (2.11), (3.68))

$$l^1_{\delta'}(\mathbb{V}) := \left\{ u = (u_{v'})_{v'\in\mathbb{V}} \in \mathbb{R}^{\mathbb{V}} \;\middle|\; |u|_{l^1_{o,\delta'}} := \sum_{v'} |u_{v'}| \exp\{-\delta'\rho(o,v')\} < \infty \right\}, \tag{4.92}$$

with $o := v \in \Lambda^-$ and $\delta' := \delta p$. Iterating it $(N-1)$-times we get that

$$I(t) \leq \sum\nolimits_{n=0}^{N} \frac{t^n}{n!}\mathbf{Q}^n c(t) + \int_0^t \int_0^{t_1} \ldots \int_0^{t_{N-1}} \mathbf{Q}^N I(t_N)\mathrm{d}t_N \ldots \mathrm{d}t_2\mathrm{d}t_1. \tag{4.93}$$

A key observation leading to the proof is that

$$\left(\mathbf{Q}^n c(t)\right)_v = 0, \quad \text{for all } n \leq N < \mathrm{dist}(v, \Lambda^c).$$

Therefore, by (4.93)

$$\begin{aligned} I_v(t) &\leq \frac{t^n ||\mathbf{Q}||^n_{\delta'}}{n!} ||I(t)||_{v,\delta p} \\ &\leq 2^{2p+1} e^{2tC^{(p)}_{4.77}} \frac{(tK)^n}{n!} \sum_{v'} (1 + |y_{v'}|^{2p} + |\tilde{y}_{v'}|^{2p}) e^{-p\delta\rho(v,v')}, \quad t \geq 0, \end{aligned} \tag{4.94}$$

whereby, having regard of (4.77), (4.87), and (4.90), we put

$$K := K^{(p)}_{4.94} := p\left(A_8 + m_{\mathbb{G}}\frac{J}{2}\right)\left(1 + m_{\mathbb{G}} e^{\delta p}\right).$$

Thus, for any given K, $M > 0$ it remains to find B, $C > 0$ such that the constraint

$$0 \le tB < \mathrm{dist}(v, \Lambda^c) - 1 \le n < \mathrm{dist}(v, \Lambda^c) \tag{4.95}$$

implies

$$\frac{(tK)^n}{n!} \le C \exp\left\{-2\left[M \mathrm{dist}(v, \Lambda^c) + tC^{(p)}_{4.77}\right]\right\}.$$

Indeed, an elementary calculation based on (4.95) and Stirling's formula $(n! \sim \sqrt{2\pi n}\,(n/e)^n\,,\, n \to \infty)$ shows that

$$\frac{(tK)^n}{n!} \le C_{4.96}\left(\frac{etK}{n}\right)^n \le C_{4.96}(B/eK)^{-\frac{1}{2}[\mathrm{dist}(v,\Lambda^c)+tB-1]}, \tag{4.96}$$

with some universal constant $C_{4.96} > 0$. The claim (i) now follows by choosing

$$B^{(p)}_{4.83} := \max\left\{e^{4M+1}K^{(p)}_{4.94}\ ,\ M^{-1}C^{(p)}_{4.77}\right\}, \quad C^{(p)}_{4.83} := 2^{2p+1}C_{4.96}(B/eK^{(p)}_{4.94})^{\frac{1}{2}}. \tag{4.97}$$

The second claim is proved analogously, whereby we use (4.78) and set $B_{4.84} := (3/2)B^{(1)}_{4.83}$, $C_{4.84} := (3/2)C^{(1)}_{4.83}$. ■

Let $\mu^{\Lambda}_{t,y}(\mathrm{d}x_\Lambda) \in \mathcal{P}(\Omega^{\mathrm{t}})$ denote the probability law of the *finite volume* solution $x^\Lambda(t,y) \in \Omega^{\mathrm{t}}$, and let $\mathbb{T}^{\Lambda,y}_t$, $t \ge 0$, be the corresponding semigroup in $L^2(\Omega_\Lambda, \mu_{\Lambda,y})$ with the generator $\mathbb{H}_{\Lambda,y}$, cf. (2.210).

Corollary 4.17 *In the situation of Proposition 4.16, for all $o \in \Lambda \subset \Delta$ such that*

$$\mathrm{dist}(\Lambda, \Delta^c) > 1 + tB_{4.84},$$

the following convergence of the finite volume projections in the Wasserstein distance holds:

$$\begin{aligned}&\mathbf{W}_\Lambda(\mathbb{P}_\Lambda\mu_{t,y}, \mathbb{P}_\Lambda\mu^{\Delta}_{t,y}) \le C_{4.98}|\Lambda|^{1/2} \\ &\times \exp\left\{\frac{1}{2}\delta \mathrm{diam}(\Lambda) - M\mathrm{dist}(\Lambda, \Delta^c)\right\}\left[\sum_v (1+|y_v|^2)e^{-\delta\rho(v,o)}\right]^{1/2}.\end{aligned} \tag{4.98}$$

For each local $f \in \mathrm{Lip}(\Omega_\Lambda)$, the dual form of this result reads as

$$\begin{aligned}&\left|(\mathbb{T}_t f)(y) - (\mathbb{T}^{\Delta,y}_t f)(y)\right| \le C_{4.98}|\Lambda|^{1/2}[f]_\Lambda \\ &\times \exp\left\{\frac{1}{2}\delta \mathrm{diam}(\Lambda) - M\mathrm{dist}(\Lambda, \Delta^c)\right\}\left[\sum_v (1+|y_v|^2)e^{-\delta\rho(v,o)}\right]^{1/2}.\end{aligned} \tag{4.99}$$

Proof. By (4.83) and the definition of the Wasserstein distance, cf. (2.169),

$$\begin{aligned}&\left[\mathbf{W}_\Lambda(\mathbb{P}_\Lambda\mu_{t,y}, \mathbb{P}_\Lambda\mu^{\Delta}_{t,y})\right]^2 := \sup_{f\in \mathrm{Lip}_1(\Omega_\Lambda)}\left(\int_\Omega f(x_\Lambda)\left[\mu_{t,y} - \mu^{\Delta}_{t,y}\right](\mathrm{d}x)\right)^2 \\ &\le \sup_{f\in \mathrm{Lip}_1(\Omega_\Lambda)} \mathbf{E}|f(x(t,y)) - f(x^\Delta(t,y))|^2 \\ &\le 2C_{4.84}\sum_{v\in\Lambda} e^{-2M\mathrm{dist}(v,\Delta^c)}\sum_{v'}(1+|y_{v'}|^2)e^{-\delta\rho(v,v')},\end{aligned} \tag{4.100}$$

which implies (4.98) with $C_{4.98} := \left(2C^{(1)}_{4.84}\right)^{1/2}$. ■

4.2.5 A-priori estimates on the invariant measures

As was demonstrated in Remark 4.12 (i), the proof of the uniqueness result for $\mu \in \mathcal{I}^{\mathrm{t}}$ in Theorem 4.9 relies on certain *moment estimates* to be hold for all *tempered invariant measures*. Such exponential bounds will be established in Theorem 4.21 below and are formally similar to those stated for the tempered Gibbs measures $\mu \in \mathcal{G}^{\mathrm{t}}$ in Theorem 2.15. In this subsection we collect a number of standard propositions which will be needed to complete the study of the Markov process (4.36) and its invariant measures. Since we could not find a universal reference with the results stated strongly enough for our purposes, a brief outline of the proofs will be given.

The first important property is the *continuous dependence* of the solution $x(t, y)$ on initial conditions y, which has to be valid in each of the Hilbert spaces Ω_δ (with the inner product $(\cdot,\cdot)_{o,\delta} = ||\cdot||^2_{o,\delta}$).

Proposition 4.18 *Let $x(t, y)$ and $x(t, \tilde{y})$ be strong solutions of the Cauchy problem (4.41) with the initial data $y, \tilde{y} \in \Omega^{\mathrm{t}}$ respectively. Then, for each $\delta \in (0, 1]$ we have (almost surely) the following estimates with some positive constants K, L, and $M^{(\delta)}$*

$$(i) \quad ||x(t,y) - x(t,\tilde{y})||_{o,\delta} \le e^{tK}||y - \tilde{y}||_{o,\delta}, \quad t \ge 0; \tag{4.101}$$

$$(ii) \quad ||x(t,y) - x(t,\tilde{y})||_{o,\delta} \le M^{(\delta)} + e^{-tL}||y - \tilde{y}||_{o,\delta}. \tag{4.102}$$

Proof. Setting $z(t) := x(t, y) - x(t, \tilde{y})$ and integrating by parts we obtain

$$\mathrm{d}|z_v(t)|^2 = (z_v(t)\ ,\ b_v(x(t,y)) - b_v(x(t,y)))\,\mathrm{d}t, \quad t > 0.$$

Applying the semi-dissipativity properties (4.76), we come to the estimates

$$\frac{\mathrm{d}}{\mathrm{d}t}|z_v(t)|^2 \le \begin{cases} \beta A_8|z_v(t)|^2 + \beta\frac{J}{2}\sum_{v'\in\partial v}|z_{v'}(t)|^2; \\ -\beta\left(A_7 - \frac{3}{2}Jm_{\mathbb{G}}\right)|z_v(t)|^2 + \beta\frac{J}{2}\sum_{v'\in\partial v}|z_{v'}(t)|^2 - \beta B_7 \end{cases}, \quad v \in \mathbb{V}. \tag{4.103}$$

Then, (i) readily follows with $K := \beta\,(A_8 + Jem_{\mathbb{G}}/2)$ from Lemma 4.22. To prove (ii) we take a weighted sum over v, use the chain rule, and arrive at

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(||z(t)||^2_{o,\delta}e^{tL}\right) \le \beta\Xi_\delta|B_7|e^{tL},\ t > 0; \quad z(0) := ||y - \tilde{y}||_{o,\delta}.$$

Here we put $L := \beta\,[A_7 - Jm_{\mathbb{G}}(3 + e)/2]$, which due to (4.20)–(4.22) is positive for all $\delta \in (0, 1)$. By the scalar Gronwall's inequality this implies (4.102) with $M^{(\delta)} := \beta\Xi_\delta|B_7|/L$. ■

An immediate sequel of the above proposition is the *Feller property* for the transition semigroup $\mathbb{T}_t$, $t \ge 0$, defined by (4.46).

Corollary 4.19 *The semigroup $\mathbb{T}_t f$, $t \geq 0$, is contractive in each of the Banach spaces $C_b(\Omega^{\mathrm{t}})$ and $C_b(\Omega_\delta)$, $\delta > 0$.*

A standard way to establish the à-priori moment estimates *for all* measures $\mu \in \mathcal{I}^{\mathrm{t}}$ is through the so-called *ultimate asymptotic bounds* for the process $x(t, y)$, which have to valid uniformly at all initial conditions $y \in \Omega^{\mathrm{t}}$. The result will depend on the parameter $P \geq 2$ describing the polynomial growth of the self-interaction in Assumption $(\mathbf{V}^*)$.

Proposition 4.20 *Under Assumptions $(\mathbf{V}^*)$, $(\mathbf{W}^*)$, the following holds:*

(i) ***Polynomial ultimate boundedness:*** *For each $p \geq 1$ and $\delta \in (0, 1]$, one finds a corresponding $C^{(p,\delta)}_{4.104} > 0$ such that simultaneously for all $o \in \mathbb{V}$ and $y \in \Omega_\delta$*

$$\mathbf{E}||x(t, y)||^{2p}_{o,\delta} \leq C^{(p,\delta)}_{4.104} + ||y||^{2p}_{o,\delta} \exp\left(-2pt\beta\Delta_U\right), \quad t \geq 0, \tag{4.104}$$

with the positive parameter Δ_U defined in (4.22). Therefore,

$$\sup_{v\in\mathbb{V}} \limsup_{t\to\infty} \left\{\mathbf{E}|x_v(t, y)|^{2p}\right\} \leq \inf_\delta C^{(p,\delta)}_{4.105} =: C^{(p)}_{4.105}. \tag{4.105}$$

(ii) ***Exponential ultimate boundedness:*** *Furthermore, there exists a constant $C_{4.106} > 0$ such that for all $\delta \in (0, 1]$ and small enough $\lambda \in (0, \lambda(\delta)]$*

$$\mathbf{E} \exp\left[\lambda||x(t, y)||^2_{o,\delta}\right] \leq C_{4.106} + \exp\left\{-2t\beta\Delta_U + \lambda||y||^2_{o,\delta}\right\}, \quad t \geq 0, \tag{4.106}$$

and hence

$$\sup_{v\in\mathbb{V}} \limsup_{t\to\infty} \left\{\mathbf{E} \exp\left[\lambda|x_v(t, y)|^2\right]\right\} \leq C_{4.106}. \tag{4.107}$$

Proof. We here prove only (ii), the proof of (i) is similar. Due to the Feller property (4.101), it suffices to verify (4.104) only for boundary condition $y \in \Omega^{\mathrm{t}}$. Consider the family of smooth functions $f_\varepsilon \in C^2_{\mathrm{b}}(\Omega_\delta)$, $\varepsilon > 0$,

$$f_\varepsilon(x) := \frac{e^{\lambda||x||^2_{o,\delta}}}{1 + \varepsilon e^{\lambda||x||^2_{o,\delta}}}, \quad x \in \Omega_\delta,$$

monotonously approximating $f(x) := e^{\lambda||x||^2_{o,\delta}}$ as $\varepsilon \to +0$. This localization procedure is needed since we do not know in advance whether the integral in the right-hand-side in (4.106) is finite. Applying Itô's formula (see e.g. [80, 83]) in the Hilbert space $\Omega_{\delta'}$ to the solution $x(t) := x(t, y) \in \Omega^{\mathrm{t}}$ of the Cauchy problem (4.41), we obtain that

$$\begin{aligned} &\mathrm{d}\mathbf{E}[f_\varepsilon(x(t))] \qquad (4.108) \\ \leq \quad &\lambda\mathbf{E}\left[\frac{(b(x(t)), x(t))_{o,\delta} + \nu\Xi_\delta\left(6\lambda||x(t)||^2_{o,\delta} + 1\right)}{\left(1 + \varepsilon e^{\lambda||x(t)|^2_{o,\delta}}\right)^2} e^{\lambda||x(t)||^2_{o,\delta}}\right] \mathrm{d}t, \end{aligned}$$

whereby according to the coercivity property (4.75)

$$(b(x), x)_{o,\delta} \leq -\beta \left[A_6 - Jm_{\mathbb{G}}(1 + e^{\delta}/2)\right] \cdot ||x||^2_{o,\delta} + \beta \Xi_\delta \left(|B_6| + JD_W/2\right). \qquad (4.109)$$

Substituting (4.109) into (4.108) and using the inequality, cf. (2.238),

$$\lambda e^{\lambda \mathrm{a}} (-\mathrm{a} + \mathrm{b}) \leq -e^{\lambda \mathrm{a}} + e^{\lambda \mathrm{b}}, \quad \lambda, \mathrm{a}, \mathrm{b} \in \mathbb{R}_+, \qquad (4.110)$$

we come to the estimate

$$\mathrm{d}\mathbf{E}\left[f_\varepsilon(x(t))\right] \leq -L\,\mathbf{E}[f_\varepsilon(x(t))]\mathrm{d}t + C_{4.111}, \quad t > 0, \qquad (4.111)$$

with the constants

$$\begin{aligned} L :=& \beta \left[A_6 - Jm_{\mathbb{G}}(1 + e^{\delta}/2)\right] - 6\lambda\nu\Xi_\delta, \\ C_{4.111} :=& \exp\left\{\lambda\nu\Xi_\delta \left[1 + \beta\left(|B_6| + JD_W/2\right)\right]\right\}. \end{aligned}$$

Note that here $L \geq 2\beta\Delta_U$, provided we assume that

$$\lambda \leq \lambda(\delta) := \beta J m_{\mathbb{G}}/(12\nu\Xi_\delta) \quad \text{and} \quad A_6 \geq a_U - Jm_{\mathbb{G}}/10.$$

Applying in (4.110) the product rule, scalar Gronwall's inequality, and Fatou's lemma, we conclude that

$$\begin{aligned} \mathbf{E}\exp\left\{\lambda|x(t,y)|^2_\delta\right\} &= \limsup_{\varepsilon \searrow +0} \mathbf{E}\left[f_\varepsilon(x_\Lambda(t))\right] \qquad (4.112) \\ &\leq C_{4.111}\frac{1 - e^{-2t\beta\Delta_U}}{2\beta\Delta_U} + \exp\left\{-2t\beta\Delta_U + \lambda||y||^2_{o,\delta}\right\}. \end{aligned}$$

This yields (4.106) with

$$C_{4.106} := (2\beta\Delta_U)^{-1}\exp\left\{\frac{1}{12}\beta J m_{\mathbb{G}}\left[1 + \beta\left(|B_6| + JD_W/2\right)\right]\right\}.$$

■

Having shown the ultimate boundedness (4.104), (4.107), we now are able to prove our main result describing the set $\mathcal{I}^{\mathrm{t}}$.

Theorem 4.21 *The set of all tempered invariant measures is nonvoid, i.e., $\mathcal{I}^{\mathrm{t}} \neq \varnothing$. With the notation of Proposition 4.20, each of the measures $\mu \in \mathcal{I}^{\mathrm{t}}$ obeys à-priori moment bounds*

$$(i) \qquad \sup_{v \in \mathbb{V}} \int_\Omega |x_v|^{2p}\mathrm{d}\mu \leq \sup_{o \in \mathbb{V}} \mathbf{E}_\mu ||x||^{2p}_{o,\delta} \leq C^{(p)}_{4.104}; \qquad (4.113)$$

$$(ii) \qquad \sup_{v \in \mathbb{V}} \int_\Omega \exp\left(\lambda|x_v|^2\right)\mathrm{d}\mu \leq \sup_{o \in \mathbb{V}} \mathbf{E}_\mu\left(\exp\lambda||x||^2_{o,\delta}\right) \leq C_{4.106}. \qquad (4.114)$$

Proof. A conventional way of proving existence of an invariant measure is to use the *Bogoliubov-Krylov time average* procedure. Let

$$\bar{\mu}_t(B) := \frac{1}{t}\int_0^t \mu_{s,0}(B)\mathrm{d}s, \quad t > 0, \quad B \in \mathcal{B}(\Omega^{\mathrm{t}}), \tag{4.115}$$

be a Cesàro mean for the probability distribution $\mu_{t,0} \in \mathcal{P}(\Omega^{\mathrm{t}})$ of the solution $x(t,0) \in \Omega^{\mathrm{t}}$ starting at $y = 0$. In virtue of (4.104) we have that for all $\delta > 0$

$$\int_\Omega ||x_v||^2_{o,\delta}\mathrm{d}\bar{\mu}_t = \frac{1}{t}\int_0^t \mathbf{E}\left[||x(s,y)||^2_\delta\right]\mathrm{d}s \le C^{(p)}_{4.104}, \quad t > 0,$$

which by Prokhorov's criterion and Lemma 3.30 implies the relative compactness of $\{\bar{\mu}_t\}_{t>0}$ in the weak topology $\mathcal{W}^{\mathrm{t}}$. By the Kolmogorov-Chapman equality and the Feller property of the semigroup $\mathbb{T}_t$, $t \ge 0$, we conclude that each $\mathcal{W}^{\mathrm{t}}$-limit point $\bar{\mu} := \lim \bar{\mu}_{t_n}$, as $t_n \to \infty$, should belong to $\mathcal{I}^{\mathrm{t}}$. Finally, the required bounds (i), (ii) follow from Proposition 4.20 by straightforward arguments based on Fatou's lemma and the ergodic theorem for invariant distributions, see e.g. [149]. ■

The à-priori bounds (4.104) and (4.113) play a crucial role in proving Proposition 4.8 describing the generator of the semigroup $\mathbb{T}_t$, $t \ge 0$.

Proof of Proposition 4.8. For each $\mu \in \mathcal{G}^{\mathrm{t}}$, the essential self-adjointness of the corresponding operator $\mathbb{H}_\mu$ on the domain $\mathcal{F}C^\infty_{\mathrm{b}}(\Omega)$ will be verified later in Theorem 4.61. So, it would suffice to check the relation

$$\lim_{t\to+0}\left\|\left(\mathbb{H}_\mu + \frac{\mathbb{T}_t - \mathbf{1}}{t}\right)f\right\|^2_{L^2(\mu)} = 0, \quad \forall f \in \mathcal{F}C^\infty_{\mathrm{b}}(\Omega). \tag{4.116}$$

To this end, we follow a standard scheme already used for similar purposes in [16, 166, 167]. By (4.45) and Itô's formula, the right-hand side in (4.116) can be rewritten as

$$\int_\Omega \left|(\mathbb{H}_\mu f)(y) - \frac{1}{t}\int_0^t \mathbf{E}(\mathbb{H}_\mu f)(x(s,y))\mathrm{d}s\right|^2 \mathrm{d}\mu(y). \tag{4.117}$$

Note that $(\mathbb{H}_\mu f)(x(t,y))$ is (almost surely) continuous in each of the variables $t \in [0,1]$ and $y \in \Omega_\delta$ due to the same property of the process $x(t,y) \in \Omega_\delta$. Furthermore, $|(\mathbb{H}_\mu f)(x)| \le C_f\left(1 + ||x||^P_{o,\delta}\right)$ and thus by (4.104)

$$\sup_{t\in[0,1]} \mathbf{E}\left|(\mathbb{H}_\mu f)(x(t,y))\right|^Q < C_{f,Q}\left((1 + ||y||^{PQ}_{o,\delta}\right), \quad \forall Q \ge 1. \tag{4.118}$$

By the uniform integrability argument (Vallée-Poussin theorem, see pages 16-17 in [190]), this implies the continuity of the mapping $[0,1] \ni t \to \mathbf{E}(\mathbb{H}_\mu f)(x(t,y))$ and the relation

$$\lim_{t\to+0}\frac{1}{t}\int_0^t \mathbf{E}(\mathbb{H}_\mu f)(x(s,y))\mathrm{d}s = (\mathbb{H}_\mu f)(y), \quad y \in \Omega^{\mathrm{t}}.$$

To pass in (4.117) to the limit under the integral with respect to μ, we may apply Lebesgue's dominated convergence theorem. This procedure is legitimate in virtue of (4.118) and the à-priori moment bound (4.113) satisfied for all $\mu \in \mathcal{G}^{\mathrm{t}} \subseteq \mathcal{I}^{\mathrm{t}}$. ■

We conclude this subsection with the formulation of an infinite dimensional version of the classical *Gronwall's inequality,* which was repeatedly used in the proofs above.

Lemma 4.22 *(see e.g. [261]) Let us given scalar sequences* $\gamma := (\gamma_v > 0)_{v\in\mathbb{V}}$, $c := (c_v \geq 0)_{v\in\mathbb{V}}$, *a matrix* $\mathbf{Q} = (Q_{vv'} \geq 0)_{\mathbb{V}\times\mathbb{V}}$, *and a family* $\{f_v(t) \geq 0\}_{v\in\mathbb{V}}$ *of measurable functions defined on a finite interval* $[0,T]$*. Suppose that*

$$f_v(t) \leq c_v + \sum\nolimits_{v'} Q_{vv'} \int_0^t f_{v'}(s)\mathrm{d}s, \quad t \in [0,T], \quad v \in \mathbb{V}, \tag{4.119}$$

with the additional assumptions

$$(i) \quad \sum\nolimits_v \gamma_v \sup_{0\leq t\leq T} f_v(t) < \infty; \quad (ii) \quad ||c||_{l^1(\gamma)} := \sum\nolimits_v \gamma_v c_v < \infty,$$
$$(iii) \quad ||\mathbf{Q}||_{l^1(\gamma)} := \sup_{v'} \sum\nolimits_{v'} Q_{vv'}\gamma_v\gamma_{v'}^{-1} < \infty. \tag{4.120}$$

Then (4.119) implies the following bound

$$\sum\nolimits_v \gamma_v \sup_{0\leq t\leq T} f_v(t) \leq ||c||_{l^1(\gamma)} e^{T||\mathbf{Q}||_{l^1(\gamma)}}. \tag{4.121}$$

Remark 4.23 In [24], Lemma 7.3, the following *regularization* effect was observed. Suppose additionally that the conditions (ii), (iii) are fulfilled for a stronger system of weights $\tilde{\gamma}_v \geq \gamma_v$. Then it also holds $\left\{\sup_{0\leq t\leq T} f_v(t)\right\}_{v\in\mathbb{V}} \in l^1(\tilde{\gamma})$ and

$$\sum\nolimits_v \tilde{\gamma}_v \sup_{0\leq t\leq T} f_v(t) \leq ||c||_{l^1(\tilde{\gamma})} e^{T||\mathbf{Q}||_{l^1(\tilde{\gamma})}}. \tag{4.122}$$

Indeed, (4.119) yields that $\sup_{0\leq t\leq T} f(t) \leq e^{T\mathbf{Q}}c$, understood as the inequality for vector-columns in $l^1(\gamma)$. Estimating the norm of $e^{T\mathbf{Q}}c$ in $l^1(\tilde{\gamma})$, we get the desired bound (4.122). Actually, such property allows us to show (even without the technical Assumption $(\mathbf{G}_0)$) that, for each fixed $y \in \Omega_\delta$ and $\delta > \delta_{\mathbb{G}}$, the corresponding solution $x(t,y)$ has to live in the same space, that means $x(\cdot,y) \in C([0,\infty) \to \Omega_\delta)$.

4.2.6 Stochastic quantization dynamics

Based on the earlier contributions [273]–[276] of the author and their further developments in the joint papers [24, 25], here we briefly describe the main ingredients of the stochastic dynamics method being applied to the quantum analog (3.1), (3.2) of the classical spin model (4.10). In physics such approach is also called the *"stochastic quantization procedure"* – a terminology going back to the pioneering article of G. Parisi and Y. Wu (1981), where they dealt with quantum fields models (see e.g. [194] for numerous references on the latter subject).

Keeping our basic Assumptions $(\mathbf{G}_0)$, $(\mathbf{V}^*)$, and $(\mathbf{W}^*)$, below we explain how to extend the main results of this section to the Euclidean Gibbs measures and the Glauber dynamics on loop spaces associated with them. A new technical issue is that such dynamics will be governed by an infinite number of *partial differential equations* of *parabolic type*, perturbed by the *space-time white noise*. So, to investigate the regularity properties of the solution $\omega(t)$ we have to perform an additional analysis of its components $\omega_v(t)$, $v \in \mathbb{V}$, in different functional spaces such as C_β^σ, L_β^r, W_β^1, etc. Note that in the most generality the method applies to the systems of multi-component spins (like that introduced in Subsection 3.2.6) interacting via many-particle potentials $W_{\{v_1,\dots,v_N\}}$ of *at most quadratic growth*, whereas in the above-mentioned papers we have restricted ourselves to a simpler case of $\mathbb{V} := \mathbb{Z}^d$, scalar $q_v \in \mathbb{R}$, and the harmonic pair interactions $W_{vv'}(q_v, q_{v'}) = -J_{vv'} q_v q_{v'}$ given by a dynamical matrix $0 \le \mathbf{J} := (J_{\ell\ell'})_{\mathbb{Z}^d \times \mathbb{Z}^d} \in \mathcal{L}(l^2(\mathbb{Z}^d))$.

So, our goal is to construct a Markov process $\omega(t) := (\omega_v(t))_{v \in \mathbb{V}}$, $t \ge 0$, which solves uniquely an infinite dimensional *SDE* with the identity diffusion matrix and the "*drift term*"

$$b(\omega) = (b_v(\omega))_{v \in \mathbb{V}}, \quad b_v(\omega) := -A\omega_v - F_v(\omega),$$

being the *logarithmic gradient* of the measures $\mu \in \mathcal{G}^{\mathrm{t}}$, see Subsection 4.5.3. More precisely, we define the weighted Banach spaces of continuous loops

$$\mathcal{C}_\delta := l_\delta^2(\mathbb{V} \to C_\beta) := \left\{ \omega \in \Omega \;\middle|\; \|\omega\|_{\mathcal{C}_\delta} := \left[\sum_v |\omega_v|_{C_\beta}^2 \exp\{-\delta \rho(\nu, o)\} \right]^{1/2} < \infty \right\}, \quad \delta > 0,$$

and the locally convex Polish space

$$\mathcal{C}^{\mathrm{t}} := \bigcap_{\delta > 0} \mathcal{C}_\delta \subset \Omega^{\mathrm{t}},$$

with the projective topology induced by the system of norms $\|\omega\|_{\mathcal{C}_\delta}$, $\delta > 0$. Then, the process $\omega(t)$, $t \ge 0$, takes values in $\mathcal{C}^{\mathrm{t}}$ and satisfies the following infinite system of stochastic partial differential equations (*SPDE*'s)

$$\begin{cases} \dfrac{\partial}{\partial t}\omega_v(t) = -\dfrac{1}{2}\left[A\omega_v(t) + F_v(\omega(t))\right] + \dot{w}_v(t), \\ \qquad\qquad v \in \mathbb{V} \quad (t > 0,\ \tau \in S_\beta). \end{cases} \tag{4.123}$$

Here $A = (-m\mathrm{d}^2/\mathrm{d}\tau^2 + a) \otimes \mathbf{Id}_\nu$ is the Laplace-Beltrami type operator on the circle S_β introduced in Subsection 3.1.3 and $F_v : \Omega \to L_\beta^2$ is the nonlinear Nemytskii-type operator acting by

$$F_v(\omega) := V_v'(\omega_\ell) + \sum\nolimits_{v'(\neq v)} \partial_{q_v} W_{vv'}(\omega_v, \omega_{v'}), \quad v \in \mathbb{V}. \tag{4.124}$$

The solution to (4.123) will always be understood in the strong probability sense, which means that we fix a probability space $(\Omega, \mathcal{F}, \mathbf{P})$ with a filtration $(\mathcal{F}_t)_{t \ge 0}$, and a family

$w(t) = (w_v(t))_{v\in\mathbb{V}}$ of independent $\mathcal{D}'(S_\beta)$-valued Brownian motions such that for any $h_1, h_2 \in \mathcal{D}(S_\beta) := C^\infty_\beta$

$$\mathbf{E}\left[(w_{v_1}(t_1), h_1)_{L^2_\beta}(w_{v_2}(t_2), h_2)_{L^2_\beta}\right] = \delta(v_1 - v_2)\,\min\{t_1, t_2\}\,(h_1, h_2)_{L^2_\beta}.$$

In the classical case when the continuous parameter $\tau \in S_\beta$ is absent, we just obtain the system of interacting diffusions (4.36). On the other hand, each single line in (4.123) is a reaction-diffusion equation with the periodic boundary conditions on $[0, \beta]$, driven by a singular random force $\dot{w}_v(t)$. Such equations, under different regularity assumptions on the random forces and drifts, are of great interest in stochastics and mathematical physics, see e.g. the monographs [77, 83, 235] devoted to this topic.

Definition 4.24 *By the* ***generalized solution*** *(in the commonly accepted sense of* PDE*'s) of the Cauchy problem to the system (4.123) with nonrandom initial data* $\xi \in \mathcal{C}^{\mathrm{t}}$ *we mean an* $(\mathcal{F}_t)$*-adapted continuous process* $\omega(t,\xi) = (\omega_v(t,\xi))_{v\in\mathbb{V}} \in \mathcal{C}^{\mathrm{t}}$ *such that, for every* $v \in \mathbb{V}$ *and* $h \in \mathcal{D}(S_\beta)$ *almost surely*

$$\begin{aligned}(\omega_v(t,\xi), h)_{L^2_\beta} &= (\zeta_v + w_v(t), h)_{L^2_\beta} \\ &- \frac{1}{2}\int_0^t \left[(\omega_v(s,\xi), Ah)_{L^2_\beta} + (F_v(\omega(s,\xi), h)_{L^2_\beta}\right] \mathrm{d}s, \quad t \geq 0.\end{aligned} \tag{4.125}$$

In the trivial case $V_v = W_{vv'} = 0$, the solution of (4.123) is given explicitly by the *Ornstein–Uhlenbeck process* $g(t,\xi) = (g_v(t,\xi))_{v\in\mathbb{V}}$,

$$g_v(t) := e^{-tA/2}\xi_v + \int_0^t e^{(t-s)A/2}\mathrm{d}w_v(s), \quad v \in \mathbb{V},\ t \geq 0. \tag{4.126}$$

Taking into account the regularity properties of the semigroup kernels, cf. (3.44),

$$\left(e^{-tA}\delta_\tau\right)(\tau') := \mathfrak{K}(t;\tau,\tau') \otimes \mathbf{Id}_\nu \in \mathbb{R}^\nu, \quad \tau,\tau' \in S_\beta,$$

(by the stochastic Fubini theorem, cf. page 109 of [83]; or alternatively by the Garsia-Rodemich-Rumsey lemma, see Remark 3.4) one can deduce from (4.126) that $g(t,\xi)$, $t \geq 0$, possesses a continuous modification in the spaces of Hölder loops $\mathcal{C}^\sigma_\delta := l^2_\delta(\mathbb{V} \to C^\sigma_\beta)$, $\delta > 0$, $\sigma \in (0, 1/2)$, and its polynomial moments are *ultimately bounded*, i.e.,

$$\limsup_{t\to\infty} \mathbf{E}|g_v(t,\xi)|^Q_{C^\alpha_\beta} =: C^{(Q,\sigma)}_{4.127} < \infty, \quad \forall Q \geq 1, \tag{4.127}$$

uniformly for all $v \in \mathbb{V}$ and initial values $g(0) := \xi \in \mathcal{C}_\delta$. Moreover, the process $g(t)$, $t \geq 0$, is *ergodic* with the *unique invariant* (and also reversible) distribution $\gamma_{\mathbb{V}}(\mathrm{d}\omega) := \prod_{v\in\mathbb{V}} \gamma(\mathrm{d}\omega_v)$, and the laws of $g(t)$ weakly converge in $\mathcal{C}^{\mathrm{t}}$ to this $\gamma_{\mathbb{V}}$ as $t \to \infty$.

A standard practice then consists of replacing (4.123) with the equivalent system of integral equations

$$\omega^i_v(t,\xi) = g^i_v(t,\xi) - \frac{1}{2}\int_0^t \mathfrak{K}\left[(t-s)/2;\tau,\tau'\right] F^i_v(\omega(s,\xi))\mathrm{d}s, \quad v \in \mathbb{V},\ 1 \leq i \leq \nu,\ t \geq 0. \tag{4.128}$$

The meaning of the so-called *mild solution* to (4.128) is similar to that in Definition 4.24, i.e., by pairing of the both sides with the test functions $h \in C_\beta^\infty$. Using the finite volume approximations (cf. (4.56) and (4.134)), in Theorem 8.2 of [24] we proved that under the above assumptions there exists the unique solution $\omega(\cdot, \xi) \in C([0,\infty) \to \mathcal{C}^{\mathrm{t}})$ starting from each initial data $\xi \in \mathcal{C}^{\mathrm{t}}$.

It is reasonable to compare the solution $\omega(t)$, $t \geq 0$, of the nonlinear problem (4.123) with the Gaussian process $g(t)$, $t \geq 0$. If their initial values coincide, i.e., $\xi := \omega(0) = g(0)$, then for the *deviation process* $\eta(t,\xi) := \omega(t,\xi) - g(t,\xi)$ solving

$$\eta_v(t,\xi) := \omega_v(t,\xi) - g_v(t,\xi) = -\int_0^t e^{-(t-s)A/2} F_v(\omega(s),\xi)\mathrm{d}s, \quad v \in \mathbb{V},\ t \geq 0,$$

some helpful *energy estimates* hold. So, by Lemma 10.1 and Remark 10.3 in [24], we have that for any $Q \geq 1$

$$\limsup_{t\to\infty} \mathbf{E}|\eta_v(t,\xi)|^Q_{C_\beta} =: C^{(Q)}_{4.129} < \infty, \tag{4.129}$$

$$\limsup_{t\to\infty} \frac{1}{t}\int_t^{2t} \mathbf{E}|\eta_v(s,\xi)|^2_{W^1_\beta}\mathrm{d}s =: C_{4.130} < \infty, \tag{4.130}$$

uniformly for all $v \in \mathbb{V}$ and initial values $\omega(0) = g(0) := \xi \in \mathcal{C}^{\mathrm{t}}$. It is important that the Sobolev space W^1_β, being defined as completion of C^∞_β for the norm $|v|_{W^1_\beta} := (Av,v)^{1/2}_{L^2_\beta}$, is compactly embedded in the Hölder spaces C^σ_β, $\sigma \in (0,1/2)$. The proof of the above bounds relies on the coercivity and semi-monotonicity properties of the mappings $F_v(\omega)$, cf. (4.75), (4.76). Combining (4.127) and (4.130), we get the following estimates for the process $\omega(t,\xi)$, $t \geq 0$, to be crucially used in the sequel: for all $v \in \mathbb{V}$ and $\xi \in \mathcal{C}^{\mathrm{t}}$

$$\limsup_{t\to\infty} \mathbf{E}|\omega_v(s,\xi)|^Q_{C_\beta} =: C^{(Q)}_{4.131} < \infty, \tag{4.131}$$

$$\limsup_{t\to\infty} \frac{1}{t}\int_t^{2t} \mathbf{E}|\omega_v(s,\xi)|^2_{C^\sigma_\beta}\mathrm{d}s =: C^{(\sigma)}_{4.132} < \infty. \tag{4.132}$$

As was further shown in [24],

$$(\mathbb{T}_t f)(\xi) := \mathbf{E}\{f(\omega(t))\,|\,\omega(0) = \xi\}, \quad \xi \in \mathcal{C}^{\mathrm{t}},$$

is a Feller transition semigroup in the space $C_{\mathrm{b}}(\mathcal{C}^{\mathrm{t}})$ of all bounded continuous functions $f : \mathcal{C}^{\mathrm{t}} \to \mathbb{R}$. Let $\mathcal{R}^{\mathrm{t}}$ and $\mathcal{I}^{\mathrm{t}}$ denote respectively the family of all tempered *reversible* and *invariant* distributions $\mu \in \mathcal{P}(\mathcal{C}^{\mathrm{t}})$ for the Markov process $\omega(t)$, $t \geq 0$, in the sense of the definitions (4.43) and (4.48). Then, similarly to (4.51), the following relation is true

$$\mathcal{G}^{\mathrm{t}} \;=\; \mathcal{R}^{\mathrm{t}} \;\subseteq\; \mathcal{I}^{\mathrm{t}} \tag{4.133}$$

(for its proof involving the Itô stochastic calculus and (*IbP*)-formulas see e.g. [119, 165]). Moreover, in our situation one can directly verify (cf. e.g. [119, 153]) that the finite volume Gibbs measures $\mu_\Lambda(\mathrm{d}\omega|\xi)$, $\Lambda \Subset \mathbb{V}$, $\xi \in \mathcal{C}^{\mathrm{t}}$, are exactly the reversible distributions for the corresponding *cut-off* dynamics $\omega^\Lambda(t) = (\omega_v^\Lambda(t,\xi))_{v\in\Lambda}$ in C_β^Λ. They solve the finite volume problems

$$\begin{cases} \dfrac{\partial}{\partial t}\omega_v^\Lambda(t) = -\dfrac{1}{2}\left[A\omega_v^\Lambda(t) + F_v^\Lambda(\omega(t))\right] + \dot{w}_v(t), \\ \qquad\qquad v \in \Lambda \quad (t > 0,\ \tau \in S_\beta), \end{cases} \tag{4.134}$$

with the initial data $\omega(0) := \xi$ and boundary conditions $\omega_v(t,\xi) := \xi_v$ for $v \in \Lambda^c$. An important point is that the bounds analogues to (4.129)–(4.132) hold also for the solutions $\omega^\Lambda(t,\xi)$, $t \geq 0$, which means that for all $v \in \Lambda \Subset \mathbb{V}$

$$\limsup_{t\to\infty} \mathbf{E}|\omega_v(s,\xi)|_{C_\beta}^Q =: C_{4.135}^{(Q)}(\xi) < \infty, \tag{4.135}$$

$$\limsup_{t\to\infty} \frac{1}{t}\int_t^{2t} \mathbf{E}|\omega_v^\Lambda(s,\xi)|_{C_\beta^\sigma}^2 \,\mathrm{d}s =: C_{4.136}^{(\sigma)}(\xi) < \infty. \tag{4.136}$$

Thus, in order to get the required information on $\mu \in \mathcal{G}^{\mathrm{t}} \subseteq \mathcal{I}^{\mathrm{t}}$ one could apply standard tools used for the long-time analysis of diffusion processes. So, by the *ergodic theorem* for invariant distributions, (4.131) and (4.132) readily imply that

$$\sup_{\mu\in\mathcal{I}^{\mathrm{t}},\ v\in\mathbb{V}} \int_\Omega |\omega_v|_{C_\beta}^Q \,\mathrm{d}\mu(\omega) \ \leq \ C_{4.131}^{(Q,\sigma)}, \tag{4.137}$$

$$\sup_{\mu\in\mathcal{I}^{\mathrm{t}},\ v\in\mathbb{V}} \int_\Omega |\omega_v|_{C_\beta^\sigma}^2 \,\mathrm{d}\mu(\omega) \ \leq \ C_{4.132}^{(\sigma)}, \tag{4.138}$$

which generalizes the corresponding result of Theorem 4.21 to the quantum case. The existence of invariant measures $\mu \in \mathcal{I}^{\mathrm{t}}$ is then a standard consequence of the estimate (4.136) for $\xi = 0$, Prokhorov's tightness criterion, and the Bogoliubov-Krylov argument (see the proof of Theorem 4.21 above). To verify the statement of Theorem 3.18 about existence of the Euclidean Gibbs measures $\mu \in \mathcal{G}^{\mathrm{t}} = \mathcal{R}^{\mathrm{t}}$, it suffices to prove the tightness in $\mathcal{C}^{\mathrm{t}}$ of the family of local kernels $\{\pi_\Lambda(\mathrm{d}\omega|0)\}_{\Lambda\Subset\mathbb{V}}$. By Prokhorov's criterion this would be a consequence of the uniform bound

$$\sup_{v\in\Lambda\Subset\mathbb{V}} \int_\Omega |\omega_v|_{C_\beta^\sigma}^2 \,\pi_\Lambda(\mathrm{d}\omega|0) < \infty \tag{4.139}$$

(see Subsection 3.2.4). Since the finite volume dynamics (4.134) are ergodic, (4.139) immediately follows from the estimate (4.137) above.

In conclusion let us briefly analyze the situation with the ergodicity result like that proved in Theorem 4.9. First of all we have to fix the *universal tangent* Hilbert space $\mathcal{H}_0 = l^2(V) \otimes L_\beta^2$, which will be used for constructing the associated *Dirichlet forms* and *operators*, as well as for defining the *Wasserstein distances* on the spaces of probability distributions. Similarly to Theorem 4.16 in the classical case, we can establish the *finite propagation* property for the dynamics (4.134). Adapted to the single spin spaces $L_\beta^2 \ni \omega_v$, it now reads as follows:

Proposition 4.25 *For each* $\delta, M > 0$ *one finds* $B_{4.140} = B_{4.140}(\delta, M) > 0$ *and* $C_{4.140} := C_{4.140}(\delta, M)$*, such that for all domains* $\Lambda \subset \Delta \subseteq \mathbb{V}$*, boundary conditions* $\xi, \tilde{\xi} \in \mathcal{C}^{\mathrm{t}}$*,* $\xi \equiv \tilde{\xi}$ *on* Λ*, and inner points* $v \in \Lambda^{-}$ *with* $\operatorname{dist}(v, \Lambda^c) \geq 1 + tB_{4.140}$*, the following estimate for the corresponding solutions hold:*

$$\begin{aligned} &\mathbf{E}|\omega_v^{\Lambda}(t,\xi) - \omega_v^{\Delta}(t,\tilde{\xi})|^2_{L^2_\beta} \\ &\leq C_{4.140} e^{-2M \operatorname{dist}(v,\Lambda^c)} \sum_{v'} (1 + |\xi_{v'}|^2_{L^2_\beta} + |\xi_{v'}|^2_{L^2_\beta}) e^{-\delta\rho(v,v')}. \end{aligned} \tag{4.140}$$

The uniform log-Sobolev inequalities for the local Gibbs distributions $\mu_{\Lambda,\xi}$ will be established in Theorem 4.56 below. The required convergence $\mu_{\Lambda,\xi} \to \mu \in \mathcal{G}^{\mathrm{t}}$ in the Wasserstein distances is implied by the Dobrushin contraction condition, see Theorem 4.38 and the proof of Theorem 3.23. So the only principal ingredient missing is the entropy estimates and *Talagrand transportation inequalities* in the loop space L^2_β, which would allow us to describe the relaxation of the finite volume dynamics (4.134) in the Wasserstein distances. We leave the latter problem as a task for the future.

4.3 Entropic control of the dynamics

This section is dedicated to the study of ergodicity properties of the finite volume Glauber dynamics (4.57) associated with the local Gibbs distributions $\mu_{\Lambda,y}$, $\Lambda \Subset \mathbb{V}$, $y \in \Omega$. Especially, we are interested in *dimension free* estimates on the convergence $\mu^{\Lambda}_{t,y} \to \mu_{\Lambda,y}$ of the transition probabilities in the Wasserstein distance. This will be realized through such analytical tools as log-Sobolev and Talagrand's inequalities which are discussed in Subsection 4.3.1. To get explicit bounds on the corresponding log-Sobolev constants $C_{\mathrm{LS}}(\Lambda, y)$, in Subsection 4.3.2 we shall apply a novel criterion for the log-Sobolev inequality suggested by F. Otto and M. Reznikoff (see Theorem 1 of [221]). In Subsection 4.3.3 we demonstrate how to estimate the relative entropy for diffusion processes by using Girsanov's formula for the corresponding probability densities. All this taken together yields the result of *Corollary 4.34*, which will be needed to perform *Step II* in the proof of Theorem 4.9.

4.3.1 Log-Sobolev and Talagrand inequalities

In this subsection we give a summary of the basic results known about the *entropy interpretation* of log-Sobolev inequalities and how they can be used to control the relaxation of the corresponding stochastic dynamics. Remind that the log-Sobolev inequality, cf. (4.143), is attached to a Markov semigroup $\mathbb{T}_t := e^{-t\mathbb{H}_\mu}$, $t \geq 0$, generated by the Dirichlet operator $\mathbb{H}_\mu$ with some symmetrizing measure μ. A key point for us will be *Talagrand's transportation inequality*, see Proposition 4.27, which allows to bound the Wasserstein distance $\mathbf{W}(\nu, \mu)$ through the relative entropy $\mathbf{H}(\nu|\mu)$, and hence to show the exponential convergence $\mathbf{W}(\nu_t, \mu) \to 0$ for the dual semigroup $\nu_t := \nu\mathbb{T}_t$, as $t \to \infty$, see Corollary 4.28. For a general introduction to log-Sobolev inequalities we refer to the standard textbooks and surveys [33, 88, 135, 137, 156, 186].

The most natural framework in finite dimensions is the Euclidean space $(\mathbb{R}^n, |\cdot|)$. Let us given a "*reference*" probability measure $\mu(\mathrm{d}x) = \frac{1}{Z}e^{-\Phi(x)}\mathrm{d}x$ with the smooth density $\Phi \in C^2(\mathbb{R}^n)$. Concerning the logarithmic derivative $b(x) := -\nabla\Phi(x) : \mathbb{R}^n \to \mathbb{R}^n$, we assume that $|b| \in L^2(\mu)$. This allows us to introduce the symmetric differential operator in (the complexification of) $L^2(\mu)$

$$\mathbb{H}_\mu f := -\left[\Delta f + (b, \nabla f)\right], \qquad f \in C_0^\infty(\mathbb{R}^n), \tag{4.141}$$

$$\mathcal{E}_\mu(f,g) := (\mathbb{H}_\mu f, g)_{L^2(\mu)} = \int (\nabla f, \nabla g)\mathrm{d}\mu, \quad f,g \in C_0^\infty(\mathbb{R}^n). \tag{4.142}$$

By $(\mathcal{E}_\mu, \mathcal{D}(\mathcal{E}_\mu))$ we denote the canonical Dirichlet form being the closure of (4.142) and by $(\mathbb{H}_\mu, \mathcal{D}(\mathbb{H}_\mu))$ respectively the self-adjoint Dirichlet operator constructed as the Friedrichs extension of (4.141) with $\mathcal{D}(\mathbb{H}_\mu^{1/2}) = \mathcal{D}(\mathcal{E}_\mu)$, see Subsection 2.3.5 (ii). Recall that the measure μ satisfies the *log-Sobolev inequality* (**LS**, for short) if there exists some $C_{\mathrm{LS}} > 0$ such that

$$\begin{aligned}(\mathbf{LS}) \quad \mathbf{Ent}_\mu(f^2) \quad : \quad &= \int f^2 \log f^2 \mathrm{d}\mu - \int f^2 \mathrm{d}\mu \cdot \log \int f^2 \mathrm{d}\mu \\ &\le \frac{2}{C_{\mathrm{LS}}} \int |\nabla f|^2 \mathrm{d}\mu = \frac{2}{C_{\mathrm{LS}}} \mathcal{E}_\mu(f,f), \quad \forall f \in \mathcal{D}(\mathbb{H}_\mu^{1/2}).\end{aligned} \tag{4.143}$$

By the fundamental result of L. Gross [133, 135], (4.143) is equivalent to the *hypercontractivity* of the semigroup $\mathbb{T}_t := e^{-t\mathbb{H}_\mu}$, $t \ge 0$, in the sense that $\mathbb{T}_t : L^{\mathrm{r}}(\mu) \to L^{\mathrm{q}}(\mu)$ is a contraction for all r, q > 1 and $t > 0$ related by $\exp(-2tC_{\mathrm{LS}}) \le (\mathrm{q}-1)/(\mathrm{r}-1)$. By the Rothaus-Simon mass gap theorem [245, 256], the log-Sobolev inequality implies the *Poincaré or spectral gap inequality* (see Subsection 2.3.5 (ii)) with the constant $C_{\mathrm{SG}} \ge C_{\mathrm{LS}}$,

$$(\mathbf{SG}) \quad \mathbf{Var}_\mu(f) := \int f^2 \mathrm{d}\mu - \left(\int f \mathrm{d}\mu\right)^2 \le \frac{1}{C_{\mathrm{SG}}} \int |\nabla f|^2 \mathrm{d}\mu, \quad \forall f \in \mathcal{D}(\mathbb{H}_\mu^{1/2}). \tag{4.144}$$

By the spectral theorem the later is equivalent to the *exponential* (or *geometrical*) L^2-*ergodicity* of the semigroup $\mathbb{T}_t$, $t \ge 0$,

$$||\mathbb{T}_t f - \mathbf{E}_\mu f||_{L^2(\mu)} \le e^{-tC_{\mathrm{SG}}} ||f||_{L^2(\mu)}, \quad \forall f \in L^2(\mu). \tag{4.145}$$

Clearly, it would be enough to check the above inequalities (4.143), (4.144) on $C_0^\infty(\mathbb{R}^n)$ or another domain which is dense in $\mathcal{D}(\mathbb{H}_\mu^{1/2})$. In this respect it is worth noting that $|b| \in L^4(\mu)$ is known as a sufficient condition in finite dimensions for the essential self-adjointness of $\mathbb{H}_\mu$ on $C_0^\infty(\mathbb{R}^n)$, see (4.331).

Definition 4.26 *For a probability measure ν, its* ***relative entropy*** *(also known as* informational divergence *or* Kullback information*) with respect to μ is given by the formula*

$$\mathbf{H}(\nu|\mu) = \begin{cases} \int \rho \log \rho \, \mathrm{d}\mu, & \text{if } \nu \ll \mu \text{ and } \rho := \mathrm{d}\nu/\mathrm{d}\mu; \\ +\infty, & \text{otherwise.} \end{cases} \tag{4.146}$$

where $\nu \ll \mu$ means that ν is absolutely continuous with respect to μ with the Radon-Nikodym derivative $\rho \in L^1(\mu)$.

An application of Jensen's inequality to the convex function $\mathbb{R}_+ \ni \rho \to \rho \log \rho$ shows that $\nu \to \mathbf{H}(\nu|\mu)$ is a convex, lower continuous functional taking its values in $\mathbb{R}_+ \cup \{+\infty\}$ and vanishing iff $\nu = \mu$ (cf. e.g. Proposition 15.5 in [122]). If μ is an *invariant measure* of some Markov process, a characteristic property of the entropy is its *monotonicity* under the associated time evolution

$$\mathbf{H}(\nu_t|\mu) \leq \mathbf{H}(\nu|\mu), \quad \forall \nu \ll \mu, \quad \forall t \geq 0, \tag{4.147}$$

where $\nu_t := \nu \mathbb{T}_t \ll \mu$ and $\rho_t := \mathrm{d}\nu_t / \mathrm{d}\mu = \mathbb{T}_t \rho \in L^1(\mu)$ (cf. e.g. Proposition 9.1 in [156] or Proposition 1 in [78]). Moreover, if μ satisfies the log-Sobolev inequality, then (4.143) can be restated as the entropy bound

$$\mathbf{H}(\nu|\mu) = \mathrm{Ent}_\mu(\rho) \leq \frac{2}{C_{\mathrm{LS}}} \int |\nabla \rho^{1/2}|^2 \mathrm{d}\mu, \tag{4.148}$$

valid for probability measures $\nu := \rho\mu$ with "nice" densities ρ such that $\rho^{1/2} \in \mathcal{D}(\mathbb{H}_\mu^{1/2})$. In turn, (4.148) is equivalent to the exponential decay of the entropy

$$\mathbf{H}(\nu_t|\mu) \leq e^{-2tC_{\mathrm{LS}}} \mathbf{H}(\nu|\mu), \qquad \forall \nu \ll \mu, \quad \forall t \geq 0, \tag{4.149}$$

(cf. page 199 of [268], page 250 of [88]). By the classical (and easy-to-check) *Csiszár-Kullback-Pinsky inequality* (cf. e.g. page 76 of [88]), the entropy always dominates (up to factor 2) the square of the total variation distance, i.e.,

$$||\mu - \nu||_{\mathrm{TV}} := \sup_{|f|_{L^\infty} \leq 1} \left| \int f \mu(\mathrm{d}x) - \int f \nu(\mathrm{d}x) \right| \leq \sqrt{2\mathbf{H}(\nu|\mu)}. \tag{4.150}$$

The much deeper fact to be used below (see Proposition 4.27) is the so-called $\mathbf{W}_2$*-transportation* or *Talagrand's inequality* , which relates the entropy and (L^2-) Wasserstein distance. Let $\mathcal{P}_p(\mathbb{R}^n)$ denote the subset of all probability measures ν on $(\mathbb{R}^n, \mathcal{B}(\mathbb{R}^n))$ having finite moments $\mathbf{E}_\nu |x|^p < \infty$ with a given $p \geq 1$. For $\nu, \tilde{\nu} \in \mathcal{P}_p(\mathbb{R}^n)$, the ($L^p$-) *Wasserstein distance* is defined as

$$\mathbf{W}_p(\nu, \tilde{\nu}) := \inf_P \left(\iint_{\mathbb{R}^{2n}} |x - \tilde{x}|^p P(\mathrm{d}x, \mathrm{d}\tilde{x}) \right)^{1/p}, \tag{4.151}$$

where the infimum is taken over all probability measures $P \in \mathcal{P}(\mathbb{R}^{2n})$ having marginal distributions ν and $\tilde{\nu}$, cf. Subsection 4.4.1. The value $[\mathbf{W}_p(\nu, \tilde{\nu})]^p$ can be viewed as the minimal cost needed to transport the measure ν into $\tilde{\nu}$, provided that the transportation cost from the point x into $\tilde{x}$ equals $|x - \tilde{x}|^p$. Note that (4.151) extends the definitions (2.169), (4.195) to arbitrary $p \geq 1$, whereby by Hölder's inequality $\mathbf{W}(\nu, \tilde{\nu}) := \mathbf{W}_1(\nu, \tilde{\nu}) \leq \mathbf{W}_p(\nu, \tilde{\nu})$. The following is the statement of Theorem 1 in [222] (see also its improvements in Corollary 3.1 in [51] and Theorem 1.1 in [286]).

Proposition 4.27 *Let*

$$\mu(\mathrm{d}x) := \frac{1}{Z} \exp\{-\Phi(x)\} (\mathrm{d}x)$$

be a probability measure from $\mathcal{P}_2(\mathbb{R}^n)$, such that $\Phi \in C^2(\mathbb{R}^n)$ and $\Phi'' \geq -c\mathbf{Id}_n$ with some $c > 0$. If μ obeys the log-Sobolev inequality (4.143), then it also satisfies the Euclidean $\mathbf{W}_2$***-transportation inequality****,* $(\mathbf{T}_2)$ *for short,*

$$\mathbf{W}_2^2(\nu,\mu) \leq \frac{2}{C_{\mathrm{LS}}}\mathbf{H}(\nu|\mu) = \frac{2}{C_{\mathrm{LS}}}\int \rho\log\rho \mathrm{d}\mu = \frac{2}{C_{\mathrm{LS}}}\int \log\rho \mathrm{d}\nu, \tag{4.152}$$

holding for all $\nu := \rho\mu \in \mathcal{P}_2(\mathbb{R}^n)$ which are absolutely continuous with respect to μ.

Note that for diffusion processes there is missing a direct probabilistic way (i.e., using the corresponding *SDE* alone) to control the convergence of $\nu_t := \nu\mathbb{T}_t \to \mu$, as $t \to \infty$, in the Wasserstein distances. Nevertheless, a combination of (4.149) and (4.152) yields to the following

Corollary 4.28 *In the situation as described above, for all t, $t_0 \geq 0$,*

$$\mathbf{W}(\nu_{t+t_0},\mu) \leq \mathbf{W}_2(\nu_{t+t_0},\mu) \leq e^{-tC_{\mathrm{LS}}}\sqrt{\frac{2}{C_{\mathrm{LS}}}\mathbf{H}(\nu_{t_0}|\mu)}. \tag{4.153}$$

Remark 4.29 (i) The metric $\mathbf{W}_p(\nu,\mu)$ (and its generalizations) is commonly used in stochastics as a natural way for measuring distance between two probability laws in a weak sense. The *transportation inequalities*

$$(\mathbf{T}_p) \quad \mathbf{W}_p(\nu,\mu) \leq \sqrt{\frac{2}{C_p}\mathbf{H}(\nu|\mu)}, \quad C_p > 0, \quad p \geq 1, \tag{4.154}$$

which compare the Wasserstein distance $\mathbf{W}_p(\nu,\mu)$ and the entropy $\mathbf{H}(\nu|\mu)$, were first introduced by K. Marton [206]–[208] and M. Talagrand [271] in connection with the measure concentration problem. In particular, Talagrand's transportation inequality says that the standard Gaussian law $\mu := \mathcal{N}(0,\mathbf{Id}_n)$ on the Euclidean space $(\mathbb{R}^n, |\cdot|)$ obeys $(\mathbf{T}_2)$ with the sharp constant $C_2 = 1$. Necessary and sufficient conditions for the transportation inequalities in the basic cases $p = 1$ and $p = 2$ were first given by S. Bobkov and F. Götze, see Theorem 3.1 in [50]. Since the works of F. Otto and C. Villani [222] and S. Bobkov, I. Gentil, and M. Ledoux [51], it is known that the log-Sobolev inequality implies $(\mathbf{T}_2)$ with $C_2 := C_{\mathrm{LS}}$. However, $(\mathbf{T}_2)$ may hold even without $(\mathbf{LS})$ as it follows from a counterexample recently constructed in [75]. Further generalizations to (L^p-) Wasserstein distances ($1 \leq p \leq 2$) on Riemannian manifolds and path spaces were discussed in [286]. Since those important contributions the domain is expanding vigorously. The main trends are: (i) to carry on relations between the transportation inequalities and various renown functional inequalities (like as the weak/super Poincaré and logarithmic (or general F-) Sobolev inequalities, as well as their interpolation of Beckner or Latala-Oleszkiewicz type); and (ii) to describe the decay of the associated operator semigroups (see e.g. [35, 73, 74, 78, 244, 296] and the references therein).

(ii) Suppose that $\mu \in \mathcal{P}_1(\mathbb{R}^n)$ satisfies just the *Poincaré inequality* (4.144), then we have the following *variance* estimates valid for all $\nu := \rho\mu \in \mathcal{P}_1(\mathbb{R}^n)$

$$\begin{aligned} \mathbf{W}^2(\nu,\mu) &\leq \frac{1}{C_{\mathrm{SG}}}\mathbf{Var}_\mu(f) = \frac{1}{C_{\mathrm{SG}}}\int(\rho^2-1)\mathrm{d}\mu = \frac{1}{C_{\mathrm{SG}}}\int(\rho-1)\mathrm{d}\nu, \\ &\mathbf{W}(\nu_{t+t_0},\mu) \leq e^{-tC_{\mathrm{SG}}}\sqrt{\frac{1}{C_{\mathrm{LS}}}\mathbf{Var}_\mu(\rho_{t_0})}. \end{aligned} \tag{4.155}$$

These estimates however cannot be of practical use for us, since for $\nu_t := \mu^{\Lambda}_{t,y}$ their right-hand sides behave as $(\mathrm{const})^{|\Lambda|}$ if $|\Lambda| \to \infty$. For comparison we note that, because of the term with $\log\rho$, the entropic control in (4.152), (4.153) furnishes a better bound like $const \cdot |\Lambda|$, cf. Lemma 4.33.

(iii) A version of the *Lyapunov function* method due to S. P. Meyn and R. L. Tweedie [212] may also provide quantitative bounds on the exponential convergence of $\nu\mathbb{T}_t \to \mu$ in the variation norm and on the convergence $\mathbb{T}_t f \to \mathbf{E}_\mu f$ in $L^p(\mu)$ (see additionally [35, 132]).

4.3.2 Otto-Reznikoff criterion and its applications

Using a sufficient criterion due to F. Otto and M. Reznikoff, in this subsection we show that the local Gibbs distributions $\mu_{\Lambda,y}$ satisfy the log-Sobolev inequality, uniformly in all finite volumes $\Lambda \Subset \mathbb{V}$ and boundary conditions $y \in \Omega$, with the log-Sobolev constant explicitly given by (4.54). This is an important analytic step towards the exponential relaxation of the corresponding finite volume dynamics, which is needed in the proof Theorem 4.9.

Recall that in the literature there are only a few classical *sufficient criteria* for the hypercontractivity, giving computable bounds on C_{LS}.

(i) **Bakry-Emery $\mathbf{\Gamma}_2$-criterion [34]:** Any *log-concave* measure μ with strictly positive density Φ such that $\Phi'' \geq C_\Phi \mathbf{Id}_n$ with $C_\Phi > 0$, obeys (**LS**) and hence (**SG**) with

$$C_{\mathrm{SG}} \geq C_{\mathrm{LS}} \geq C. \tag{4.156}$$

(ii) **Perturbation result of Holley-Stroock** [146]**:** Assume that probability measures $\mu, \tilde{\mu}$ are related by $\tilde{\mu}(\mathrm{d}x) := e^{-U(x)}\mu(\mathrm{d}x)$ where U is bounded. Then, if μ satisfies (**LS**) with the constant $C_{\mathrm{LS}}(\mu)$, then $\tilde{\mu}$ does so with

$$C_{\mathrm{LS}}(\tilde{\mu}) \geq C_{\mathrm{LS}}(\mu) \cdot \exp\left\{-2\mathbf{Osc}U\right\}, \quad \mathbf{Osc}\,U := \sup_{\mathbb{R}^n} U - \inf_{\mathbb{R}^n} U. \tag{4.157}$$

Note that in some papers this result is stated with the wrong factor $\exp\{-\mathbf{Osc}\,U\}$.

(iii) **Tensorisation [133]:** The product measure $\tilde{\mu}(\times_{k=1}^N \mathrm{d}x) := \times_{k=1}^N \mu_k(\mathrm{d}x_k)$ satisfies (**LS**) with the constant

$$C_{\mathrm{LS}}(\tilde{\mu}) \geq \min_{1\leq k\leq N} C_{\mathrm{LS}}(\mu_k). \tag{4.158}$$

It is well known that (ii) and (iii), being straightforwardly applied to spin systems with non-convex potentials, give us that $C_{\mathrm{LS}}(\mu_{\Lambda,y}) \sim (\mathrm{const})^{|\Lambda|}$ as $|\Lambda| \to \infty$. To get *dimension free* estimates for Gibbs measures, one should proceed in a more refined way by using the so-called *Markov tensorisation* based on weak dependence of $\pi_\Lambda(\mathrm{d}x|y)$ on y_v as $\mathrm{dist}(v, \Lambda) \to \infty$ (like that in Dobrushin's uniqueness theorem). This idea was realized both in the *Stroock-Zegarlinski iterative method* [270, 294, 295] (see also [137, 249]) and in the *Lu-Yau martingale method* (developed further in [52, 53, 186, 289, 291]). Such methods typically work in a *perturbative regime*, when the inverse temperature or strength of inter-particle interaction asymptotically tends to zero, and are not aimed to produce concrete bounds on the critical values.

In contrary to the above papers, our considerations will be based on a *principally new criterion* for the log-Sobolev inequality on $\mathbb{R}^n$ suggested by *F. Otto and M. Reznikoff*, Theorem 1 in [221], which gives a simple *explicit bound* on the (**LS**) constant improving all previous results. Furthermore, this criterion is best suited to the local Gibbs measures

$$\mu_{\Lambda,y}(\mathrm{d}x_\Lambda) := \frac{1}{Z} \exp\{-\beta H_\Lambda(x_\Lambda|y)\}(\mathrm{d}x_\Lambda), \tag{4.159}$$

since it can be formulated directly in terms of the log-Sobolev constants $C_{\mathrm{LS}}(v, y)$ of the one-point conditional distributions $\mu_{v,y}(\mathrm{d}x_v)$ on $\mathbb{R}^\nu$ and the off-diagonal terms $\partial^2_{x_v x_{v'}} H(x_\Lambda|y) \in \mathcal{L}(\mathbb{R}^\nu)$ of the Hessian $H''_\Lambda(x_\Lambda|y)$. For $v, v' \in \Lambda \Subset \mathbb{V}$, $v \neq v'$, let us introduce the quantities

$$s_v := \inf\{C_{\mathrm{LS}}(v, y) \mid y \in \Omega\} \geq 0, \tag{4.160}$$

$$h_{vv'} := \sup\left\{ \left|\partial^2_{x_v x_{v'}} H(x_\Lambda|y)\right|_{\mathcal{L}(\mathbb{R}^\nu)} \;\middle|\; x_\Lambda \in \Omega_\Lambda,\, y \in \Omega \right\} \leq \infty. \tag{4.161}$$

Consider a symmetric matrix $A = (A_{vv'})_{\Lambda\times\Lambda}$ with the entries

$$A_{vv'} := -h_{vv'},\ v \neq v', \quad \text{and} \quad A_{vv} := s_v, \tag{4.162}$$

and assume that in the sense of quadratic forms on $\mathbb{R}^\Lambda \times \mathbb{R}^\Lambda$

$$A \geq C(\Lambda) \cdot \mathbf{Id}_\Lambda, \quad \text{with some } C(\Lambda) > 0. \tag{4.163}$$

Theorem 1 of [221] then yields that each of $\mu_{\Lambda,y}$ obeys the log-Sobolev inequality (4.143) on smooth enough functions $f : \Omega^{\nu|\Lambda|} \to \mathbb{R}$, with the constant independent on $y \in \Omega$

$$C_{\mathrm{ULS}}(\Lambda) \geq C(\Lambda). \tag{4.164}$$

In particular, under the conditions (4.13), (4.17)–(4.19) on the potentials $W_{vv'}$ and $V_v := U_v + Q_v$, we have by (4.156), (4.157) that for all $\mu_{v,y}$

$$C_{\mathrm{LS}}(v, y) \geq \beta e^{-2\beta\delta_Q}(a_U - Jm_{\mathbb{G}}), \tag{4.165}$$

whereas the operator norms of the off-diagonal terms for $v, v \in \Lambda$, $v \sim v'$,

$$\partial^2_{x_v x_{v'}} H(x_\Lambda|y) := \partial^2_{x_v x_{v'}} W_{vv'}(x_v, x_{v'}),$$

are uniformly bounded by J. Hence, the log-Sobolev constant in (4.164) can be estimated as follows:

Theorem 4.30 *Let the interaction parameters fulfill the relation (4.53). Then, the family of local Gibbs distributions $\mu_{\Lambda,y}$, $\Lambda \Subset \mathbb{V}$, $y \in \Omega$, satisfies the log-Sobolev inequality (4.143) with the uniform constant (coinciding with that one in (4.54)),*

$$\begin{aligned} C_{\mathrm{ULS}} :=& \inf \{ C_{\mathrm{LS}}(\Lambda, y) \mid \Lambda \Subset \mathbb{V},\ y \in \Omega \} \\ \geq& C_{4.166} := \beta \left[(a_U - Jm_{\mathbb{G}}) e^{-2\beta\delta_Q} - Jm_{\mathbb{G}} \right]. \end{aligned} \tag{4.166}$$

Remark 4.31 (i) The bound in (4.164) is *sharp* for Gaussian or product measures. As for the spectral gap estimates (cf. Subsections 2.3.5 (iii) and 4.5.1), an analogous abstract criterion has been proven by M. Ledoux, cf. Proposition 3.1 in [186] (which in turn was inspired by the earlier results of B. Helffer [141] obtained via the Witten-Laplacian approach). In the above notation for $\nu = 1$, it gives the validity of the Poincaré inequality (4.144) for $\mu_{\Lambda,y}(\mathrm{d}x_\Lambda)$ with the constant $C_{\mathrm{SG}}(\Lambda) \geq \inf_{x_\Lambda, y} C(\Lambda, x_\Lambda, y)$, where $C(\Lambda, x_\Lambda, y)$ is the lower bound in $\mathbb{R}^\Lambda$ of the symmetric operator $A(x_\Lambda, y)$ with matrix elements

$$A_{vv'}(x_\Lambda, y) := -\partial^2_{x_v x_{v'}} H(x_\Lambda|y),\ \ v \neq v', \quad \text{and} \quad A_{vv}(x_\Lambda, y) := s_v. \tag{4.167}$$

We emphasize that, when applied to the spin model (4.10), both criteria independently establish the *same computable bound* on the log-Sobolev (cf. 4.166)) and spectral gap (cf. (2.214), (4.266)) constants. This indicates that this bound might be best one could expect in this setting. An alternative approach to the spectral gap estimates, which is based on the *Efron-Stein inequality* for variances, is discussed in Subsections 2.3.5 (iii) and 4.5.1.

(ii) The recent paper of G. Blower and F. Bolley (cf. [49], Theorem 1.3) contains a similar sufficient condition for the global (**LS**) in terms of the one-point conditional measures $\mu_{v,y}(dx_v)$ and the off-diagonal blocks $\partial_{x_v}\partial_{x_{v'}} H(x_\Lambda|y)$ of the Hessian matrix.

(iii) In Subsection 4.5.2 we shall apply the Otto-Reznikoff criterion in a more comprehensive situation of the Gibbs measures on *loop spaces*. It is remarkable that we will get a perfect analog of Theorem 4.30, with the *same* bound (4.166) on the log-Sobolev constant.

Now, Theorem 4.30 enables us to apply Talagrand's inequality (4.152) and its Corollary 4.28 to the measures $\mu := \mu_{\Lambda,y}$ and $\nu_t := \mu^\Lambda_{t,y}$ with $t \geq t_0 > 0$. We have to consider only positive t_0, since the initial law of the solution process $(x^\Lambda_v(t,y))_{v\in\Lambda} \in \Omega_\Lambda$ (starting from the point y_Λ at $t = 0$) is the Dirac measure $\nu_0 := \delta_{\{y_\Lambda\}}$. As a result we obtain the exponential relaxation of the finite volume dynamics (4.57) in the Wasserstein metric $\mathbf{W}_\Lambda$ on Ω_Λ, which will be crucially used in the proof of Theorem 4.9. Note that the additional factor $|\Lambda|$ in the right-hand side in (4.168) is coming from the non-Euclidean distance $|x_\Lambda|_\Lambda := \sum_{v\in\Lambda}|x_v| \le |\Lambda|^{1/2}|x_\Lambda|$ on the underlying configuration space $\Omega_\Lambda := \mathbb{R}^{\nu|\Lambda|}$.

Corollary 4.32 *Under the assumptions of Theorem 4.9, for all $\Lambda \Subset \mathbb{V}$, $y \in \Omega$, and $t \ge t_0 > 0$,*

$$\mathbf{W}_\Lambda(\mathbb{P}_\Lambda\mu^\Lambda_{t,y}\,,\,\mu_{\Lambda,y}) \le e^{-(t-t_0)C_{\mathrm{ULS}}}\sqrt{\frac{2|\Lambda|}{C_{\mathrm{ULS}}}\mathbf{H}(\mu^\Lambda_{t_0,y}|\mu_{\Lambda,y})}. \tag{4.168}$$

In the next subsection we show that the right-hand side in (4.168) is finite and depends polynomially on $y \in \Omega^{\mathrm{t}}$.

4.3.3 How to estimate the entropy at $t = t_0$

Recall that in our notation, $\mu_{\Lambda,y} \in \mathcal{P}(\Omega_\Lambda)$ is the local Gibbs measure with boundary condition $y \in \Omega^{\mathrm{t}}$, cf. (4.159), and $\mu^\Lambda_{t,y} \in \mathcal{P}(\Omega^{\mathrm{t}})$, $t \ge 0$, respectively the probability laws of the cut-off dynamics (4.56). This subsection will be devoted to proving the following statement.

Theorem 4.33 *With the previous hypotheses there exists $t^0 := t^0(\beta)$ such that, for each $\delta > 0$ and a corresponding $C^{(\delta)}_{4.169} := C^{(\delta)}_{4.169}(t_0) > 0$,*

$$\begin{aligned}\mathbf{H}(\mathbb{P}_\Lambda\mu^\Lambda_{t',y}\,|\,\mu_{\Lambda,y}) \le \mathbf{H}(\mathbb{P}_\Lambda\mu^\Lambda_{t,y}\,|\,\mu_{\Lambda,y}) &\le |\Lambda|C^{(\delta)}_{4.169} \\ &\times\left[(1+\ln(1/t) + e^{\delta\,\mathrm{diam}(\Lambda)}\sum_{v\in\mathbb{V}}(1+|y_v|^P)e^{-\delta\rho(v,o)}\right],\end{aligned} \tag{4.169}$$

uniformly for all $0 < t \le t^0 \le t' < \infty$, $y \in \Omega^{\mathrm{t}}$, and $o \in \Lambda \Subset \mathbb{V}$. Recall that $P \ge 2$ describes the polynomial growth of V's in Assumption $(\mathbf{V}^*)$.

A key idea of the proof (originated by A. Ramirez and S. R. S. Varadhan in [241, 242] and then implemented to unbounded continuous spins with the harmonic interaction in Theorem 3.2.7 of [249] and in Lemma 3.4 of [290]) is to use *Girsanov's transform* for getting an explicit expression of the Radon-Nikodym density $\mathrm{d}\mu^\Lambda_{t,y}/\mathrm{d}\mu_{\Lambda,y}$. Further examples of calculating the relative entropy for diffusion processes can be found in the recent papers [74, 296]. The above lemma is a refinement of the results of [249, 290], in so far as we consider more general interactions and obtain better bounds on $\mathbf{H}(\mu^\Lambda_{t,y}|\mu_{\Lambda,y})$ based on the conditioning free representations (4.177), (4.180).

Proof. We conventionally divide the proof into several steps.

(i) Let us compare the path measures $P^x_{[0,T]}$ and $P^w_{[0,T]}$ induced on $C([0,T],\Omega_\Lambda)$ respectively by the processes $x^\Lambda(t,y)$ and $w_\Lambda(t) + y_\Lambda$, $0 \le t \le T < \infty$, both starting from y_Λ

at $t = 0$. A version of Girsanov's transform, see Theorems 7.6 and 7.7 in [190], says that:
(a) these measures are equivalent, i.e., $P^x_{[0,T]} \sim P^w_{[0,T]}$, if and only if

$$\begin{aligned} \mathbf{P}\left(\int_0^T |\nabla H_{\Lambda,y}(w_\Lambda(t)+y_\Lambda)|^2 \, \mathrm{d}t < \infty\right) = 1, \\ \mathbf{P}\left(\int_0^T \left|\nabla H_{\Lambda,y}(x^\Lambda(t,y))\right|^2 \mathrm{d}t < \infty\right) = 1; \end{aligned} \tag{4.170}$$

(b) the corresponding Radon-Nikodym derivative is given ($\mathbf{P}$-a.s.) by

$$\begin{aligned} \frac{\mathrm{d}P^x_{[0,T]}}{\mathrm{d}P^w_{[0,T]}}(w_\Lambda(\cdot)) := I_{\Lambda,y}(T) := \exp\Bigg\{ -\frac{\beta}{2}\int_0^T (\nabla H_{\Lambda,y}(w_\Lambda(t)+y_\Lambda), \mathrm{d}w_\Lambda(t)) \\ - \frac{\beta^2}{8}\int_0^T |\nabla H_{\Lambda,y}(w_\Lambda(t)+y_\Lambda)|^2 \, \mathrm{d}t \Bigg\}; \end{aligned} \tag{4.171}$$

(c) the process $(I_{\Lambda,y}(t), \mathcal{F}^w_t)$, $0 \le t \le T$, is a continuous martingale. Here $\mathcal{F}^w_t (\subseteq \mathcal{F}_t)$ denotes the smallest σ-algebra generated by the events

$$\{\omega \in \Omega \mid w_\Lambda(s,\omega) \in B_\Lambda \in \mathcal{B}(\Omega_\Lambda), \ s \le t\}.$$

Note that by the construction (cf. Subsection 6.1.3 in [190]), the process (4.171) has to be a supermartingale with respect to the initial filtration $(\mathcal{F}_t, \ 0 \le t \le T)$. Obviously, the both conditions in (4.170) are satisfied under the assumptions made on the potentials. Now we take advantage of the fact that our dynamics is of a gradient type and exclude the stochastic integral from (4.171). Using Ito's formula (cf. Theorem 4.5 in [190]), one can rewrite (4.171) as

$$\begin{aligned} I_{\Lambda,y}(T) = \exp\Bigg\{ \frac{\beta}{2} H_{\Lambda,y}(y_\Lambda) - \frac{\beta}{2} H_{\Lambda,y}(w_\Lambda(T)+y_\Lambda) \\ - \frac{1}{2}\int_0^T \Phi_{\Lambda,y}(w_\Lambda(t)+y_\Lambda)\mathrm{d}t \Bigg\}, \end{aligned} \tag{4.172}$$

with

$$\Phi_{\Lambda,y}(x_\Lambda) := \frac{\beta^2}{4}|\nabla H_{\Lambda,y}(x_\Lambda)|^2 - \frac{\beta}{2}\Delta H_{\Lambda,y}(x_\Lambda). \tag{4.173}$$

This enables us to derive an explicit formula for the densities

$$\rho^\Lambda_{t,y}(x_\Lambda) := \frac{\mathrm{d}\mathbb{P}_\Lambda \mu^\Lambda_{t,y}}{\mathrm{d}\mu_{\Lambda,y}}(x_\Lambda).$$

Let us recall that for all $t > 0$

$$\mathbf{P}\{w_\Lambda(t) \in B_\Lambda\} = (2\pi t)^{-\nu|\Lambda|/2} \int_{B_\Lambda} \exp\left(-\frac{1}{2t}\sum_{v\in\Lambda} x_v^2\right) \mathrm{d}x_\Lambda, \quad B_\Lambda \in \mathcal{B}(\Omega_\Lambda). \tag{4.174}$$

Thus, from (4.171) and (4.174) we get the following representation (**P**-a.s.) for the transition density of the process $x_\Lambda(t,y)$

$$p^\Lambda_{t,y}(x_\Lambda) \; : \; = \frac{\mathrm{d}\mathbb{P}_\Lambda \mu^\Lambda_{t,y}(x_\Lambda)}{\mathrm{d}x_\Lambda} = (2\pi t)^{-\nu|\Lambda|/2} \exp\left\{ -\frac{1}{2t} \sum_{v\in\Lambda} (x_v - y_v)^2 \right\} \times \mathbf{E}\left(I_{\Lambda,y}(t) \mid w_\Lambda(t) + y_\Lambda = x_\Lambda \right), \tag{4.175}$$

where in the last line there appears the conditional expectation of $I_{\Lambda,y}(t)$ at a given value of $w_\Lambda(t)+y_\Lambda$. Next, we exclude the conditioning in (4.175) proceeding in a same way as described (for the scalar case $\nu = |\Lambda| = 1$) in Section 13 of [127]. Namely, one may prove that

$$\mathbf{E}\left(I_{\Lambda,y}(t) \mid w_\Lambda(t) + y_\Lambda = x_\Lambda \right) = G_{\Lambda,y}(x_\Lambda) \exp\left\{ \frac{\beta}{2} H_{\Lambda,y}(y_\Lambda) - \frac{\beta}{2} H_{\Lambda,y}(x_\Lambda) \right\}, \tag{4.176}$$

where

$$G_{\Lambda,y}(x_\Lambda, t) := \mathbf{E} \exp\left\{ -\frac{t}{2} \int_0^1 \Phi_{\Lambda,y}\left[u x_\Lambda + (1-u) y_\Lambda + w_\Lambda(tu) - u w_\Lambda(t) \right] \mathrm{d}u \right\}. \tag{4.177}$$

This yields us that

$$p^\Lambda_{t,y}(x_\Lambda) = (2\pi t)^{-\nu|\Lambda|/2} G_{\Lambda,y}(x_\Lambda, t) \times \exp\left\{ -\frac{1}{2t} \sum_{v\in\Lambda} (x_v - y_v)^2 + \frac{\beta}{2} H_{\Lambda,y}(y_\Lambda) - \frac{\beta}{2} H_{\Lambda,y}(x_\Lambda) \right\}, \tag{4.178}$$

and hence

$$\rho^\Lambda_{t,y}(x_\Lambda) = p^\Lambda_{t,y}(x_\Lambda) \cdot \frac{\mathrm{d}x_\Lambda}{\mu_{\Lambda,y}(\mathrm{d}x_\Lambda)} = Z_{\Lambda,y}\, (2\pi t)^{-\nu|\Lambda|/2} G_{\Lambda,y}(x_\Lambda, t) \times \exp\left\{ -\frac{1}{2t} \sum_{v\in\Lambda} (x_v - y_v)^2 + \frac{\beta}{2} H_{\Lambda,y}(y_\Lambda) + \frac{\beta}{2} H_{\Lambda,y}(x_\Lambda) \right\}. \tag{4.179}$$

Finally, from (4.179) we get the explicit expression for the entropy

$$\begin{aligned} \mathbf{H}(\mu^\Lambda_{t,y} | \mu_{\Lambda,y}) &:= \int_{\Omega_\Lambda} \rho^\Lambda_{t,y}(x_\Lambda) \log \rho^\Lambda_{t,y}(x_\Lambda) \mu_{\Lambda,y}(\mathrm{d}x_\Lambda) \\ &= \mathbf{E}\left[\log \rho^\Lambda_{t,y}(x_\Lambda(t,y)) \right] = \\ &= \ln Z_{\Lambda,y} - \frac{1}{2}\nu|\Lambda| \, \log(2\pi t) - \frac{1}{2t} \sum_{v\in\Lambda} (x_v - y_v)^2 + \frac{\beta}{2} H_{\Lambda,y}(y_\Lambda) \\ &\quad + \mathbf{E}\left[\frac{\beta}{2} H_{\Lambda,y}(x^\Lambda(t,y)) + \log G_{\Lambda,y}(x^\Lambda(t,y), t) \right]. \end{aligned} \tag{4.180}$$

(ii) Our next task will be to get an upper bound on the right-hand side in (4.179) and (4.180). To this end we need the following estimates resulting from Assumptions ($\mathbf{V}^*$), ($\mathbf{W}^*$):

$$\begin{aligned} H_{\Lambda,y}(x_\Lambda) &\geq (A_V - m_{\mathbb{G}}J)\sum_{v\in\Lambda}|x_v|^2 - \frac{1}{2}m_{\mathbb{G}}J\sum_{v\in\partial^+\Lambda}|y_{v'}|^2 \\ &\quad +|\Lambda|\left(B_V - \frac{1}{2}m_{\mathbb{G}}JC_W\right); \\ H_{\Lambda,y}(x_\Lambda) &\leq C_V\sum_{v\in\Lambda}|x_v|^P + m_{\mathbb{G}}J\sum_{v\in\Lambda}|x_v|^2 + \frac{1}{2}m_{\mathbb{G}}J\sum_{v\in\partial^+\Lambda}|y_{v'}|^2 \\ &\quad +|\Lambda|\left(C_V + \frac{1}{2}m_{\mathbb{G}}JC_W\right); \\ \ln Z_{\Lambda,y} &\leq \frac{\beta}{2}m_{\mathbb{G}}J\sum_{v\in\partial^+\Lambda}|y_{v'}|^2 + \frac{1}{2}\nu|\Lambda|\cdot\{\ln\pi - \ln[\beta(A_V - m_{\mathbb{G}}J)]\} \\ &\quad -\beta|\Lambda|\left(B_V - \frac{1}{2}m_{\mathbb{G}}JC_W\right). \end{aligned} \tag{4.181}$$

To estimate $\Phi_{\Lambda,y}(x_\Lambda)$, cf. (4.173), we use the elementary inequality

$$(\mathrm{a} + \sum_{v\in\Lambda}\mathrm{b}_v)^2 \geq \frac{1}{2}\mathrm{a}^2 - 2|\Lambda|^2\sum_{v\in\Lambda}\mathrm{b}_v^2, \quad \mathrm{a}, \mathrm{b}_v \in \mathbb{R}.$$

This yields us that

$$\begin{aligned} \Phi_{\Lambda,y}(x_\Lambda) &\geq \sum_{v\in\Lambda}\left[\frac{1}{8}\beta^2\left|\nabla V_v(x_v)\right|^2 - \frac{1}{2}\beta\Delta V_v(x_v) - \gamma|x_v|^2\right] \\ &\quad -\frac{1}{2}\gamma\sum_{v\in\partial^+\Lambda}|y_{v'}|^2 - |\Lambda|\left(\frac{1}{2}\beta m_{\mathbb{G}}J + \frac{1}{2}\gamma D_W\right) \\ &\geq -\left[\alpha\sum_{v\in\Lambda}|x_v|^2 + \frac{1}{2}\gamma\sum_{v\in\partial^+\Lambda}|y_{v'}|^2 + \frac{1}{2}\varrho|\Lambda|\right], \end{aligned} \tag{4.182}$$

with

$$\begin{aligned} \gamma &:= \beta^2 m_{\mathbb{G}}^3 J^2, \quad \alpha := \gamma + \frac{1}{2}\beta K, \\ \varrho &:= \gamma D_W + \beta\left(L + m_{\mathbb{G}}J\right), \end{aligned} \tag{4.183}$$

and certain $K := K_\vartheta \geq 0$, $L := L_\vartheta \in \mathbb{R}$ corresponding to $\vartheta := \beta$ in (4.41). Thus, by Jensen's inequality applied to (4.177)

$$\begin{aligned} G_{\Lambda,y}(x_\Lambda, t) &\leq \exp\left\{t\left[2\alpha\left(\sum_{v\in\Lambda}|x_v|^2 + \sum_{v'\in\Lambda^+}|y_{v'}|^2\right) + \frac{1}{4}\varrho|\Lambda|\right]\right\} \\ &\quad \times\int_0^1 \mathbf{E}\exp\left\{\alpha t|w_\Lambda(tu) - uw_\Lambda(t)|^2\right\}\mathrm{d}u. \end{aligned} \tag{4.184}$$

Note that the random variable $w_\Lambda(tu) - uw_\Lambda(t)$ is distributed according to the normal law $\mathcal{N}(0; tu(1-u))$, which implies that for all $t < \sqrt{2/\alpha}$

$$\begin{aligned}
&\mathbf{E}\exp\left\{\alpha t\left|w_\Lambda(tu) - uw_\Lambda(t)\right|^2\right\} \qquad (4.185)\\
&= \left(\frac{1}{\sqrt{2\pi tu(1-u)}}\int_{\mathbb{R}}\exp\left\{-s^2\left(\frac{1}{2tu(1-u)} - \alpha t\right)\right\}\mathrm{d}s\right)^{\nu|\Lambda|}\\
&= \left(\frac{1}{2} - \alpha t^2 u(u-1)\right)^{-\nu|\Lambda|/2} \le \left(\frac{1}{2} - \frac{1}{4}\alpha t^2\right)^{-\nu|\Lambda|/2} < \infty.
\end{aligned}$$

Putting together (4.179)–(4.185),we get the following estimate

$$\begin{aligned}
\log\rho^\Lambda_{t,y}(x_\Lambda) \le{}& \frac{1}{2}\nu|\Lambda|\left\{C_{4.186} + \log(1/t)\right\}\\
&+ \frac{1}{2}\beta C_V\left[\sum_{v\in\Lambda}|x_v|^P + \sum_{v'\in\Lambda^+}|y_{v'}|^P\right]\\
&+ \left(\beta m_{\mathbb{G}} J + 2\sqrt{\alpha}\right)\left[\sum_{v\in\Lambda}|x_v|^2 + \sum_{v'\in\Lambda^+}|y_{v'}|^2\right], \qquad (4.186)
\end{aligned}$$

which is valid for all $t \le t^0(\beta) := 1/\sqrt{\alpha}$, where $\alpha := \alpha(\beta)$ was defined in (4.183). The constant $C_{4.186} := C_{4.186}(\beta)$ here does not depend on t, x_Λ, y and might be written explicitly in terms of β and the parameters in ($\mathbf{V}^*$), ($\mathbf{W}^*$).

(iii) Having shown (4.186), we are able to estimate the entropy on the interval $t \in (0, t^0]$ as

$$\begin{aligned}
&\mathbf{H}(\mathbb{P}_\Lambda\mu^\Lambda_{t,y}\,|\,\mu_{\Lambda,y}) = \mathbf{E}\log\rho^\Lambda_{t,y}(x^\Lambda(t,y))\\
&\le C_{4.187}\left\{|\Lambda|^+\left[1 + \log(1/t)\right] + \sum_{v\in\Lambda}\sup_{0\le t\le t_0}\mathbf{E}|x^\Lambda_v(t,y)|^P + \sum_{v'\in\Lambda^+}|y_{v'}|^P\right\}, \qquad (4.187)
\end{aligned}$$

with some $C_{4.187} := C_{4.187}(\beta) > 0$ which is the same for all x_Λ, y, and Λ. Substituting into the last line the uniform bound (4.77) on $\mathbf{E}|x^\Lambda_v(t,y)|^P$, we finally can rewrite (4.187) as

$$\mathbf{H}(\mathbb{P}_\Lambda\mu^\Lambda_{t,y}\,|\,\mu_{\Lambda,y}) \le C^{(\delta)}_{4.188} m_{\mathbb{G}}|\Lambda|\left\{1 + \log(1/t) + e^{\delta\mathrm{diam}(\Lambda)}\sum_{v\in\mathbb{V}}\left(1 + |y_v|^P\right)e^{-\delta\rho(v,o)}\right\}, \qquad (4.188)$$

with a proper constant $C^{(\delta)}_{4.188} := C^{(\delta)}_{4.188}(\beta) > 0$. On the other hand,

$$\mathbf{H}(\mathbb{P}_\Lambda\mu^\Lambda_{t',y}\,|\,\mu_{\Lambda,y}) \le \mathbf{H}(\mathbb{P}_\Lambda\mu^\Lambda_{t,y}\,|\,\mu_{\Lambda,y}), \quad \text{for all } t' \ge t,$$

because of the monotonicity principle for the relative entropy, cf. (4.147). ■

Combining both Theorems 4.30 and 4.33, we obtain the precise bound on the exponential relaxation which is used in proving Theorem 4.9, cf. (4.60).

Corollary 4.34 *Under the hypotheses of Theorem 4.9, it holds for all $\delta > 0$, $t \geq 1$ and any $y \in \Omega^{\mathrm{t}}$, $o \in \Lambda \Subset \mathbb{V}$,*

$$\mathbf{W}_\Lambda(\mathbb{P}_\Lambda \mu^\Lambda_{t,y}\,,\,\mu_{\Lambda,y}) \leq \exp\{\delta \mathrm{diam}(\Lambda) - tC_{\mathrm{ULS}}\}\, C^{(\delta)}_{4.189}|\Lambda| \tag{4.189}$$

$$\times \sum_v (1+|y_v|^{P/2}) e^{-\delta\rho(v,o)}, \tag{4.190}$$

with some $C^{(\delta)}_{4.189} > 0$ (depending on β and the other parameters in Assumptions $(\mathbf{V}^*)$*,* $(\mathbf{W}^*)$*).*

Remark 4.35 (i) The explicit representation (4.177), (4.178) allows for studying the *regularity* properties of the transition densities $p^\Lambda_{t,y}(x_\Lambda)$ with respect to the variables t, y, x_Λ. Under the assumptions imposed on V_v, $W_{vv'}$, one may directly check (cf. [127], Section 14) that the above constructed $p^\Lambda_{t,y}(x_\Lambda)$ continuously depend on $t < t^0$, $y \in \Omega^{\mathrm{t}}$, $x_\Lambda \in \Omega_\Lambda$, and satisfy the Chapman-Kolmogorov identity (4.44) in the form

$$p^\Lambda_{t+\Delta t,y}(x_\Lambda) = \int_{\Omega_\Lambda} p^\Lambda_{t,y}(\tilde{x}_\Lambda) p^\Lambda_{\Delta t,\tilde{x}_\Lambda \times y_{\Lambda^c}}(x_\Lambda) \mathrm{d}\tilde{x}_\Lambda, \quad \forall t,\ t+\Delta t < t_0.$$

Furthermore, (4.177), (4.178) provides a straightforward way to prove that the transition densities $p^\Lambda_{t,y}(x_\Lambda)$ are fundamental solutions to the Kolmogorov equation

$$\frac{\partial p^\Lambda_{t,y}(x_\Lambda)}{\partial t} = \frac{1}{2}\Delta p^\Lambda_{t,y}(x_\Lambda) - \frac{1}{2}\left(\nabla H_{\Lambda,y}(x_\Lambda), \nabla p^\Lambda_{t,y}(x_\Lambda)\right), \quad t < t_0. \tag{4.191}$$

Concerning the recent developments in the so-called *analytic approach* to diffusion processes, in which the transition densities $p^\Lambda_{t,y}(x_\Lambda)$ are dealt with as weak solutions to the parabolic problem (4.191), see [57, 58] and the references therein.

(ii) There is one more justification for the formula (4.178) by means of the *inverse ground state transform*

$$L^2(\Omega_\Lambda, \mu_{\Lambda,y}(\mathrm{d}x_\Lambda)) \xrightarrow{\mathbf{U}} L^2(\Omega_\Lambda, \mathrm{d}x_\Lambda), \quad \mathbf{U}f := (Z_{\Lambda,y})^{-1/2}\, e^{-H_{\Lambda,y}/2} f.$$

The Dirichlet operator $\mathbf{H}_{\Lambda,y}$ is then *unitary equivalent* to the Schrödinger operator $\Delta + \Phi_{\Lambda,y}$ acting in $L^2(\Omega_\Lambda, \mathrm{d}x_\Lambda)$, where the potential $\Phi_{\Lambda,y}$ is given by (4.173). It is well known (see e.g. Proposition 5.3 in [73]) that $\liminf_{|x_\Lambda|\to\infty} \Phi_{\Lambda,y} > 0$ is a sufficient condition for the presence of spectral gap for $\Delta + \Phi_{\Lambda,y}$. Then, the Feynman-Kac formula applied to $\exp\{-t(\Delta + \Phi_{\Lambda,y})/2\} f$ immediately leads to the following identity with $I_{\Lambda,y}(t)$ defined by (4.172)

$$\exp\{-t\mathbf{H}_{\Lambda,y}/2\} f(y_\Lambda) = \mathbf{E}\left[f(w_\Lambda(t) + y_\Lambda) I_{\Lambda,y}(t)\right], \quad f \in C_{\mathrm{b}}(\Omega_\Lambda),$$

(see also Proposition 2.1 in [295]). After conditioning with respect to given $w_\Lambda(t) + y_\Lambda = x_\Lambda$, one gets the formula (4.178) for $p^\Lambda_{t,y}(x_\Lambda)$ being the integral kernel of the operator $\exp\{-t\mathbf{H}_{\Lambda,y}/2\}$.

4.4 Rate of convergence in Dobrushin's criterion

In this section we establish *computable* estimates in Wasserstein distance on the rate of convergence $\mu_{\Lambda,y} \to \mu \in \mathcal{G}^{\mathrm{t}}$ in the thermodynamic limit $\Lambda \nearrow \mathbb{V}$. The results obtained, in particular *Corollary 4.40*, will allow us to complete *Step III* in the proof of Theorem 4.9. This will be done in the framework of Dobrushin's uniqueness criterion, which presumes that the local Gibbs specification $\{\pi_\Lambda\}_{\Lambda \Subset \mathbb{V}}$ satisfies the *weak dependence* condition $||\mathbf{D}||_0 < 1$. We stress that the previously known results, see e.g. [91, 94, 176], were restricted to the systems with bounded spins, and hence could not cover the case of tempered (i.e., growing) boundary conditions $y \in \Omega^{\mathrm{t}}$. To this end, in Subsection 4.4.2 we suggest slightly stronger conditions on Dobrushin's matrix $\mathbf{D}$, see (4.205), (4.240), which makes possible to extend the above criterion to unbounded spin systems on *graphs* (Theorems 4.37 and 4.38) and to the lattice systems with interactions of *infinite range* (Theorem 4.46). Among other results we mention a positive answer given in Subsection 4.4.1 to the *measurability problem* for optimal couplings, which arose e.g. in the original works of R. Dobrushin and remained so far open.

4.4.1 Measurability problem for the Wasserstein distance

Here we collect some useful facts about the Wasserstein distances. An important *new observation*, see Items (v), (vi), concerns the *measurability* of solutions to the mass transportation problem, which is known to be not unique solvable in general. In some recent papers concerning the applications to mathematical physics (see e.g. [32, 111]) such measurability was mentioned as a *long-standing open problem*. We shall provide a simple proof of the measurability result, which is based on the fundamental selection theorem for multifunctions. Note that this property is crucial for proving uniqueness criteria for Gibbs fields via the so-called reconstruction procedure suggested by R. Dobrushin (see Subsection 4.4.2).

(i) **Definition of $\mathbf{W}_{\rho,p}$:** Let (X, ρ) be a Polish space. For $p \geq 1$, let $\mathcal{P}_p(X)$ denote the subset of all probability measures ν on $(X, \mathcal{B}(X))$ having finite moments

$$\mathbf{E}_\nu \left[\rho(x, x_0)\right]^p < \infty, \tag{4.192}$$

for some (and hence for all) $x_0 \in X$. For a pair $\nu, \tilde{\nu} \in \mathcal{P}_p(X)$, we define the ($L^p$-) *Wasserstein distance*, cf. (4.151),

$$\mathbf{W}_{\rho,p}(\nu, \tilde{\nu}) := \inf_{P \in \Pi(\nu,\tilde{\nu})} \left[\int_{X^2} [\rho(x, \tilde{x})]^p P(\mathrm{d}x, \mathrm{d}\tilde{x}) \right]^{1/p}, \tag{4.193}$$

where the infimum is taken over all couplings $P \in \Pi(\nu, \tilde{\nu})$, i.e., probability measures $P \in \mathcal{P}(X \times X)$ with the marginal distributions ν and $\tilde{\nu}$. It can be shown that $(\mathcal{P}_p(X), \mathbf{W}_{\rho,p})$ becomes itself a *Polish space* (cf. Theorem 6.1 in [281]), whereby the convergence $\mathbf{W}_{\rho,p}(\nu, \nu_n) \to 0$, as $n \to \infty$, is equivalent to the weak convergence of the measures $\nu_n \to \nu$ combined with the convergence of their moments (4.192) (cf. Theorem 5.4 2 in [238] or Theorem 6.8 in [281]). Since $\mathcal{P}_p(X)$ is closed as a subset in $(\mathcal{P}(X), \mathcal{W})$, it also can be considered as the Polish space equipped by the weak topology $\mathcal{W}$.

(ii) **Optimal couplings:** Actually, the infimum in (4.195) is always attained at some (either *unique* or at *infinite many*) $P \in \Pi(\nu, \tilde{\nu})$ (cf. Theorem 4.1 in [281]). Such minimizing couplings will be called *optimal*, whereby their set will be denoted by $\Pi^*(\nu, \tilde{\nu})$. Each $P \in \Pi^*(\nu, \tilde{\nu})$ can be looked upon as a solution to the *mass transportation problem* with the cost function $[\rho(x, \tilde{x})]^p$ between ν and $\tilde{\nu}$ [239]. The set $\Pi^*(\nu, \tilde{\nu})$ is a convex compact in $\mathcal{P}(X \times X)$ endowed with the corresponding topology of weak convergence (cf. Corollary 5.20 in [281]).

Below we restrict our considerations to the case $p = 1$. When no clarity is lost, we shall omit the subscript ρ specifying the metric.

(iii) **Kantorovich-Rubinstein dual relation** (cf. Theorem 2.5.6 in [238] and Theorem 5.9 in [281]) says that the following two definitions of the (L^1-) Wasserstein distance are equivalent for any pair of $\nu, \tilde{\nu} \in \mathcal{P}_1(X)$:

$$\mathbf{W}(\nu, \tilde{\nu}) := \inf_{P \in \Pi(\nu, \tilde{\nu})} \int_{X^2} \rho(x, \tilde{x}) P(\mathrm{d}x, \mathrm{d}\tilde{x}) \tag{4.194}$$

$$= \sup_{f \in \mathrm{Lip}_1(X, \rho)} \left| \int_X f(x)[\nu - \tilde{\nu}](\mathrm{d}x) \right|, \tag{4.195}$$

where

$$\mathrm{Lip}_1(X, \rho) := \left\{ f : X \to \mathbb{R} \,\middle|\, [f] := \sup_{x \neq \tilde{x}} \frac{|f(x) - f(\tilde{x})|}{\rho(x, \tilde{x})} \leq 1 \right\}. \tag{4.196}$$

(iv) **Stability of optimal couplings** (see Theorem 5.19 in [281])**:** Let $\nu^{(N)}, \tilde{\nu}^{(N)} \in \mathcal{P}_1(X)$ converge weakly, as $N \to \infty$, to $\nu, \tilde{\nu} \in \mathcal{P}_1(X)$ respectively. Let $P^{(N)} \in \Pi^*(\nu^{(N)}, \tilde{\nu}^{(N)})$ be a corresponding sequence of optimal couplings. Then, there exists a subsequence $P^{(N_M)}$ which converges weakly, as $M \to \infty$, to an optimal coupling $P \in \Pi^*(\nu, \tilde{\nu})$.

(v) **Measurable selection:** Consider the product space $X \times X$ with the metric

$$\tilde{\rho}[(x_1, x_2), (\tilde{x}_1, \tilde{x}_2)] := \rho(x_1, \tilde{x}_1) + \rho(x_2, \tilde{x}_2), \quad (x_1, x_2),\ (\tilde{x}_1, \tilde{x}_2) \in X \times X. \tag{4.197}$$

In a similar way one may endow $\mathcal{P}_1(X \times X)$ with the Wasserstein metric $\mathrm{W}_{\tilde{\rho}}$ (or with the topology of weak convergence $\mathcal{W}$). Then

$$\mathcal{P}_1(X) \times \mathcal{P}_1(X) \ni (\nu, \tilde{\nu}) \to \Pi^*(\nu, \tilde{\nu}) \subset \mathcal{P}_1(X \times X) \tag{4.198}$$

can be viewed as a *multifunction* taking values in nonempty closed subsets $\Pi^*(\nu, \tilde{\nu})$ of the Polish space $(\mathcal{P}_1(X \times X), \mathrm{W}_{\tilde{\rho}})$ (or $(\mathcal{P}_1(X \times X), \mathcal{W})$). Endowing all the metric spaces with the corresponding Borel σ-algebras, one may ask the following

Question *Whether there exists a measurable selection P of the random set Π^*, which is a Borel function*

$$\mathcal{P}_1(X) \times \mathcal{P}_1(X) \ni (\nu, \tilde{\nu}) \to P(\nu, \tilde{\nu}) \in \Pi^*(\nu, \tilde{\nu}) \ \ ? \tag{4.199}$$

Answer to this problem is *always positive* as we show now. By the *fundamental selection theorem* (see Theorems III.6 and III.8 in [72] or Theorem 2.13 in [217]), the following (strong) measurability of the multifunction Π^* is sufficient for the existence of such P: for any closed set $F \subseteq \mathcal{P}_1(X \times X)$,

$$(\Pi^*)^{-1}(F) := \{(\nu, \tilde{\nu}) \mid \Pi^*(\nu, \tilde{\nu}) \cap F \neq \varnothing\} \tag{4.200}$$

should be a Borel set in $\mathcal{P}_1(X) \times \mathcal{P}_1(X)$. Actually, we can prove that the set (4.200) will be closed too. Let $(\nu^{(N)}, \tilde{\nu}^{(N)})_{N\in\mathbb{N}} \subset (\Pi^*)^{-1}(F)$ be a fundamental sequence in $\mathcal{P}_1(X) \times \mathcal{P}_1(X)$, and let us denote its limit by $(\nu, \tilde{\nu})$. For each $N \in \mathbb{N}$, there exists at least one $P^{(N)} \in \Pi^*(\nu^{(N)}, \tilde{\nu}^{(N)}) \cap F$. By Item (iii) one finds a subsequence $P^{(N_M)}$, $M \in \mathbb{N}$, converging weakly to an optimal coupling $P \in \Pi^*(\nu, \tilde{\nu})$. Since F is closed, this means that the limit point $(\nu, \tilde{\nu})$ also belongs to $(\Pi^*)^{-1}(F)$. ■

An immediate corollary of the foregoing item is the next useful statement:

(vi) **Measurable dependence on a parameter:** Let the marginals $\nu_\lambda, \tilde{\nu}_\lambda \in \mathcal{P}_1(X)$ vary in a measurable way with respect to some abstract parameter λ. Then there exists a *measurable realization* of the optimal coupling

$$\lambda \to P_\lambda \in \Pi^*(\nu_\lambda, \tilde{\nu}_\lambda) \subset \mathcal{P}_1(X \times X).$$

Hence, the Wasserstein distance $\mathbf{W}\,(\nu_\lambda, \tilde{\nu}_\lambda)$ also *measurably depends* on this λ.

In the *DLR* approach all this applies to the mappings $y \to \pi_\Lambda(\mathrm{d}x|y) \in (\mathcal{P}(\Omega), \mathcal{W})$ which by construction are known to be measurable, see Remark 2.2 (ii).

Corollary 4.36 *For any local specification $\{\pi_\Lambda\}_{\Lambda \Subset \mathbb{V}}$ obeying (4.201), the Wasserstein distance $\mathbf{W}_\Lambda\left(\mu_{\Lambda,y}, \mu_{\Lambda,\tilde{y}}\right)$ is a measurable function of $(y, \tilde{y}) \in \Omega \times \Omega$.*

4.4.2 Dobrushin's contraction technique for unbounded spins

Relying on the Dobrushin contraction technique (already mentioned in Subsection 2.3.5 (i)), here we establish the *rate of convergence*, as $\Lambda \nearrow \mathbb{V}$, of the local conditional distributions $\mu_{\Lambda,y}(\mathrm{d}x_\Lambda)$, $y \in \Omega^{\mathrm{t}}$, to the unique Gibbs measure $\mu \in \mathcal{G}^{\mathrm{t}}$. For unbounded spins, such convergence is naturally expressed in terms of the Wasserstein distance. All the constants will be written down *explicitly*, which allows to analyze their dependence on the geometry of the underlying graph and on the choice of boundary conditions $y \in \Omega^{\mathrm{t}}$. Similar estimates in the total variation distance $||\cdot||_{\mathrm{TV}}$ are long known for the discrete spin systems (see e.g. [90, 91, 95, 96, 176]) and are based on Dobrushin's reconstruction procedure for Gibbs states. For the translation invariant lattice systems with continuous but *bounded* spins, the convergence of local Gibbs distributions in the Wasserstein metric was first established by R. Dobrushin and S. Shlosman in [94]. However, this was done *without justification* of the reconstruction procedure and the related measurability problems, which must be clarified if the spin spaces are not more finite. So, one of our aims will be to fill this gap in the original proofs of R. Dobrushin. Another peculiarity of our case is that we need to consider all boundary conditions $y \in \Omega^{\mathrm{t}}$. They are typically growing and hence do not belong to $l^\infty(\mathbb{V})$, which means that the standard arguments based on the $l^\infty(\mathbb{V})$-contractivity of Dobrushin's matrix do not work. To overcome this problem, we shall introduce proper weighted norms on Ω^{t} and then apply the contractivity principles with respect to them. Such setting allows us to to extend Dobrushin's criterion to unbounded spin systems on *graphs* (Theorems 4.37 and 4.38) and to the lattice systems with interactions of *infinite range* (Theorem 4.46).

(i) Rate of convergence in Dobrushin's criterion

As the problems discussed are of independent interest, we turn back to the most general setting introduced at the beginning of Chapter 4. Let us suppose that the graph $\mathbb{G}(\mathbb{V},\mathbb{E})$ satisfies the regularity Assumption $(\mathbf{G}_\delta)$ with some $\delta_{\mathbb{G}} \geq 0$. The subset of tempered configurations is given by $\Omega^{\mathrm{t}} := \bigcap_{\delta>\delta_{\mathbb{G}}} \Omega_\delta$, cf. (4.25), (4.28). In what follows, $\mathbf{W}$ stands for the (L^1-) Wasserstein probability distance on Ω_Λ. Recall that we consider each Ω_Λ as a Banach space with the norm $|x_\Lambda|_\Lambda := \sum_{v\in\Lambda}|x_v|$ (to distinguish it from the Euclidean one $|x_\Lambda|$ in $\mathbb{R}^{\nu|\Lambda|}$). For $\Lambda \Subset \mathbb{V}$, the related sets $\Lambda^\pm$ and $\partial^\pm\Lambda$ were introduced in (4.8).

Let us given a local specification $\{\pi_\Lambda\}_{\Lambda\Subset\mathbb{V}}$ such that for all $y \in \Omega$ and $v \in \Lambda \Subset \mathbb{V}$

$$\int_\Omega |x_v|\pi_\Lambda(\mathrm{d}x|y) < \infty. \tag{4.201}$$

Define the Dobrushin interdependence matrix $\mathbf{D} = (D_{vv'})_{\mathbb{V}\times\mathbb{V}}$ with entries

$$D_{vv'} := \sup_{\substack{y,\tilde{y}\in\Omega \\ y=\tilde{y} \text{ off } v'}} \left\{ \frac{\mathbf{W}\left(\mu_v(\mathrm{d}x_v|y), \mu_v(\mathrm{d}x_v|\tilde{y})\right)}{|y_{v'} - \tilde{y}_{v'}|} \right\}, \quad v \sim v'. \tag{4.202}$$

Assume that the above matrix is $l^\infty(\mathbb{V})$-contractive,.which means $||\mathbf{D}||_0 < 1$. Then (cf. Subsection 2.3.4 (i)), Dobrushin's criterion, Theorem 4 in [91], immediately implies the uniqueness of $\mu \in \mathcal{G}$ such that $\sup_v \mathbf{E}_\mu|x_v| < \infty$. However, it says nothing about the speed of convergence $\mu_{\Lambda,y} \to \mu$ as $\Lambda \nearrow \mathbb{V}$ for a set $y \in \Omega$ of full measure μ. Since this is not strong enough for our purposes, we introduce the weighted Banach spaces $l^1_\delta(\mathbb{V})$, $l^1_\delta(\mathbb{V}) \subset \Omega$ with norms

$$\begin{aligned} ||x||_{l^1_\delta} \ &: \ = \sum_v |x_v| \exp\{-\delta\rho(\nu, o)\}, \\ ||x||_{l^\infty_\delta} \ &: \ = \sup_v \left[|x_v| \exp\{-\delta\rho(\nu, o)\}\right], \quad \text{for some } o \in \mathbb{V}. \end{aligned} \tag{4.203}$$

Note that

$$||\mathbf{D}||_{\mathcal{L}(l^\infty_\delta)} = ||\mathbf{D}^{\mathrm{t}}||_{\mathcal{L}(l^1_\delta)} \leq \sup_v \sum_{v'\in\partial v} D_{vv'} \exp\{\delta\rho(\nu, v')\} \leq ||\mathbf{D}||_0 \exp\delta. \tag{4.204}$$

Theorem 4.37 *Let a specification $\Pi = \{\pi_\Lambda\}_{\Lambda \Subset \mathbb{V}}$ satisfy the following*

Contraction Condition $(\mathbf{D}_{\mathbb{G}})$:
$$||\mathbf{D}||_0 := \sup_v \sum_{v' \in \partial v} D_{vv'} < \exp(-\delta_{\mathbb{G}}) < 1. \tag{4.205}$$

Then, for each $o \in \mathbb{V}$ and $\delta \in (\delta_{\mathbb{G}}, -\log ||\mathbf{D}||_0)$, the class of all Gibbs measures $\mu \in \mathcal{G}$ obeying the exponential growth restriction on their moment sequence

$$\sum_v \mathbf{E}_\mu |x_v| \cdot \exp\{-\delta\rho(o,v)\} < \infty \tag{4.206}$$

consists of at most one element.

Theorem 4.38 *In the same situation, for each $\delta \in (\delta_{\mathbb{G}}, -\log ||\mathbf{D}||_0)$, $o \in \Lambda \subseteq \Delta \Subset \mathbb{V}$, and arbitrarily chosen boundary conditions $y, \tilde{y} \in \Omega^{\mathrm{t}}$, we have:*

(i) The computable bound on the Wasserstein distance between the corresponding finite volume projections

$$\begin{aligned} & \mathbf{W}\left(\mathbb{P}_\Lambda \mu_{\Delta,y}, \mathbb{P}_\Lambda \mu_{\Delta,\tilde{y}}\right) \\ \leq\ & (1 - ||\mathbf{D}||_0 \exp\delta)^{-1} |\Lambda| \sum_{v \in \partial^+\Delta} |y_v - \tilde{y}_v| \exp\{-\delta \mathrm{dist}(\nu, \Lambda)\}; \end{aligned} \tag{4.207}$$

(ii) Convergence as $\Delta \nearrow \mathbb{V}$ to the (unique) $\mu \in \mathcal{G}^{\mathrm{t}}$

$$\begin{aligned} & \mathbf{W}\left(\mathbb{P}_\Lambda \mu_{\Delta,y}, \mathbb{P}_\Lambda \mu\right) \leq (1 - ||\mathbf{D}||_0 \exp\delta)^{-1} |\Lambda| \exp\{\delta \mathrm{diam}(\Lambda)\} \\ & \times \sum_{v \in \partial^+\Delta} (\mathbf{E}_\mu |x_v| + |y_v|) \exp\{-\delta\rho(\nu, o)\}, \end{aligned} \tag{4.208}$$

provided that the quantities $||y||_{l^1_{o,\delta}}$ and $C^{(o,\delta)}_{4.206}(\mu)$ are finite.

Remark 4.39 Both Dobrushin's Conditions, see $(\mathbf{D}_1)$ in Subsection 2.2.1 and $(\mathbf{D}_2)$ in Subsection 2.3.4, are related as follows. Suppose additionally that

$$\sup_{v \in \mathbb{V}} \int_\Omega |x_v| \pi_v(\mathrm{d}x|0) =: C_{4.209} < \infty, \tag{4.209}$$

then by the definition of the Dobrushin coefficients (4.202) we have that for each $y \in \Omega$

$$\begin{aligned} \int_\Omega |x_v| \pi_v(\mathrm{d}x|y) &\leq \left| \int_\Omega (|x_v| - |\tilde{x}_v|)\, \pi_v(\mathrm{d}x|y) \pi_v(\mathrm{d}\tilde{x}|0) \right| + \int_\Omega |x_v| \pi_v(\mathrm{d}x|0) \\ &\leq C_{4.209} + \sum_{v' \in \partial v} D_{vv'} |y_{v'}|. \end{aligned}$$

Thus $(\mathbf{D}_2)$ implies $(\mathbf{D}_1)$ with the compact function $h(x_v) := |x_v|$ and contractive matrix $\mathbf{D} = (D_{vv'})_{\mathbb{V} \times \mathbb{V}}$, which in turn guarantees existence of $\mu \in \mathcal{G}$ with $\sup_{v \in \mathbb{V}} \mathbf{E}_\mu |x_v| < \infty$.

For the graphs satisfying the stronger Assumption $(\mathbf{G}_0)$ with $\delta_{\mathbb{G}} = 0$, instead of $(\mathbf{D}_{\mathbb{G}})$ it suffices to require the usual Dobrushin's condition $||\mathbf{D}||_0 < 1$. The results of Theorems 4.37 and 4.38 then hold for all $\delta < -\log ||\mathbf{D}||_0$ and $y, \tilde{y} \in \Omega^{\mathrm{t}} := \bigcap_{\delta>0} \Omega_\delta$. Below we formulate this precisely in the form needed for proving the ergodicity result of Theorem 4.9. Remind that by the claim (i) of Theorem 4.18 we have the uniform bound

$$\sup_{\mu\in\mathcal{G}^{\mathrm{t}}} \sup_{v\in\mathbb{V}} \int_\Omega |x_v| \mathrm{d}\mu \leq C_{4.210} < \infty. \tag{4.210}$$

Corollary 4.40 *Under the hypotheses of Theorem 4.9, for all positive $\delta < -\log ||\mathbf{D}||_0$ and $y \in \Omega^{\mathrm{t}}$, $o \in \Lambda \subseteq \Delta \Subset \mathbb{V}$,*

$$\begin{aligned} \mathbf{W}\left(\mathbb{P}_\Lambda \mu_{\Delta,y}, \mathbb{P}_\Lambda \mu\right) \;&\leq\; \left(1 - ||\mathbf{D}||_0 \exp\delta\right)^{-1} |\Lambda| \exp\left\{\delta \mathrm{diam}(\Lambda) - \frac{\delta}{2}\mathrm{dist}(\Lambda, \Delta^c)\right\} \\ &\quad\times \sum_v \left(C_{4.210} + |y_v|\right) \exp\left\{-\frac{\delta}{2}\rho(\nu, o)\right\}. \end{aligned} \tag{4.211}$$

The proof of Theorems 4.37, 4.38 are based on the following statement (motivated by Lemmas 2.1, 2.2 and Theorem 3.1 in [94], which however dealt with translation invariant interactions and bounded spins only).

Lemma 4.41 *Supposing that $(\mathbf{D}_{\mathbb{G}})$ holds, let us fix any $\Delta \Subset \mathbb{V}$ and consider a pair of probability measures μ, $\tilde{\mu} \in \mathcal{P}_1(\Omega)$ such that*

$$\pi_v \mu = \mu, \quad \pi_v \tilde{\mu} = \tilde{\mu}, \quad \textit{for all} \;\; \nu \in \Delta^-. \tag{4.212}$$

Then, the Wasserstein distance between their projections $\mathbb{P}_\Lambda\mu$, $\mathbb{P}_\Lambda\tilde{\mu}$ on volumes $\Lambda \subseteq \Delta$ can be estimated by

$$\begin{aligned} \mathbf{W}(\mathbb{P}_\Lambda\mu, \mathbb{P}_\Lambda\tilde{\mu}) \;&\leq\; \left(1 - ||\mathbf{D}||_0 \exp\delta\right)^{-1} |\Lambda| \\ &\quad\times \sum_{v\in\partial^-\Delta} \exp\left\{-\delta \mathrm{dist}(\nu, \Lambda)\right\} \int_\Omega |x_v| \left[\mu(\mathrm{d}x) + \tilde{\mu}(\mathrm{d}x)\right], \end{aligned} \tag{4.213}$$

with any positive $\delta < -\log ||\mathbf{D}||_0$.

Proof. By Remark 2.32 (ii), there exists an optimal coupling $P \in \Pi^*(\mathbb{P}_\Delta\mu, \mathbb{P}_\Delta\tilde{\mu})$ such that

$$\mathbf{W}(\mathbb{P}_\Delta\mu, \mathbb{P}_\Delta\tilde{\mu}) = \sum_{v\in\Delta} \int_{[\Omega_\Delta]^2} |x_v - \tilde{x}_v| P(\mathrm{d}x_\Delta, \mathrm{d}\tilde{x}_\Delta). \tag{4.214}$$

Setting

$$M_v \;:\; = \int_{[\Omega_\Delta]^2} |x_v - \tilde{x}_v| P(\mathrm{d}x_\Delta, \mathrm{d}\tilde{x}_\Delta), \tag{4.215}$$

$$M_v \;\leq\; \int_\Omega |x_v| \left[\mu(\mathrm{d}x) + \tilde{\mu}(\mathrm{d}x)\right], \quad v \in \Delta, \tag{4.216}$$

we first show the following estimate in terms of the Dobrushin coefficients (4.202)

$$M_v \leq \sum_{v' \in \partial v} D_{vv'} M_{v'}, \quad \text{for all } \ \nu \in \Delta^-. \tag{4.217}$$

Fixed some vertex $\nu \in \Delta^-$, let us apply to $P(\mathrm{d}x_\Delta, \mathrm{d}\tilde{x}_\Delta)$ the following *reconstruction procedure* originally discovered for discrete spins by R. Dobrushin in [90, 91]. The result of this reconstruction would be a new measure $\tilde{P} \in \mathcal{P}(\Omega_\Delta \times \Omega_\Delta)$ with the same marginals $\mathbb{P}_\Delta \mu$ and $\mathbb{P}_\Delta \tilde{\mu}$, which we define as follows. Let

$$\Omega^2 \ni (y, \tilde{y}) \to \pi_v(\mathrm{d}x_v \mathrm{d}\tilde{x}_v | y, \tilde{y}) \in \Pi^*(\mu_{v,y}, \mu_{v,\tilde{y}}) \subset \mathcal{P}_1(\mathbb{R}^{2\nu}) \tag{4.218}$$

be a *measurable* mapping such that

$$\int_{\mathbb{R}^{2\nu}} |x_v - \tilde{x}_v| \pi_v(\mathrm{d}x_v \mathrm{d}\tilde{x}_v | y, \tilde{y}) = \mathrm{W}\left(\mu_{v,y}, \mu_{v,\tilde{y}}\right). \tag{4.219}$$

Since $\Omega \ni y \to \mu_{v,y}(\mathrm{d}x_v) \in \mathcal{P}_1(\mathbb{R}^\nu)$ is continuous, such measurable version of the optimal coupling between $\mu_{v,y}$ and $\mu_{v,\tilde{y}}$ does exist by Remark 2.32 (v),(vi) (or in a general situation, by Remark 2.2 (ii)). Then $\tilde{P}$ is uniquely determined by the duality

$$\int_{[\Omega_\Delta]^2} f(x_\Delta, \tilde{x}_\Delta) \tilde{P}(\mathrm{d}x_\Delta, \mathrm{d}\tilde{x}_\Delta) : \tag{4.220}$$
$$= \int_{[\Omega_\Delta]^2} \left(\int_{\mathbb{R}^{2\nu}} f(x_v \times y_{\Delta \setminus \{v\}}, \tilde{x}_v \times \tilde{y}_{\Delta \setminus \{v\}}) \pi_v(\mathrm{d}x_v \mathrm{d}\tilde{x}_v | y, \tilde{y}) \right) P(\mathrm{d}y_\Delta, \mathrm{d}\tilde{y}_\Delta),$$

which holds on all bounded uniformly continuous functions $f \in C_\mathrm{b}^\mathrm{u}(\Omega_\Delta \times \Omega_\Delta)$. Note that for such f the integral over $\mathbb{R}^{2\nu}$ in the right-hand side in (4.220) is a *measurable* mapping of $(y, y) \in \Omega^2$, which makes the above definition correct. By (4.220) it is obvious that $\tilde{P} \in \Pi(\mathbb{P}_\Delta \mu, \mathbb{P}_\Delta \tilde{\mu})$, whereas $\tilde{P}$ and P coincide on the σ-algebra generated by the events $B' \times B''$ with $B', B'' \in \mathcal{B}(\Omega_{\Delta \setminus \{v\}})$. Setting

$$\tilde{M}_{v'} := \int_{[\Omega_\Delta]^2} |x_{v'} - \tilde{x}_{v'}| \tilde{P}(\mathrm{d}x_\Delta, \mathrm{d}\tilde{x}_\Delta), \quad v' \in \Delta,$$

we get by (4.219) and (4.220) that

$$M_{v'} = \tilde{M}_{v'}, \text{ for all } v' \neq v, \tag{4.221}$$

whereby by (4.202)

$$\begin{aligned} \tilde{M}_v \quad &\leq \quad \sum_{v' \in \partial v} D_{vv'} \int_{[\Omega_\Delta]^2} |y_{v'} - \tilde{y}_{v'}| \tilde{P}(\mathrm{d}x_\Delta, \mathrm{d}\tilde{x}_\Delta) \\ &= \quad \sum_{v' \in \partial v} D_{vv'} \tilde{M}_{v'}. \end{aligned} \tag{4.222}$$

On the other hand,

$$\sum_{v'\in\Delta}\tilde{M}_{v'} \geq \mathbf{W}(\mathbb{P}_\Delta\mu, \mathbb{P}_\Delta\tilde{\mu}) = \sum_{v'\in\Delta} M_{v'},$$

which together with (4.221) and (4.222) implies the required estimate (4.217)

$$M_v \leq \tilde{M}_v \leq \sum_{v'\in\partial v} D_{vv'}M_{v'}, \quad \text{for all} \ \ \nu \in \Delta^-.$$

For $\Lambda \subseteq \Delta$ and $\delta \in (\delta_{\mathbb{G}}, -\log||\mathbf{D}||_0)$, let us introduce the system of weights

$$c_v := c_v(\Lambda,\delta) := \exp\left\{-\delta \mathrm{dist}(\nu,\Lambda)\right\}, \quad \nu \in \mathbb{V}. \tag{4.223}$$

Note that

$$c_v = 1 \text{ if } \nu\in\Lambda, \ \text{ and } \ c_{v'}c_v^{-1} \leq \exp\left\{\delta\rho(v,v')\right\}. \tag{4.224}$$

Elementary rearrangements then show that

$$\begin{aligned}\sup_{v\in\Delta}\left\{c_vM_v\right\} &\leq \sup_{v'\in\Delta}\left\{c_{v'}M_{v'}\right\}\cdot\sup_{v\in\Delta}\left(\sum_{v'\in\partial v} D_{vv'}c_vc_{v'}^{-1}\right) + \sum_{v\in\partial^-\Delta} c_vM_v \qquad (4.225)\\ &\leq ||\mathbf{D}||_0\exp\delta\cdot\sup_{v\in\Delta}\left\{c_vM_v\right\} + \sum_{v\in\partial^-\Delta}c_vM_v.\end{aligned}$$

Finally,

$$\begin{aligned}\mathbf{W}(\mathbb{P}_\Lambda\mu,\mathbb{P}_\Lambda\tilde{\mu}) &\leq |\Lambda|\sup_{v\in\Delta}\left\{c_vM_v\right\} \qquad (4.226)\\ &\leq \left(1-||\mathbf{D}||_0\exp\delta\right)^{-1}|\Lambda|\sum_{v\in\partial^-\Delta}c_vM_v,\end{aligned}$$

which was needed to prove in (4.213). ■

In a similar way we can establish the *dual* result:

Corollary 4.42 *If the transposed matrix* $\mathbf{D}^{\mathrm{t}}$ *satisfies Assumption* $(\mathbf{D}_{\mathbb{G}})$, *then respectively*

$$\begin{aligned}\mathbf{W}(\mathbb{P}_\Lambda\mu,\mathbb{P}_\Lambda\tilde{\mu}) &\leq \left(1-||\mathbf{D}^{\mathrm{t}}||_0\exp\delta\right)^{-1} \qquad (4.227)\\ &\quad\times\sum_{v\in\partial^-\Delta}\exp\left\{-\delta\mathrm{dist}(\nu,\Lambda)\right\}\int_\Omega|x_v|\left[\mu(\mathrm{d}x)+\tilde{\mu}(\mathrm{d}x)\right],\end{aligned}$$

for all positive $\delta < -\log||\mathbf{D}^{\mathrm{t}}||_0$.

Proof. Instead of (4.225) and (4.226), we now finish as follows:

$$\begin{aligned}\sum_{v\in\Delta}c_vM_v &\leq \sum_{v'\in\Delta}c_{v'}M_{v'}\left(\sum_{v\in\Delta^-\cap\partial v'}D_{vv'}c_vc_{v'}^{-1}\right)+\sum_{v\in\partial^-\Delta}c_vM_v \qquad (4.228)\\ &\leq ||\mathbf{D}^T||_0\exp\delta\cdot\sum_{v\in\Delta}c_vM_v+\sum_{v\in\partial^-\Delta}c_vM_v,\end{aligned}$$

and hence

$$\mathbf{W}(\mathbb{P}_\Lambda\mu, \mathbb{P}_\Lambda\tilde{\mu}) \le \sum_{v\in\Lambda} M_v \le \sum_{v\in\Delta} c_v M_v \le \left(1 - ||\mathbf{D}^T||_0 \exp\delta\right)^{-1} \sum_{v\in\partial^-\Delta} c_v M_v. \tag{4.229}$$

■

Proof of Theorem 4.38. (i) Let us apply Lemma 4.41 to the measures $\mu := \pi_\Delta(\mathrm{d}x|y)$ and $\tilde{\mu} := \pi_\Delta(\mathrm{d}x|\tilde{y})$, which by (2.26) are consistent with the one-point specification kernels π_v for all $v \in \Delta$. Repeating the previous arguments, we conclude from (4.229) that for all $\Lambda \subseteq \Delta$

$$\mathbf{W}\left(\mathbb{P}_\Lambda\pi_\Delta(\mathrm{d}x|y), \mathbb{P}_\Lambda\pi_\Delta(\mathrm{d}x|\tilde{y})\right) \le \left(1 - ||\mathbf{D}||_0 \exp\delta\right)^{-1} |\Lambda| \sum_{v\in\partial^+\Delta} c_v M_v. \tag{4.230}$$

Since $\pi_\Delta(B|y) \equiv \mathbf{1}_{\{y_{\Delta^c}\in B\}}$ for all $B \in \mathcal{B}(\Omega_{\Delta^c})$, one easily observes that

$$M_v := \int_{[\Omega_\Delta]^2} |x_v - \tilde{x}_v| P(\mathrm{d}x_\Delta, \mathrm{d}\tilde{x}_\Delta) = |y_v - \tilde{y}_v|, \quad v \in \partial^+\Delta,$$

which together with (4.230) yields the result.

(ii) Fix a positive $\delta < -\log||\mathbf{D}||_0$, and let the quantities $\|y\|_{o,\delta}$, $C^{(o,\delta)}_{4.206}$, $\mathbf{\Xi}_\delta$ be finite. Applying Lemma 4.41 and taking into account (4.7), (4.206), we get that

$$\begin{aligned} &\mathbf{W}\left(\mathbb{P}_\Lambda\pi_\Delta(\mathrm{d}x|y), \mathbb{P}_\Lambda\mu\right) \le \left(1 - ||\mathbf{D}||_0 \exp\delta\right)^{-1} |\Lambda| \\ &\times \sum_{v\in\partial^+\Delta} \exp\left\{-\delta \mathrm{dist}(\nu, \Lambda)\right\} \cdot \left(\mathbf{E}_\mu|x_v| + |y_v|\right) \\ \le\ &\left(1 - ||\mathbf{D}||_0 \exp\delta\right)^{-1} |\Lambda| \exp\left\{\delta \mathrm{diam}(\Lambda)\right\} \\ &\times \sum_{v\in\partial^+\Delta} \left(\mathbf{E}_\mu|x_v| + |y_v|\right) \exp\left\{-\delta\rho(\nu, o)\right\}, \end{aligned} \tag{4.231}$$

which completes the proof. ■

Proof of Theorem 4.37. Suppose there exist at least two different $\mu, \tilde{\mu} \in \mathcal{G}$ obeying (4.206). Then Lemma 4.41 says that for all $\delta \in (\delta_{\mathbb{G}}, -\log||\mathbf{D}||_0)$, $\Lambda \Subset \mathbb{V}$, and any cofinal sequence $\Lambda_N \nearrow \mathbb{V}$

$$\begin{aligned} &\mathbf{W}(\mathbb{P}_\Lambda\mu, \mathbb{P}_\Lambda\tilde{\mu}) \le \left(1 - ||\mathbf{D}||_0 \exp\delta\right)^{-1} |\Lambda| \exp\left\{\delta \mathrm{diam}(\Lambda)\right\} \\ &\times \sum_{v\in\partial^+\Lambda_N} \left(\mathbf{E}_\mu|x_v| + \mathbf{E}_{\tilde{\mu}}|x_v|\right) \exp\left\{-\delta\rho(\nu, o)\right\} \to 0, \quad \text{as } N \to \infty. \end{aligned}$$

Thus, all finite volume projections $\mathbb{P}_\Lambda\mu$ and $\mathbb{P}_\Lambda\tilde{\mu}$ coincide, which means that $\mu = \tilde{\mu}$. ■

Proceeding in the same way but replacing Lemma 4.41 by Corollary 4.42, we prove the following

Corollary 4.43 *The statement of Theorem 4.37 remains true if, instead of* $\mathbf{D}$, *the transposed matrix* $\mathbf{D}^{\mathrm{t}}$ *obeys Assumption* $(\mathbf{D}_{\mathbb{G}})$.

Remark 4.44 The above proof reduced the uniqueness problem of $\mu \in \mathcal{G}$ to some kind of an optimization problem for the Wasserstein distance, cf. (4.214). Such approach seems to be much simpler as usual inductive schemes of [90, 91, 122, 176, 178, 176] where one has move step by step over all points of $\mathbb{V}$ in order to construct the unique limit of $\pi_\Lambda(\mathrm{d}x|y)$ as $\Lambda \nearrow \mathbb{V}$.

We briefly discuss some further applications. An immediate sequel of Theorem 4.38 is the following *mixing property* for correlations calculated with respect to $\mu \in \mathcal{G}^{\mathrm{t}}$.

Corollary 4.45 *Let* $f : \Omega_\Delta \to \mathbb{R}$, $g : \Omega_\Lambda \to \mathbb{R}$ *be measurable cylinder functions with disjoint supports* $\Delta \cap \Lambda = \varnothing$, *such that* f *is globally bounded with* $||f||_{L^\infty} < \infty$ *and* g *is Lipschitz-continuous with* $[g]_\Lambda < \infty$. *Then*

$$\begin{aligned} |\mathbf{Cov}_\mu(f;g)| \;\le\;& 2\left(1 - ||\mathbf{D}||_0 \exp\delta\right)^{-1} |\Lambda| \cdot ||f||_{L^\infty}[g]_\Lambda \\ & \times \sum_{v \in \partial^+\Delta} \mathbf{E}_\mu |x_v| \cdot \exp\left\{-\delta \mathrm{dist}(\nu, \Lambda)\right\}. \end{aligned} \tag{4.232}$$

Proof. Without loss of generality assume that $\mathbf{E}_\mu g = 0$. Making use of the *DLR* equation (2.34) and the dual relation (3.58), we may write

$$\mathbf{Cov}_\mu(f;g) = \int_\Omega f(x_\Delta)\left[\int_\Omega g(y_\Lambda)\pi_\Delta(\mathrm{d}y|x) - \int_\Omega g(y_\Lambda)\mu(\mathrm{d}y)\right],$$

and hence

$$|\mathbf{Cov}_\mu(f;g)| \le ||f||_{L^\infty}[g]_\Lambda \int_\Omega \mathbf{W}\left(\mathbb{P}_\Lambda \mu_{\Delta,y}, \mathbb{P}_\Lambda \mu\right) \mathrm{d}\mu(y). \tag{4.233}$$

The statement now follows by substituting the upper bound for $\mathbf{W}\left(\mathbb{P}_\Lambda \mu_{\Delta,y}, \mathbb{P}_\Lambda \mu\right)$ which was derived in (4.231). ■

(ii) Modification for interactions of infinite range

An important situation, which we have already faced in Subsections 2.1.1 and 3.1.4, is when the interaction has *infinite range* and hence the local energies $H_\Lambda(x|y)$ may be divergent for some $y \in \Omega$. To keep the *DLR* picture consistent, we conventionally put $\pi_\Lambda(\mathrm{d}x|y) \equiv 0$ as $y \notin \Omega^{\mathrm{t}}$ and then consider only those Gibbs measures $\mu \in \mathcal{G}^t$ which are supported by the tempered configurations $y \in \Omega^{\mathrm{t}}$. Clearly, such models *do not fit* into the standard framework of Dobrushin's uniqueness criterion [91, 108, 122], which in advance expects that $\pi_\Lambda(\mathrm{d}x|y)$ are *probability* kernels continuously depending on $y \in \Omega$ in the product topology $\mathcal{T}(\Omega)$. Below we suggest a proper modification of the Dobrushin theorem for the interactions of possibly infinite range, which allows to cover (both the classical and the quantum) spin systems living on a lattice $\mathbb{Z}^d$ or on a more general indexing set $\mathbb{L}$.

Let us given a local specification $\Pi = \{\pi_\Lambda\}_{\Lambda \Subset \mathbb{L}}$ of the *measure* kernels $\pi_\Lambda(\mathrm{d}x|y)$ (in the sense of Remark 2.2 (i)) with the countable indexing set $\mathbb{L}$ and the Polish single spin space $(X, \rho) \ni x_\ell$, such that for all $y \in \Omega := X^{\mathbb{L}}$

$$\int_\Omega \rho(x_\ell, o)\pi_\ell(\mathrm{d}x|y) < \infty. \tag{4.234}$$

Consider the Polish space (Ω_w, ρ_w) of "*tempered*" configurations

$$\Omega_w := \left\{ x \in \Omega \;\middle|\; \sum_\ell \rho(x_\ell, o) \cdot w(\ell_0, \ell) < \infty \right\} \tag{4.235}$$

$$\text{with} \quad \rho_w(x, \tilde{x}) := \sum_\ell \rho(x_\ell, \tilde{x}_\ell) \cdot w(\ell_0, \ell), \tag{4.236}$$

constructed, for some fixed $\ell_0 \in \mathbb{L}$ and $o \in X$, with the help of a weight mapping $w : \mathbb{L} \times \mathbb{L} \to (0, +\infty)$ described by Definition 3.6. Additionally, we assume that

$$\mathbf{\Xi}_w := \sup_\ell \sum_{\ell'} w(\ell, \ell') < \infty. \tag{4.237}$$

For all $\Lambda \Subset \mathbb{L}$, let we know that $\pi_\Lambda(\mathrm{d}x|y) = 0$ if $y \notin \Omega_w$. Define now the Dobrushin matrix $\mathbf{D} = (D_{\ell\ell'})_{\mathbb{L}\times\mathbb{L}}$ with the entries

$$D_{\ell\ell'} := \sup_{\substack{y, \tilde{y} \in \Omega_w \\ y = \tilde{y} \text{ off } \ell'}} \left\{ \frac{\mathbf{W}_\rho\left(\mu_{\ell,y}\,,\, \mu_{\ell,\tilde{y}}\right)}{\rho(y_{\ell'}, \tilde{y}_{\ell'})} \right\}, \quad \ell \neq \ell', \tag{4.238}$$

where the supremum is taken over the *tempered* $y, \tilde{y} \in \Omega^{\mathrm{t}}$ only

Theorem 4.46 *In order that there is at most one "tempered" Gibbs measure $\mu \in \mathcal{G}$ such that*

$$\sum_\ell \mathbf{E}_\mu\left[\rho(x_\ell, o)\right] \cdot w(\ell_0, \ell) < \infty, \tag{4.239}$$

the fulfillment of the following is sufficient:

Contraction Condition $(\mathbf{D}_w)$:

$$||\mathbf{D}||_w := \sup_\ell \sum_{\ell'} D_{\ell\ell'}\left[w(\ell, \ell')\right]^{-1} < 1. \tag{4.240}$$

Proof. For any $\mu, \tilde{\mu} \in \mathcal{G}$ obeying (4.239), provided such exist, let us estimate the Wasserstein distance

$$\mathbf{W}_w\left(\mu, \tilde{\mu}\right) := \inf_{P \in \Pi(\mu, \tilde{\mu})} \int_{\Omega^2} \rho_w(x, \tilde{x}) P(\mathrm{d}x, \mathrm{d}\tilde{x}). \tag{4.241}$$

Let $P \in \Pi^*(\mu, \tilde{\mu})$ be an optimal coupling, which means

$$\mathbf{W}_w(\mathbb{P}\mu, \mathbb{P}_\Lambda\tilde{\mu}) = \sum_\ell M_\ell w(\ell_0, \ell) \tag{4.242}$$

with

$$M_\ell := \int_{\Omega^2} \rho(x_\ell, \tilde{x}_\ell) P(\mathrm{d}x, \mathrm{d}\tilde{x}), \quad \ell \in \mathbb{L}.$$

Similarly to the proof of Lemma 4.41, a main issue is to show that the Dobrushin condition (4.238) implies that

$$M_\ell \leq \sum_{\ell'(\neq \ell)} D_{\ell\ell'} M_{\ell'}, \quad \text{for all } \ \ell \in \mathbb{L}. \tag{4.243}$$

Note that by the above construction, cf. (4.206), the vector $M := (M_\ell)_{\ell \in \mathbb{L}}$ belongs to the Banach space

$$l_w^\infty(\mathbb{L}) := \left\{ s \in \mathbb{R}^{\mathbb{V}} \,\middle|\, ||s||_{l_w^\infty} := \sup_\ell \{|s_v| w(\ell_0, \ell)\} < \infty \right\}. \tag{4.244}$$

Since the matrix $\mathbf{D} = (D_{vv'})_{\mathbb{V}\times\mathbb{V}}$ generates a bounded operator in l_δ^∞ whose norm does not exceed $||\mathbf{D}||_w < 1$ (cf. Remark 2.1), we get from (4.243) that

$$M_\ell \leq (\mathbf{D}M)_\ell \leq (\mathbf{D}^N M)_\ell, \quad ||M||_{l_\delta^\infty} \leq ||\mathbf{D}^N M||_{l_\delta^\infty}, \quad N \in \mathbb{N}.$$

This immediately implies that $M \equiv 0$ provided

$$r_{\mathrm{sp}}(\mathbf{D}) := \lim_{N\to\infty} ||\mathbf{D}^N||_w^{1/N} < 1.$$

In order to check (4.243) we apply to $P(\mathrm{d}x_\Lambda, \mathrm{d}\tilde{x}_\Lambda)$ Dobrushin's reconstruction at a fixed point ℓ. Recall that the justification of such procedure was done in the proof of Lemma 4.41. As a result we get a new measure $\tilde{P} \in \Pi(\mu, \tilde{\mu})$, which is uniquely determined by the duality

$$\int_{X^2} f(x, \tilde{x}) \tilde{P}(\mathrm{d}x, \mathrm{d}\tilde{x}) : \tag{4.245}$$
$$= \int_{\Omega^2} \left(\int_{X^2} f(x_\ell \times y_{\{\ell\}^c}, \tilde{x}_\ell \times \tilde{y}_{\{\ell\}^c}) \pi_\ell(\mathrm{d}x_\ell \mathrm{d}\tilde{x}_\ell | y, \tilde{y}) \right) P(\mathrm{d}y, \mathrm{d}\tilde{y}),$$

holding on all cylinder uniformly continuous functions $f \in C_{\mathrm{b}}^{\mathrm{u}}(\Omega_\Lambda \times \Omega_\Lambda)$ with $\Lambda \Subset \mathbb{L}$. Here $\Omega^2 \ni (y, \tilde{y}) \to \pi_\ell(\mathrm{d}x_\ell \mathrm{d}\tilde{x}_\ell | y, \tilde{y}) \in \Pi^*(\mu_{\ell,y}, \mu_{\ell,\tilde{y}})$ is a measurable solution to the optimization problem

$$\int_{X^2} \rho(x_\ell, \tilde{x}_\ell) \pi_\ell(\mathrm{d}x_\ell \mathrm{d}\tilde{x}_\ell | y, \tilde{y}) = \mathbf{W}_\rho \left(\mu_{\ell,y}, \mu_{\ell,\tilde{y}} \right). \tag{4.246}$$

By (4.238) and (4.246) we get that

$$M_{\ell'} = \tilde{M}_{\ell'} := \int_{\Omega^2} |x_{\ell'} - \tilde{x}_{\ell'}| \tilde{P}(\mathrm{d}x, \mathrm{d}\tilde{x}), \ \text{for all } \nu' \neq v, \tag{4.247}$$

$$\tilde{M}_\ell \leq \sum_{\ell'(\neq \ell)} D_{\ell\ell'} \int_{\Omega^2} |y_{\ell'} - \tilde{y}_{\ell'}| \tilde{P}(\mathrm{d}x, \mathrm{d}\tilde{x}) = \sum_{\ell'(\neq \ell)} D_{\ell\ell'} \tilde{M}_{\ell'}. \tag{4.248}$$

On the other hand,

$$\sum_{\ell' \in \mathbb{L}} \tilde{M}_{\ell'} w(\ell_0, \ell') \geq \mathbf{W}_w(\mu, \tilde{\mu}) = \sum_{\ell' \in \mathbb{L}} M_{\ell'} w(\ell_0, \ell'),$$

which together with (4.247) and (4.248) yields the required estimate

$$M_\ell \leq \tilde{M}_\ell \leq \sum_{\ell'(\neq \ell)} D_{\ell\ell'} M_{\ell'}. \tag{4.249}$$

■

Going through the previous proof and using a contractivity argument in the Banach space

$$l^1_w(\mathbb{L}) := \left\{ s \in \mathbb{R}^{\mathbb{V}} \,\middle|\, ||s||_{l^\infty_w} := \sum_\ell |s_v| w(\ell_0, \ell) < \infty \right\}, \tag{4.250}$$

we get the following:

Corollary 4.47 *In the statement of Theorem 4.46 one can replace* $\mathbf{D}$ *by the transposed matrix* $\mathbf{D}^T$.

Our last result here can be viewed as a dual form of *Dobrushin's comparison theorem* (Theorem 3 in [91]; see also Theorem 2.1 in [176] and Theorem 3.7 in [108]). On the other hand, the estimate we are going to prove constitutes the counterpart for the Efron-Stein-Wu inequality (4.258) for variances of weakly dependent Gibbs fields. We present only the statement for $\mu \in \mathcal{G}^{\mathrm{t}}$, its finite volume version can be founded in [288], Proposition 4.2.

Corollary 4.48 *Assume that* $||\mathbf{D}^{\mathrm{t}}||_w < 1$. *Let* μ *be the (unique) Gibbsian measure and* $\tilde{\mu} \in \mathcal{P}(\Omega)$ *be an arbitrary probability measure, both satisfying the temperedness condition (4.239). Then, the following estimate in the Wasserstein distance holds*

$$\mathbf{W}_w(\mu, \tilde{\mu}) \leq \left(1 - ||\mathbf{D}^{\mathrm{t}}||_w\right)^{-1} \sum_\ell w(\ell_0, \ell) \int_\Omega \mathbf{W}_\rho\left(\mu_{\ell,\tilde{y}}, \tilde{\mu}_{\ell,\tilde{y}}\right) \tilde{\mu}(\mathrm{d}\tilde{y}), \tag{4.251}$$

where $\tilde{\mu}_{\ell,\tilde{y}}$ *are (regular) one-point conditional distributions of* $\tilde{\mu}$ *under knowing* $\tilde{y}_{\{\ell\}^c}$.

Proof. First we note that the mapping $(y, \tilde{y}) \to \mathbf{W}\left(\mu_{\ell,y}, \tilde{\mu}_{\ell,\tilde{y}}\right)$ is measurable by Corollary 4.36. Proceeding similarly to the proof of Theorem 4.46 and Corollary 4.48, let us look at

$$\mathbf{W}_w(\mu, \tilde{\mu}) := \inf_{P \in \Pi(\mu,\tilde{\mu})} \int_{\Omega^2} \rho_w(x, \tilde{x}) P(\mathrm{d}x, \mathrm{d}\tilde{x}) = \sum_\ell m_\ell w(\ell_0, \ell). \tag{4.252}$$

Performing the reconstruction $\tilde{P}$ of P at the point ℓ, we obtain by the triangle inequality and (4.249) that

$$\begin{aligned} M_\ell &\leq \tilde{M}_\ell \leq \int_{\Omega^2} \mathbf{W}_\rho\left(\mu_{\ell,y}, \tilde{\mu}_{\ell,\tilde{y}}\right) P(\mathrm{d}y_\Lambda, \mathrm{d}\tilde{y}_\Lambda) \\ &\leq \sum_{\ell'(\neq \ell)} D_{\ell\ell'} M_{\ell'} + \int_\Omega \mathbf{W}_\rho\left(\mu_{\ell,\tilde{y}}, \tilde{\mu}_{\ell,\tilde{y}}\right) \tilde{\mu}(\mathrm{d}\tilde{y}). \end{aligned}$$

After summing over $\ell \in \mathbb{L}$ with the weights $w(\ell_0, \ell)$, we get the result. ■

4.5 Analysis of the Dirichlet operators

This Section is devoted to the comprehensive study of the Dirichlet operators associated with Gibbs measures, both in the *classical* and in the *quantum* cases. We shall focus on the following issues:

– A novel abstract approach to the spectral gap estimates via the Efron-Stein-Wu inequalities for weakly dependent Markov fields (Subsection 4.5.1);

– Poincaré and log-Sobolev inequalities on loop spaces, with precise estimates on the size of spectral gap and the log-Sobolev constants (Subsection 4.5.2);

– Analytical approach to the Euclidean Gibbs measures, which is based on their integration by parts description (Subsection 4.5.3);

– Essential self-adjointness of the Dirichlet operators (Subsection 4.5.4).

4.5.1 Spectral gap and Efron-Stein-Wu inequalities

Here we give an *elegant new proof* of the *Efron-Stein inequality* for variances, which recently was extended by L. Wu [288] to *weakly dependent* Markov fields obeying Dobrushin's Contraction Condition $(\mathbf{D_2})$. A classical version of this inequality stated for a family of independent random variables is well known in statistics (see e.g. Section 2.5 of [209]). Recall that in Subsection 2.3.5 (iii) we have already applied such generalized Efron-Stein-Wu inequality to the interacting spin systems on a lattice. Its important consequence is the uniform, in volumes and boundary conditions, spectral gaps estimates for the probability kernels $\pi_\Lambda(\mathrm{d}x|y)$ of the Gibbs specification, cf. Proposition 2.39. This seems to be the *shortest way* (also in comparison to the Γ_2-approach of M. Ledoux [186], cf. Remark 4.31 (i)) of getting the global spectral gap estimates by using à-priori information about the one-point conditional distributions. Moreover, without an extra technical effort this method also covers the case of infinite dimensional (i.e., loop) spin spaces, see Subsection 4.5.2 below.

For simplicity we keep the former notation, however all things obviously apply to any Polish spin space (X,ρ) taken instead of $(\mathbb{R}^\nu, |\cdot|)$. Given a *finite* indexing set Λ, let us consider a probability measure $\nu_\Lambda(\mathrm{d}x) \in \mathcal{P}(\Omega_\Lambda)$ with a family of its regular *conditional distributions* $\nu_{v,y}(\mathrm{d}x_v) := \nu_v(\mathrm{d}x_v|y_\Lambda)$, $v \in \Lambda$, $y_\Lambda \in \Omega_\Lambda$, subject to fixed $y_{\Lambda\setminus v}$. Furthermore, we assume that

$$\int_{\Omega_\Lambda} |x_v|^2 \nu_\Lambda(\mathrm{d}x) < \infty, \quad \text{for all } v \in \Lambda. \tag{4.253}$$

Define the *Dobrushin's interdependence matrix* $\mathbf{D}_\Lambda := (D^\Lambda_{vv'})_{\Lambda\times\Lambda}$, cf. (2.174),

$$D^\Lambda_{vv'} := \sup_{\substack{y,\tilde{y}\in\Omega_\Lambda \\ y=\tilde{y} \text{ off } v'}} \left\{ \frac{\mathbf{W}\left(\nu_{v,y}\,,\,\nu_{v,\tilde{y}}\right)}{|y_{v'} - \tilde{y}_{v'}|} \right\}, \quad v \neq v', \quad v,\, v' \in \Lambda, \tag{4.254}$$

(with zero diagonal $D^\Lambda_{vv} = 0$) and suppose that its *spectral radius* (i.e., $\sup_n |\lambda_n|$ taken over all eigenvalues $\lambda_n \in \mathbb{C}$, $1 \le n \le |\Lambda|$) is strictly smaller than one, i.e.,

$$r_{\mathrm{sp}}(\mathbf{D}_\Lambda) < 1. \tag{4.255}$$

In particular, (4.255) is surely implied by the contraction condition in $l^\infty(\Lambda)$

$$||\mathbf{D}_\Lambda||_0 := \sup_{v\in\Lambda} \sum_{v'(\neq v)} D^\Lambda_{vv'} < 1. \tag{4.256}$$

For $v \in \Lambda$, $x \in \Omega_\Lambda$, and $f \in L^2(\nu_\Lambda)$, set

$$\begin{aligned} \mathbf{Var}_{\nu_\Lambda} f &:= \int_{\Omega_\Lambda} f^2(x)\, \mathrm{d}\nu_\Lambda(x) - \left[\int_{\Omega_\Lambda} f(x)\, \mathrm{d}\nu(x)\right]^2, \\ \mathbf{Var}_{v,x} f &:= \mathbf{E}_{v,x} f^2 - (\mathbf{E}_{v,x} f)^2, \\ \mathbf{E}_{v,x} f &:= \int_{\Omega_\Lambda} f(\tilde{x}_v \times x_{\Lambda\setminus v})\, \mathrm{d}\nu_{v,x}(\tilde{x}_v). \end{aligned} \tag{4.257}$$

Proposition 4.49 *(cf. Theorem 2.1 in [288]). If (4.255) holds, then for all functions $f \in L^2(\nu_\Lambda)$*

$$\mathbf{Var}_{\nu_\Lambda} f \le [1 - r_{\mathrm{sp}}(\mathbf{D}_\Lambda)]^{-1} \sum_{v\in\Lambda} \int_{\Omega_\Lambda} \mathbf{Var}_{v,x} f \;\, \mathrm{d}\nu_\Lambda(x). \tag{4.258}$$

Simplified proof of Proposition 4.49. In $L^2(\nu_\Lambda)$ let us consider the following bounded symmetric operator

$$Hf := \sum_{v\in\Lambda} [f - \mathbf{E}_{v,x} f]. \tag{4.259}$$

As was observed by L. Wu, (4.258) is equivalent to the spectral gap for H with the constant

$$C_{\mathrm{SG}}(H) := 1 - r_{\mathrm{sp}}(\mathbf{D}_\Lambda), \tag{4.260}$$

which means

$$\mathrm{Var}_{\nu_\Lambda} f \le [1 - r_{\mathrm{sp}}(\mathbf{D}_\Lambda)]^{-1} (Hf, f)_{L^2(\nu_\Lambda)}, \quad \forall f \in L^2(\nu_\Lambda). \tag{4.261}$$

From here on we proceed in a different way as compared with the original proof in [288]. Let us introduce the Banach space B of all Lipschitz continuous functions on Ω_Λ (factorized modulo constants) with the norm

$$\begin{aligned} ||f||_B &:= ||\delta(f)||_{l^1(\Lambda)} = \sum_{v\in\Lambda} \delta_v(f), \\ \delta_v(f) &:= \sup_{x=\tilde{x} \text{ off } v} \frac{|f(x) - f(\tilde{x})|}{|x_v - \tilde{x}_v|}, \quad \delta(f) = \{\delta_v(f)\}_{v\in\Lambda}. \end{aligned} \tag{4.262}$$

In virtue of (4.253), B is densely embedded into the Hilbert space $\mathcal{H} := L^2(\nu_\Lambda) \ominus \{1\}$. Since $H1 = 0$, both operators $H : B \to B$ and $H : \mathcal{H} \to \mathcal{H}$ are well-defined and bounded. According to the definitions (4.254), (4.257), and (4.262),

$$\delta_v(\mathbf{E}_{v,x}(f)) \le \begin{cases} 0, & v = v', \\ \delta_v(f) + D^\Lambda_{v'v}\delta_{v'}(f), & v \neq v'. \end{cases}$$

Herefrom, calculating Hf by means of (4.259), we find that for any $\lambda \geq 0$

$$\delta_v[(\lambda\mathbf{Id} + H)f] \geq (\lambda+1)\delta_v(f) - \sum_{v'(\neq v)} D^{\Lambda}_{v'v}\delta_{v'}(f) = [((\lambda+1)\mathbf{Id} - \mathbf{D}^{\mathrm{t}}_{\Lambda})\delta(f)]_v.$$

The latter can be understood as the inequality for vector-columns in $\mathbb{R}^{\Lambda}$

$$\delta[(\lambda\mathbf{Id} + H)f] \geq \left[(\lambda+1)\mathbf{Id} - \mathbf{D}^{\mathrm{t}}_{\Lambda})\right] \delta(f). \tag{4.263}$$

Take $\lambda > \max\{||H||_{\mathcal{L}(B)}, ||H||_{\mathcal{L}(\mathcal{H})}\}$, so that the resolvent $(\lambda\mathbf{Id} + H)^{-1}$ is well defined through the Neumann series $\lambda^{-1}\sum_{N=0}^{\infty}(-1/\lambda)^N H^N$ in the both spaces. Recall that by the spectral theorem

$$C_{\mathrm{SG}}(H) := \inf_{||f||_{H_0}=1} (Hf, f)_{H_0} = \limsup_{N\in\mathbb{N}} \left\|(\lambda\mathbf{Id} + H)^{-N}\right\|^{-1/N}_{\mathcal{L}(H_0)} - \lambda. \tag{4.264}$$

On the other hand, the matrix $\mathbf{C} := [(\lambda+1)\mathbf{Id}_{\Lambda} - \mathbf{D}_{\Lambda}]^{-1}$ can be represented through the Neumann series converging in $l^1(\Lambda)$,

$$\mathbf{C} = (\lambda+1)^{-1}\sum_{N=0}^{\infty}\left(\frac{1}{\lambda+1}\mathbf{D}\right)^N,$$

and thus all its entries $C_{vv'}$ are nonnegative. Set $f := (\lambda\mathbf{1} + H)^{-1}g$, then (4.263) may be rewritten for each $g \in B$ as

$$\delta[(\lambda\mathbf{Id} + H)^{-1}g] \leq \left[(\lambda+1)\mathbf{Id}_{\Lambda} - \mathbf{D}^{\mathrm{t}}_{\Lambda}\right]^{-1}\delta(g) = \mathbf{C}^{\mathrm{t}}\delta(g),$$

and hence by the iteration

$$\delta[(\lambda\mathbf{Id} + H)^{-N}g] \leq \left[(\lambda+1)\mathbf{Id}_{\Lambda} - \mathbf{D}^{\mathrm{t}}_{\Lambda}\right]^{-N}\delta(g) = \left(\mathbf{C}^{\mathrm{t}}\right)^N\delta(g), \quad N \in \mathbb{N}.$$

The above inequality for the vectors ensures the estimate for their norms

$$||(\lambda\mathbf{Id} + H)^{-N}g||_B \leq \left\|\mathbf{C}^N\right\|_0 ||g||_B, \tag{4.265}$$

where we have used that $||\left(\mathbf{C}^{\mathrm{t}}\right)^N||_{l^1(\Lambda)} = ||\mathbf{C}^N||_{l^{\infty}(\Lambda)} = \left\|\mathbf{C}^N\right\|_0$.

Now we employ the following *general fact* from the operator theory (for its proof see e.g. Lemma 3.1 in [213]): Let B be a separable Banach space, which is continuously and densely embedded in some Hilbert space $\mathcal{H}$. Then any bounded self-adjoint operator $\mathbf{A} : \mathcal{H} \to \mathcal{H}$ such that $\mathbf{A}B \subseteq B$ certainly satisfies $||\mathbf{A}||_{\mathcal{L}(\mathcal{H})} \leq ||\mathbf{A}||_{\mathcal{L}(B)}$. In the context of (4.265) this means that

$$||(\lambda\mathbf{1} + H)^{-N}||_{\mathcal{L}(\mathcal{H})} \leq ||(\lambda\mathbf{1} + H)^{-N}||_{\mathcal{L}(B)} \leq \left\|\mathbf{C}^N\right\|_0.$$

Finally, we observe that

$$\limsup_{N\in\mathbb{N}}\left\|(\lambda\mathbf{1} + H)^{-N}\right\|^{-1/N}_{\mathcal{L}(\mathcal{H})} \geq \limsup_{N\in\mathbb{N}}\left\|\mathbf{C}^N\right\|_0^{-1/N} = r_{\mathrm{sp}}^{-1}(\mathbf{C}) = \lambda + 1 - r_{\mathrm{sp}}(\mathbf{D}_{\Lambda}),$$

which in virtue of (4.264) gives the result (4.260)

$$C_{\rm SG}(H) \geq 1 - r_{\rm sp}\left(\mathbf{D}_\Lambda\right) \geq 1 - ||\mathbf{D}_\Lambda||_0 > 0.$$

■

Now let us apply the above proposition to the local Gibbs specification dealt with in the Theorem 4.9. We get the following complement to the hypercontractivity result of Theorem 4.30, whereby both (the Otto-Reznikoff and the Efron-Stein-Wu) criteria yield independently the *same upper bound* on $C_{\rm ULS}$ and $C_{\rm USG}$.

Theorem 4.50 *Let the interaction parameters fulfill the relation (4.53). Then, the family of conditional distributions $\mu_{\Lambda,y}(\mathrm{d}x_\Lambda)$, $\Lambda \Subset \mathbb{L}$, $y \in \Omega$, satisfies the Poincaré inequality (4.143) with the uniform constant (coinciding with that one $C_{\rm ULS}$ in (4.166))*

$$\begin{aligned} C_{\rm USG} &:= \inf\left\{C_{\rm SG}(\Lambda, y) \mid \Lambda \Subset \mathbb{V},\ y \in \Omega\right\} \\ &\geq C_{4.166} := \beta\left[(a_U - Jm_{\mathbb{G}})e^{-2\beta\delta_Q} - Jm_{\mathbb{G}}\right]. \end{aligned} \tag{4.266}$$

Proof. Fixed Λ and y, let us calculate the coefficients $D_{vv'}^{\Lambda,y}$ in (4.254) corresponding to the measure $\nu_\Lambda(\mathrm{d}x_\Lambda) := \mu_{\Lambda,y}(\mathrm{d}x_\Lambda)$ and its family of one-point conditional distributions $\nu_{v,z}(\mathrm{d}x_v) := \nu_v(\mathrm{d}x_v|z_\Lambda) = \mu_{v,z_\Lambda\times y_{\Lambda^c}}(\mathrm{d}x_\Lambda)$, $z_\Lambda \in \Omega_\Lambda$. By the construction it is obvious that $D_{vv'}^{\Lambda,y} \leq D_{vv'}$, $v, v' \in \Lambda$, so that the norm $||\mathbf{D}_{\Lambda,y}||_0$ of every finite volume matrix $\mathbf{D}_{\Lambda,y} := (D_{vv'}^{\Lambda,y})_{\Lambda\times\Lambda}$ is dominated by the global norm $||\mathbf{D}||_0$ in Dobrushin's Contraction Condition $(\mathbf{D_2})$ from Subsection 2.3.4. As follows from (4.165), each $\mu_{v,y}(\mathrm{d}x_v)$ obeys the Poincaré inequality (4.144) with the constant

$$C_{\rm SG}(v,y) \geq C_{\rm LS}(v,y) \geq C_{4.267} := \beta e^{-2\beta\delta_Q}(a_U - Jm_{\mathbb{G}}). \tag{4.267}$$

Similarly to the proof of Theorems 2.33 and 2.34, this implies that $D_{vv'} \leq C_{\rm SG}^{-1}\cdot\beta Jm_{\mathbb{G}}$ as $v \sim v'$, and hence

$$r_{\rm sp}(\mathbf{D}) \leq ||\mathbf{D}||_0 \leq \beta e^{2\beta\delta_Q} Jm_{\mathbb{G}}(a_U - Jm_{\mathbb{G}})^{-1}. \tag{4.268}$$

Applying successively the consistency property (2.26) and the one-point Poincaré inequalities to the right-hand-side in (4.258), we get that

$$\begin{aligned} &\mathrm{Var}_{\mu_{\Lambda,y}} f \leq [1 - ||\mathbf{D}||_0]^{-1}\sum_{v\in\Lambda}\int_{\Omega_\Lambda}\mathrm{Var}_{\mu_{v,z_\Lambda\times y_{\Lambda^c}}} f(x_v\times z_{\Lambda\setminus\{v\}})\ \mathrm{d}\mu_{\Lambda,y}(z_\Lambda) \\ \leq\ &[C_{4.267}(1 - ||\mathbf{D}||_0)]^{-1} \\ &\times\sum_{v\in\Lambda}\int_{\Omega_\Lambda}\int_{\Omega_v}|\partial_{x_v} f(x_v\times z_{\Lambda\setminus\{v\}})|^2\mathrm{d}\mu_{v,z_\Lambda\times y_{\Lambda^c}}(x_v)\mathrm{d}\mu_{\Lambda,y}(z_\Lambda) \\ =\ &C_{4.166}^{-1}\int_{\Omega_\Lambda}|\nabla f(z_\Lambda)|^2_{\mathbb{R}^{\nu|\Lambda|}}\ \mathrm{d}\mu_{\Lambda,y}(z_\Lambda), \quad \forall f \in C_0^\infty(\mathbb{R}^{\nu|\Lambda|}). \end{aligned} \tag{4.269}$$

In the last line in (4.269) we have used (4.267), (4.268) to check that $C_{4.267}(1-||\mathbf{D}||_0) \geq C_{4.166}$. As will be shown in the next subsection, cf. Proposition 4.62, $C_0^\infty(\mathbb{R}^{\nu|\Lambda|})$ is the domain of essential self-adjointness for the associated Dirichlet operator $\mathbb{H}_{\Lambda,y}$ in $L^2(\mu_{\Lambda,y})$. Thus, (4.269) extends by continuity to all $f \in \mathcal{D}(\mathbb{H}_\mu^{1/2})$. ■

Remark 4.51 Using tightness of the specification $\{\pi_\Lambda\}_{\Lambda\Subset\mathbb{V}}$ (cf. Proposition 2.7), from Theorems 4.30 and 4.50 one readily obtains the validity of log-Sobolev and Poincaré inequalities, with the same constant C_{ULS}, for the (unique) Gibbs measure $\mu \in \mathcal{G}^{\mathrm{t}}$. The associated (infinite dimensional) Dirichlet operator $\mathbb{H}_\mu$ will be considered in Subsection 4.5.4.

4.5.2 Poincaré and Sobolev inequalities on loop spaces

In this subsection we prove the log-Sobolev and Poincaré inequalities for the local Euclidean Gibbs measures $\mu_{\Lambda,\xi}$ on the loop spaces $\Omega_\Lambda := [C_\beta]^\Lambda$, $\Lambda \Subset \mathbb{L}$. We stress that these results are *entirely new* for the quantum anharmonic systems with non-convex interactions. They will be obtained by means of the Efron-Stein-Wu and Otto-Reznikoff criteria (see Subsections 4.3.2 and 4.5.1), applied to a proper cylinder approximation of the initial path measures. In the special case of strictly convex interactions such functional inequalities have been established in [16, 191]. The log-Sobolev inequality for the one-particle loop measures $\mu_{\ell,\xi}$, similar to that stated in Theorem 4.52 if $\nu = 1$, was first obtained in the joint paper [20], however below we shall offer a new and considerably simpler proof.

For the sake of concreteness, we place ourselves again in the situation of the uniqueness Theorem 3.23, see Subsections 3.2.1 and 3.2.5.

A general strategy is to start with the uniform one-point estimates for the measures $\mu_{\ell,\xi}$. Let us introduce the spaces of smooth functions on the tangent Hilbert space L^2_β. Fix conventionally the orthobasis in L^2_β

$$\begin{aligned} \mathrm{bas}(L^2_\beta) := \left\{ h_{(k,i)} := \varphi_k \otimes e^i, \quad k \in \mathbb{Z}, \quad 1 \le i \le \nu \right\}, \\ A h_{(k,i)} = \lambda_k h_{(k,i)}, \quad \lambda_k = 2m(\pi k/\beta)^2 + a, \end{aligned} \tag{4.270}$$

consisting of the eigenvectors of the operator A which was introduced in Subsection 3.1.3. Here $(\varphi_k)_{k\in\mathbb{Z}}$ is the complete orthonormal system (3.40) of trigonometric functions on S_β and $(e^i)_{i=1}^\nu$ is the standard base of the Euclidean space $\mathbb{R}^\nu$. By $\mathcal{F}C^k_{\mathrm{b}} := \mathcal{F}C^k_{\mathrm{b}}(L^2_\beta)$ for $k \in \mathbb{N} \cup \{0, +\infty\}$ we denote the set of all *cylinder functions* $f : L^2_\beta \to \mathbb{R}$ which can be represented as

$$f(\upsilon) = \phi_L\left((\upsilon, h_1)_{L^2_\beta}, ..., (\upsilon, h_L)_{L^2_\beta} \right), \quad \upsilon \in L^2_\beta, \tag{4.271}$$

with some $\phi_L \in C^k_{\mathrm{b}}(\mathbb{R}^L)$, $l_j \in \mathrm{bas}(L^2_\beta)$, and $1 \le j \le L \in \mathbb{N}$. We shall use the symbol $\nabla f(\upsilon) \in L^2_\beta$ for the *gradient* realization of the Frechét derivative $f'(\upsilon) \in \mathcal{L}(L^2_\beta \to \mathbb{R})$, that is

$$\nabla f(\upsilon) := \sum_{1\le j\le L} \partial_{x_j}\phi_L\left((\upsilon, h_1)_{L^2_\beta}, ..., (\upsilon, h_L)_{L^2_\beta} \right) h_j.$$

Given $\ell \in \mathbb{L}$ and $\xi \in \Omega^{\mathrm{t}}$, on the domain $\mathcal{F}C^\infty_{\mathrm{b}}$ we define the canonical *pre-Dirichlet form* associated with the measure $\mu_{\ell,\xi}$

$$\mathcal{E}_{\mu_{\ell,\xi}}(f,g) := \int_{L^2_\beta} (\nabla f(\omega_\ell), \nabla g(\omega_\ell))_{L^2_\beta} \mathrm{d}\mu_{\ell,\xi}(\omega_\ell), \quad f, g \in \mathcal{F}C^\infty_{\mathrm{b}}. \tag{4.272}$$

Clearly, $(\mathcal{E}_\gamma, \mathcal{F}C_{\rm b}^\infty)$ is closable on $L^2(\mu_{\ell,\xi}) := L^2(L^2_\beta, \mu_{\ell,\xi})$; its closure will be denoted by $\Big(\mathcal{E}_{\mu_{\ell,\xi}}, \mathcal{D}(\mathcal{E}_{\mu_{\ell,\xi}})\Big)$. For a comprehensive introduction to the theory of infinite dimensional Dirichlet forms and additional references see e.g. [26, 199].

Theorem 4.52 *Under assumptions of Theorem 3.23, for all $f \in \mathcal{F}C_{\rm b}^\infty$ the following log-Sobolev inequality is true*

$$\begin{aligned}\mathbf{Ent}_{\mu_{\ell,\xi}} f &:= \int_{L^2_\beta} f^2 \log f^2 \mathrm{d}\mu_{\ell,\xi} - \int_{L^2_\beta} f^2 \mathrm{d}\mu_{\ell,\xi} \cdot \log \int_{L^2_\beta} f^2(\omega_\ell) \mathrm{d}\mu_{\ell,\xi} \\ &\le \frac{2}{C_{\rm LS}} \int_{L^2_\beta} |\nabla f(\omega_\ell)|^2_{L^2_\beta} \mathrm{d}\mu_{\ell,\xi}(\omega_\ell) = \frac{2}{C_{\rm LS}} \mathcal{E}_{\mu_{\ell,\xi}}(f,f), \end{aligned} \tag{4.273}$$

with the log-Sobolev coefficient, which is the same for all $\ell \in \mathbb{L}$, $\xi \in \Omega^{\rm t}$,

$$C_{\rm LS} := (a + a_U + a_W ||\mathbf{J}||_0)\, e^{-2\beta\delta_Q}. \tag{4.274}$$

Proof. Using the Holley-Stroock argument (4.157) to control the bounded perturbation Q and extracting the quadratic potential $(a_U + a_W||\mathbf{J}||_0)\,|q|^2/2$ from $V_\ell := U_\ell + Q_\ell$, we can reduce the problem to studying the following measure on the space of continuous loops $\upsilon \in C_\beta$

$$\gamma(\mathrm{d}\upsilon) = (1/Z) \exp\left\{ - \int_0^\beta U(\upsilon(\tau))\mathrm{d}\tau \right\} \chi(\mathrm{d}\upsilon). \tag{4.275}$$

Here χ is the Gaussian measure associated with a single harmonic oscillator with the rigidity $\tilde{a} = a + a_U + a_W||\mathbf{J}||_0$, cf. (3.41), and $U \in C^2(\mathbb{R}^\nu)$ is a convex function such that $U''(q) \ge 0$ for all $q \in \mathbb{R}^\nu$. Then, the statement would follow, if γ satisfies the log-Sobolev inequality (4.273) with the constant $C_{\rm LS} := \tilde{a}$. To this end, we shall construct a proper cylinder approximation of the density term in (4.275). For any $\upsilon \in L^2_\beta$ we define its *Fourier* and *Cesàro* partial sums (cf. [103]) by

$$\mathbb{S}^i_K(\upsilon) := \sum_{1\le i\le \nu} \sum_{|k|\le K} (\upsilon^i, \varphi_n)_{L^2_\beta} \varphi_k \otimes e_i\,, \quad \mathbb{M}^i_N(\upsilon) := \frac{1}{N+1} \sum_{K=0}^N \mathbb{S}^i_K(\upsilon). \tag{4.276}$$

As well known, the Fourier series does not converge uniformly for $\tau \in S_\beta$ as we need. But fortunately, by Fejér's theorem (cf. Item 6.1.1 in [103]), for any $\upsilon \in C_\beta$

$$\sup_{N\in\mathbb{N}} |\mathbb{M}_N(\upsilon)|_{C_\beta} \le |\upsilon|_{C_\beta} \quad \text{and} \quad \lim_{N\to\infty} |\upsilon - \mathbb{M}_N(\upsilon)|_{C_\beta} = 0. \tag{4.277}$$

Consider the corresponding sequence of probability measures on C_β

$$\gamma_N(\mathrm{d}\upsilon) = (1/Z_N) \exp\left\{ - \int_0^\beta U\left[\mathbb{M}_N(\upsilon)\right](\tau)\mathrm{d}\tau \right\} \chi(\mathrm{d}\upsilon), \quad N \in \mathbb{N}. \tag{4.278}$$

Employing (4.277) and the below boundedness of U, by Lebesgue's dominated convergence theorem we conclude that $\gamma_N \to \gamma$, as $N \to \infty$, in the weak topology on C_β. Having chosen the orthobasis (4.270) in L^2_β, we obtain the isomorphism

$$L^2_\beta \ni \upsilon(\cdot) \mapsto x := (x^i_k)_{k\in\mathbb{Z},\ 1\le i\le\nu} \in l^2(\mathbb{Z}^\nu).$$

It transforms the measure γ_N into a product-measure on $l^2(\mathbb{Z}^\nu)$ of the form

$$\mu_N(\mathrm{d}x) := \frac{1}{Z_N}\exp\{-\mathcal{U}_N(x_{-N},...,x_N)\}\prod_{k\in\mathbb{Z}}\sqrt{\frac{\lambda_k}{2\pi}}e^{-\lambda_k|x_k|^2/2}\mathrm{d}x_k, \tag{4.279}$$

with $\lambda_k := 2m(\pi k/\beta)^2 + \tilde{a}/2$ (cf. (3.39)). The potential $\mathcal{U}_N \in C^2(\mathbb{R}^{\nu(2N+1)})$ is given by

$$\mathcal{U}_N(x_{-N},...,x_N) := \int_0^\beta U\left(\frac{1}{N+1}\sum_{K=0}^{N}\sum_{|k|\le K} x_k\varphi_k(\tau)\right)\mathrm{d}\tau. \tag{4.280}$$

By the construction this function is convex on $\mathbb{R}^{\nu(2N+1)}$ which implies, via the Bakry-Emery criterion (4.156) and the tensorisation property (4.158), the log-Sobolev inequality (4.273) for each $\gamma_N(\mathrm{d}\upsilon)$ with the same $C_{\mathrm{LS}} := \tilde{a}$. Since $f^2\log f^2 \in \mathcal{F}C_{\mathrm{b}}$, its validity for γ follows by taking the limit $N \to \infty$. ■

There are two important sequels of the above theorem. The first one standardly claims validity of the uniform Poincaré inequalities for the measures $\mu_{\ell,\xi}$. The second one, which is known as the variance estimate for Lipschitz continuous functions, was crucially used in proving Theorem 3.23.

Corollary 4.53 *(**Rothaus-Simon mass gap theorem**, cf. [245, 256]) For all $f \in \mathcal{F}C_{\mathrm{b}}^\infty$ such that $f \perp 1$ in $L^2(\mu_{\ell,\xi})$, we have*

$$\mathcal{E}_{\mu_{\ell,\xi}}(f,f) = \int_{L^2_\beta} |\nabla f(\omega_\ell)|^2_{L^2_\beta}\mathrm{d}\mu_{\ell,\xi}(\omega_\ell) \ge C_{\mathrm{LS}}||f||^2_{L^2(\mu_{\ell,\xi})}. \tag{4.281}$$

Corollary 4.54 *(**Variance estimate**) For all $f \in \mathrm{Lip}(L^2_\beta)$ such that*

$$[f]_{\mathrm{Lip}} := \sup_{\upsilon\ne\tilde{\upsilon}}\frac{|f(\upsilon)-f(\tilde{\upsilon})|}{|\upsilon-\tilde{\upsilon}|_{L^2_\beta}} < \infty,$$

the following inequality holds:

$$\mathbf{Var}_{\mu_{\ell,\xi}}f := \int_{L^2_\beta}\left[f(\omega_\ell) - \mathbf{E}_{\mu_{\ell,\xi}}f\right]^2\mathrm{d}\mu_{\ell,\xi}(\omega_\ell) \le \frac{1}{C_{\mathrm{LS}}}[f]^2_{\mathrm{Lip}}. \tag{4.282}$$

Proof. The spectral gap inequality (4.281) can be rewritten for all $f \in \mathcal{F}C_{\mathrm{b}}^\infty$ as

$$\begin{aligned}\int_{L^2_\beta}\left[f(\omega_\ell) - \mathbf{E}_{\mu_{\ell,\xi}}f\right]^2\mathrm{d}\mu_{\ell,\xi}(\omega_\ell) &\le \frac{1}{C_{\mathrm{LS}}}\int_{L^2_\beta}|\nabla f(\omega_\ell)|^2_{L^2_\beta}\mathrm{d}\mu_{\ell,\xi}(\omega_\ell)\\ \le \frac{1}{C_{\mathrm{LS}}}\sup_{\omega_\ell\in L^2(\nu)}|\nabla f(\omega_\ell)|^2_{L^2_\beta} &= \frac{1}{C_{\mathrm{LS}}}[f]^2_{\mathrm{Lip}}.\end{aligned} \tag{4.283}$$

To extend (4.283) to general $f \in \mathrm{Lip}(L^2_\beta)$ we shall use their cylinder approximation $f_N(\upsilon) := f(\mathbb{S}_N(\upsilon))$, cf. (4.276). Since $\mathbf{E}_{\mu_{\ell,\xi}}|\omega_\ell|^2 < \infty$, it is obvious that $[f_N]_{\mathrm{Lip}} \le [f]_{\mathrm{Lip}}$

and $|f - f_N|_{L^2(\mu_{\ell,\xi})} \to 0$ as $N \to \infty$. So, it would suffice to prove (4.283) for all cylinder Lipschitz functions $f_N(\upsilon) := \phi_L\left((\upsilon, h_1)_{L^2_\beta}, ..., (\upsilon, h_L)_{L^2_\beta}\right)$ of the form (4.271) with some $\phi_L \in \mathrm{Lip}(\mathbb{R}^L)$ and $L \in \mathbb{N}$. But for every Lipschitz function $\phi_L(s_1, ..., s_L)$ on $\mathbb{R}^L$ there exists a sequence $(\phi_{L,M})_{n\in\mathbb{N}} \subset C_0^\infty(\mathbb{R}^L)$ such that $[\phi_{L,M}]_{\mathrm{Lip}} \leq [\phi_L]_{\mathrm{Lip}}$ and $\phi_{L,M} \to \phi_L$ pointwise on $\mathbb{R}^L$ as $M \to \infty$. Here we can e.g. use general properties of regularization by convolutions $\phi_{L,M} := \rho_M * \phi_L$ in Hölder spaces, see the proof of Theorem 4.61 above or Subsection 1.3.2 in [272]. And finally, $f_{N,M}(\upsilon) := \phi_{L,M}\left((\upsilon, h_1)_{L^2_\beta}, ..., (\upsilon, h_L)_{L^2_\beta}\right)$ gives us the desired approximation of f_N as $M \to \infty$. ■

For $\Lambda \Subset \mathbb{L}$, let $\mathcal{F}C_{\mathrm{b}}^\infty(\Omega_\Lambda)$ denote the set of all smooth *cylinder* functions $f : \Omega_\Lambda \to \mathbb{R}$ which can be represented as

$$f(\omega_\Lambda) = \phi_L\left((\omega_{\ell_1}, h_1)_{L^2_\beta}, ..., (\omega_{\ell_L}, h_L)_{L^2_\beta}\right), \tag{4.284}$$

with some $\phi_L \in C_{\mathrm{b}}^\infty(\mathbb{R}^\Lambda)$, $\ell_j \in \Lambda$, $h_j \in \mathrm{bas}(L^2_\beta)$, and $1 \leq j \leq L \in \mathbb{N}$. Define the *gradient*

$$\nabla f(\omega) := (\nabla_\ell f(\omega))_{\ell\in\mathbb{L}} \in l^2(\mathbb{L} \to L^2_\beta), \tag{4.285}$$
$$\nabla_\ell f(\omega) := \sum_{j:\, \ell_j=\ell} \partial_{x_{\ell_j}}\phi_L\left((\omega_{\ell_1}, h_1)_{L^2_\beta}, ..., (\omega_{\ell_L}, h_L)_{L^2_\beta}\right) h_{\ell_j} \in L^2_\beta, \quad \ell \in \mathbb{L}.$$

Applying now the *Efron-Stein-Wu inequality* for variances to the Euclidean measures $\mu_{\Lambda,\xi}$, cf. Proposition 4.49, we in a *short way* get the quantum analog of Theorems 2.39, 4.50.

Theorem 4.55 *Let the interaction parameters fulfill the relation (3.106). Then, the family of conditional distributions $\mu_{\Lambda,\xi}(\mathrm{d}\omega_\Lambda)$, $\Lambda \Subset \mathbb{L}$, $\xi \in \Omega^{\mathrm{t}}$, satisfies the Poincaré inequalities on all smooth cylinder functions $f \in \mathcal{F}C_{\mathrm{b}}^\infty(\Omega_\Lambda)$ such that $\mathbf{E}_{\mu_{\Lambda,\xi}} f = 1$*

$$\begin{aligned}\mathcal{E}_{\mu_{\Lambda,\xi}}(f, f) \quad : &= \int_{\Omega_\Lambda} \sum_{\ell\in\Lambda} |\nabla_\ell f(\omega)|^2_{L^2_\beta} \mathrm{d}\mu_{\mu_{\Lambda,\xi}}(\omega) \\ &= \int_{\Omega_\Lambda} |\nabla f(\omega)|^2_{l^2(\mathbb{L}\to L^2_\beta)} \mathrm{d}\mu_{\Lambda,\xi}(\omega_\Lambda) \geq C_{\mathrm{USG}}||f||^2_{L^2(\mu_{\Lambda,\xi})},\end{aligned} \tag{4.286}$$

with the uniform constant (coinciding with that one for the classical case in (2.221))

$$C_{\mathrm{USG}} \geq C_{4.287} := \beta\left[(a_U + a_W||\mathbf{J}||_0)\, e^{-2\beta\delta_Q} - b_W||\mathbf{J}||_0\right]. \tag{4.287}$$

Proof. Using the same arguments that proved Theorem 4.50 and substituting there the upper bound (3.171) for $||\mathbf{D}||_0$, we get the required estimate

$$\begin{aligned}
&\mathbf{Var}_{\mu_{\Lambda,\xi}} f \leq [1-||\mathbf{D}||_0]^{-1} \sum_{v\in\Lambda} \int_{\Omega_\Lambda} \mathbf{Var}_{\mu_{\ell,\eta_\Lambda \times \xi_{\Lambda^c}}} f(\omega_\ell \times \eta_{\Lambda\setminus\{\ell\}})\, \mathrm{d}\mu_{\Lambda,y}(\eta_\Lambda) \\
\leq\ & [C_{4.267}(1-||\mathbf{D}||_0)]^{-1} \\
&\times \sum_{v\in\Lambda} \int_{\Omega_\Lambda} \int_{\Omega_\ell} |\nabla_\ell f(x_\ell \times \eta_{\Lambda\setminus\{\ell\}})|^2 \mathrm{d}\mu_{\ell,\eta_\Lambda\times\xi_{\Lambda^c}}(\omega_\ell) \mathrm{d}\mu_{\Lambda,\xi}(\eta_\Lambda) \\
=\ & [C_{4.166}]^{-1} \int_{\Omega_\Lambda} |\nabla f(\eta_\Lambda)|^2_{l^2(\mathbb{L}\to L^2_\beta)} \mathrm{d}\mu_{\Lambda,\xi}(\eta_\Lambda), \quad \forall f \in \mathcal{F}C_{\mathrm{b}}^\infty(\Omega_\Lambda).
\end{aligned}$$

■

With some extra effort we can prove even the stronger property of *hypercontractivity*, which extends the result of Theorem 4.30 to the loop spaces. Like as in the classical case, cf. Theorem 4.30, the proof will rely on the *Otto-Reznikoff criterion* presented in Subsection 4.3.2.

Theorem 4.56 *Under the same assumptions, the local conditional distributions* $\mu_{\Lambda,\xi}(\mathrm{d}\omega_\Lambda)$, $\Lambda \Subset \mathbb{L}$, $\xi \in \Omega^{\mathrm{t}}$, *obey the log-Sobolev inequalities*

$$\mathbf{Ent}_{\mu_{\Lambda,\xi}} f \leq \frac{2}{C_{\mathrm{ULS}}} \int_\Omega \sum_{\ell\in\Lambda} |\nabla_\ell f(\omega)|^2_{L^2_\beta} \mathrm{d}\mu(\omega) = \mathcal{E}_{\mu_{\Lambda,\xi}}(f,f), \quad f \in \mathcal{F}C_{\mathrm{b}}^\infty(\Omega_\Lambda), \quad (4.288)$$

with the uniform constant $C_{\mathrm{ULS}} \geq C_{4.287}$.

Proof. We begin similarly to the proof of Theorem 4.61 by extracting the quadratic terms $(a_U + a_W||\mathbf{J}||_0)\,|\omega_\ell|^2_{L^2_\beta}/2$ and setting $\tilde{a} := a + a_U + a_W||\mathbf{J}||_0$. This gives rise to the following assumptions on the potentials, cf. (3.101)–(3.105),

$$\begin{aligned}
&V_\ell := U_\ell + Q_\ell, \quad U_\ell'' \geq 0, \quad \mathbf{Osc}(Q_\ell) \leq \delta_Q < \infty, \\
&\partial^2_{q_\ell} W_{\ell\ell}(q_\ell, q_\ell) \geq 0, \quad |\partial^2_{q_\ell q_{\ell'}} W_{\ell\ell'}(q_\ell, q_{\ell'})|_{\mathcal{L}(\mathbb{R}^\nu)} \leq J_{\ell\ell'} b_W.
\end{aligned} \quad (4.289)$$

Fixed $\Lambda \Subset \mathbb{L}$ and $\xi \in \Omega^{\mathrm{t}}$, we then consider the approximation of $\mu_{\Lambda,\xi}(\mathrm{d}\omega_\Lambda)$ by means of the Cesàro partial sums

$$\mu_N(\mathrm{d}\omega_\Lambda) := (1/Z_N)\exp\left\{-I_\Lambda\left(\mathbb{M}_N^\Lambda(\omega_\Lambda)\ \mid \xi\right)\right\}\chi_\Lambda(\mathrm{d}\omega_\Lambda),$$

where $I_\Lambda(\omega_\Lambda|\xi)$ is the Euclidean energy functional defined in (3.51), (3.54). Passing to the isomorphism

$$L^2_\beta \ni \omega_\Lambda(\cdot) \mapsto x_\Lambda := (x_{\ell,k})_{\ell\in\Lambda,\, k\in\mathbb{Z}} \in l^2(\mathbb{Z}^{\nu|\Lambda|}), \quad x_{\ell,k} \in \mathbb{R}^\nu,$$

we transform the measure μ_N into the following classical Gibbs distribution on the

Hilbert space $l^2(\mathbb{Z}^{\nu|\Lambda|})$

$$\mu_N(\mathrm{d}x) := \frac{1}{Z_N} \prod_{\ell\in\Lambda,\ |k|>N} \sqrt{\frac{\lambda_k}{2\pi}} e^{-\lambda_k |x_{\ell,k}|^2/2} \mathrm{d}x_{\ell,k}$$
$$\times \prod_{\ell\in\Lambda} \left\{ \exp\left[-\mathcal{H}_{\ell,N}(x_{\ell,-N},...,x_{\ell,N};\xi)\right] \prod_{|k|\le N} \sqrt{\frac{\lambda_k}{2\pi}} e^{-\lambda_k |x_{\ell,k}|^2/2} \mathrm{d}x_{\ell,k} \right\}$$
$$\times \exp\left\{ -\frac{1}{2} \sum_{\ell,\ell'\in\Lambda} \mathcal{W}_{\ell\ell',N}(x_{\ell,-N},...,x_{\ell,N}; x_{\ell',-N},...,x_{\ell',N}) \right\}, \tag{4.290}$$

with the interactions defined by

$$\mathcal{U}_{\ell,N}(x_{\ell,-N},...,x_{\ell,N}) := \int_0^\beta U_\ell \left(\sum_{|k|\le N} x_{\ell,k}\psi_k(\tau) \right) \mathrm{d}\tau,$$
$$\mathcal{W}_{\ell\ell',N}(x_{\ell,-N},...,x_{\ell,N}\,;\,\xi_{\ell'}) := \int_0^\beta W_{\ell\ell'} \left(\sum_{|k|\le N} x_{\ell,k}\psi_k(\tau)\,,\,\xi_{\ell'} \right) \mathrm{d}\tau,$$
$$\mathcal{W}_{\ell\ell',N}(x_{\ell,-N},...,x_{\ell',N}) := \int_0^\beta W_{\ell\ell',N} \left(\sum_{|k|\le N} x_{\ell,k}\psi_k(\tau)\,,\, \sum_{|j|\le N} x_{\ell,j}\psi_j(\tau) \right) \mathrm{d}\tau,$$
$$\mathcal{H}_{\ell,N}(x_{\ell,-N},...,x_{\ell,N}\,;\,\xi) := \frac{1}{2} \sum_{\ell,\ell'\in\Lambda} \mathcal{U}_{\ell,N}(x_{\ell,-N},...,x_{\ell,N})$$
$$+ \sum_{\ell'\in\Lambda^c} \mathcal{W}_{\ell\ell',N}(x_{\ell,-N},...,x_{\ell,N}\,;\,\xi_{\ell'}). \tag{4.291}$$

For convenience we here introduced the functions $\psi_k : S_\beta \to \mathbb{R}$,

$$\psi_k(\tau) := \left(1 - \frac{|k|}{N+1}\right) \varphi_k(\tau), \quad \tau \in S_\beta. \tag{4.292}$$

By the tensorisation argument (4.158), one-point estimates (4.273), (4.274), and Otto-Reznikoff theorem we have that $C_{\mathrm{LS}}(\Lambda,\xi)$ will be not smaller than the lower bound of the symmetric matrix $\mathbf{A} := (A_{\ell\ell'})_{\Lambda\times\Lambda}$ with the entries

$$A_{\ell\ell} := \tilde{a}\beta e^{-2\beta\delta_Q}, \quad A_{\ell\ell'} := -\sup_{x_\ell, x_{\ell'}} \left| \partial^2_{x_\ell x_{\ell'}} \mathcal{W}_{\ell\ell',N}(x_{\ell,-N},...,x_{\ell',N}) \right|_{\mathcal{L}(\mathbb{R}^{2N+1})}.$$

Observe that by (4.289), (4.291), and (4.292) we have that for any finite sequence

$c_k \in \mathbb{R}^\nu$, $|k| \leq N$,

$$\left| \sum_{k,j} \left\langle \partial^2_{x_{\ell,k} x_{\ell',j}} \mathcal{W}_{\ell\ell',N}(x_{\ell,-N}, ..., x_{\ell',N}) c_k \ , \ c_j \right\rangle_{\mathbb{R}^\nu} \right| \tag{4.293}$$
$$= \left| \int_0^\beta \sum_{k,j} \left\langle \partial^2_{q_\ell q_{\ell'}} W_{\ell\ell'} \left(\sum_{|k|\leq N} x_{\ell,k}\psi_k(\tau) \right) c_k \psi_k(\tau) \ , \ c_j \psi_j(\tau) \right\rangle_{\mathbb{R}^\nu} \mathrm{d}\tau \right|$$
$$\leq J_{\ell\ell'} \int_0^\beta \left| \sum_k c_k \psi_k(\tau) \right|^2 \mathrm{d}\tau \leq J_{\ell\ell'} \sum_k \left(1 - \frac{|k|}{N+1} \right)^2 |c_k|^2 \leq J_{\ell\ell'} \sum_k |c_k|^2,$$

which implies $A_{\ell\ell'} \geq -b_W J_{\ell\ell'}$. Then obviously

$$\inf_{\Lambda,\xi} C_{\mathrm{LS}}(\Lambda,\xi) \geq \tilde{a}\beta e^{-2\beta\delta_Q} - b_W ||\mathbf{J}||_0,$$

which completes the proof. ■

The validity of the uniform log-Sobolev inequalities (4.288) would be an important step towards establishing the *pointwise ergodicity* of the stochastic dynamics on loop spaces associated with the Euclidean Gibbs states $\mu \in \mathcal{G}^{\mathrm{t}}$ (see the motivating discussion in Subsection 4.2.3)

4.5.3 Integration by parts description of Gibbs states

Here we briefly discuss the main ingredients of the so-called *analytical approach* to the Euclidean Gibbs measures, which was developed in the joint papers [10]–[13] with S. Albeverio, Yu. Kondratiev, and M. Röckner. The aim is twofold: (i) to illustrate some striking applications of stochastic analysis in quantum statistical physics; and (ii) to prepare a background for studying the corresponding Dirichlet operators on loop spaces in Subsection 4.5.4.

A basic idea of the analytical approach is to use an *alternative characterization* of Gibbs measures in terms their *Radon–Nikodym* and *logarithmic derivatives*, instead of the traditional one through the local specification $\Pi = \{\pi_\Lambda\}_{\Lambda \Subset \mathbb{L}}$ and the *DLR* equation (3.86). Such alternative descriptions of Gibbs measures have long been known for a number of specific models in statistical mechanics and field theory (see e.g. [87, 109, 115, 117, 119, 144, 153, 157, 218, 247]). Both for the classical and for the quantum lattice systems, a complete characterization of $\mu \in \mathcal{G}^{\mathrm{t}}$ as quasi-invariant measures with the prescribed *Radon-Nikodym derivatives* has first been proven in [17, 18]. Assuming that the interaction potentials $V_\ell, W_{\ell\ell'}$ are differentiable, it was further shown in [10]–[13] that the later description of $\mu \in \mathcal{G}^{\mathrm{t}}$ is equivalent to their characterization as *differentiable measures* satisfying *integration by parts formulas*. In the earlier articles of the same authors [22, 23] this alternative approach to studying Gibbs measures has been realized in the (much simpler) situation of classical lattice systems, see also Proposition 2.37 above. However, the abstract scheme which was suggested there is not directly applicable to the quantum case. The reason is that for Euclidean Gibbs states we have to perform not only a "*lattice analysis*" (depending on the stability properties of the interaction potentials V_ℓ, $W_{\ell\ell'}$), but also a separate and rather non-trivial "*single spin space analysis*" (taking into account the spectral properties of the operator A which describes the "quantumness" of the system).

To present the main statements we confine ourselves to the quantum model (3.1), (3.2), which was studied in Chapter 3. Additionally to the basic Assumptions (**V**), (**W**), ($\mathbf{J}_\alpha$), we suppose that $V_\ell \in C^2(\mathbb{R}^\nu)$, $W_{\ell\ell'} \in C^2(\mathbb{R}^{2\nu})$ satisfy the following conditions, which are typical for the method.

Assumption ($\mathbf{W_8}$) *For all* $\ell \neq \ell'$ *and* $q_\ell, q_{\ell'} \in \mathbb{R}^\nu$, *it holds*

$$|\partial_{q_\ell}^{(n)} W(q_\ell, q_{\ell'})| \leq \frac{1}{2} J_{\ell\ell'}(C_W + |q_\ell|^{R-n} + |q_{\ell'}|^{R-n}), \quad n = 0, 1, 2. \tag{4.294}$$

Assumption ($\mathbf{V_8}$) *Functions* V_ℓ *and their derivatives are polynomially bounded, which means that for some* $Q \geq 2$ *and* $C_V > 0$

$$|V_\ell^{(n)}(q_\ell)| \leq C_V \,(1 + |q_\ell|)^Q\,, \quad n = 0, 1, 2. \tag{4.295}$$

Moreover, there exist $A_8 > ||\mathbf{J}||_0 R/2$ *and corresponding* $B_8 > 0$, $C_8 \in \mathbb{R}$, *such that for all* $\ell \in \mathbb{L}$ *and* $q_\ell \in \mathbb{R}^\nu$

$$(V_\ell'(q_\ell), q_\ell) \geq A_8 |q_\ell|^R + B_8 \sum_{n=1,2} |\nabla^{(n)} V(q_\ell)|\,(1 + |q_\ell|^n) + C_8. \tag{4.296}$$

We start with the *flow description* of $\mu \in \mathcal{G}^{\mathrm{t}}$ in terms of their Radon–Nikodym derivatives under the local shift transformations of the underlying configuration space. Let us consider

$$\mathcal{H}_0 := l^2(\mathbb{L} \to L^2_\beta) \cong l^2(\mathbb{L}) \otimes L^2_\beta \tag{4.297}$$

with inner product $(\omega, \omega)_0 = ||\omega||_0^2 := \sum_\ell |\omega_\ell|^2_{L^2_\beta}$ as a *tangent* Hilbert space to Ω. Similarly to (4.270), we fix the canonical orthobasis

$$\mathrm{bas}(\mathcal{H}_0) := \left\{ h_{(\ell,k,i)} := e_\ell \otimes \varphi_k \otimes e^i \;\mid \ell \in \mathbb{L},\; k \in \mathbb{Z},\; 1 \leq i \leq \nu \right\}, \tag{4.298}$$

where $e_\ell := (\delta_{\ell\ell'})_{\ell' \in \mathbb{L},}$ $e^i := (\delta_{ii'})_{1 \leq i' \leq \nu}$, and $h_{(k,i)} := \varphi_k \otimes e^i$, $Ah_{(k,i)} = \lambda_k h_{(k,i)}$.

Proposition 4.57 *(see [10]–[13], [17, 18]). Let* $\mathcal{P}^{\mathrm{t}}_a$ *denote the set of all probability measures* $\mu \in \mathcal{P}^{\mathrm{t}}(\Omega)$ *which satisfy the temperedness condition (3.77) and are quasi-invariant with respect to the shifts*

$$\omega \mapsto \omega + \theta h_{(\ell,k,i)}, \quad \textit{for all} \;\; \theta \in \mathbb{R},\; h_{(\ell,k,i)} \in \mathrm{bas}(\mathcal{H}_0),$$

with the Radon–Nikodym derivatives

$$a_{(\ell,k,i)}(\theta, \omega) := \exp\left\{ -\theta (Ah_{(k,i)}, \omega_\ell)_{L^2_\beta} - \frac{\theta^2}{2} (Ah_{(k,i)}, h_{(k,i)})_{L^2_\beta} - \mathcal{I}_{\mathrm{rel}}(\omega | \theta h_{(\ell,k,i)}) \right\}, \tag{4.299}$$

where

$$\mathcal{I}_{\mathrm{rel}}(\omega|\theta h_{(\ell,k,i)}) := \int_0^\beta \left[V_\ell(\omega_\ell + \theta h_{(k,i)}) - V_\ell(\omega_\ell)\right] \mathrm{d}\tau \tag{4.300}$$
$$+ \int_0^\beta \sum_{\ell'(\neq \ell)} \left[W_{\ell\ell'}(\omega_\ell, \omega_{\ell'}) - W_{\ell\ell'}(\omega_\ell + \theta h_{(k,i)}, \omega_{\ell'})\right] \mathrm{d}\tau$$

is the corresponding relative interaction. Then $\mathcal{G}^{\mathrm{t}} = \mathcal{M}^{\mathrm{t}}_a$.

In applications it is more convenient to use not the flow characterization itself, but its *infinitesimal* form which we shall describe now. To this end we define the *partial logarithmic derivatives* of the measures $\mu \in \mathcal{M}^{\mathrm{t}}_a$ along directions $h_{(\ell,k,i)}$ by

$$b_{(\ell,k,i)}(\omega) := \frac{\partial}{\partial\theta} a_{\theta h_{(\ell,k,i)}}(\omega)\Big|_{\theta=0} \tag{4.301}$$
$$= -\left(A h_{(k,i)}, \omega_\ell\right)_{L^2_\beta} - \left(F_\ell(\omega), h_{(k,i)}\right)_{L^2_\beta}, \quad \omega \in \Omega^{\mathrm{t}}.$$

Here $F_\ell : \Omega^{\mathrm{t}} \to L^{R'}_\beta$ (with $1/R + 1/R' = 1$) is the nonlinear Nemytskii-type operator acting by

$$F_\ell(\omega) := V'_\ell(\omega_\ell) + \sum\nolimits_{\ell'(\neq\ell)} \partial_{q_\ell} W_{\ell\ell'}(\omega_\ell, \omega_{\ell'}). \tag{4.302}$$

The *logarithmic gradient* of the measure $\mu \in \mathcal{M}^{\mathrm{t}}_a$ is a vector field $b := (b_\ell)_{\ell\in\mathbb{L}}$ with the components

$$\Omega^{\mathrm{t}} \ni \omega \to b_\ell(\omega) := \sum_{k,i} b_{(\ell,k,i)}(\omega) \cdot h_{(k,i)} = -A\omega_\ell - F_\ell(\omega) \in W^{-2}_\beta, \tag{4.303}$$

where, by definition, the Sobolev spaces $W^{\pm 2}_\beta$ are completion of C^∞_β for the norms $|v|_{W^{\pm 2}_\beta} := |A^{\pm 1} v|_{L^2_\beta}$. As well known, the embeddings $W^2_\beta \hookrightarrow L^R_\beta$ and $L^{R'}_\beta \hookrightarrow W^{-2}_\beta$ are compact for any $R \geq 2$.

For each direction $h_{(\ell,k,i)}$, we denote by $C^1_{\mathrm{dec}}(\Omega^{\mathrm{t}}; h_{(\ell,k,i)})$ the set of all functions $f : \Omega^{\mathrm{t}} \to \mathbb{R}$ which are bounded and continuous together with their partial derivatives $\partial_{h_{(\ell,k,i)}} f$ and satisfy the decay condition

$$\sup_{\omega\in\Omega^{\mathrm{t}}} \left| f(\omega)\left(1 + |\omega_\ell|_{L^1_\beta} + |F_\ell(\omega)|_{L^1_\beta}\right)\right| < \infty. \tag{4.304}$$

By the above construction, $f b_{(\ell,k,i)} \in L^\infty(\mu)$ for all such f and any $\mu \in \mathcal{P}^{\mathrm{t}}$, even though we do not know *à-priori whether* $b_{(\ell,k,i)}(\omega) \in L^1(\mu)$. For smooth interaction potentials (as they are in our case), the flow characterization of $\mu \in \mathcal{G}^{\mathrm{t}}$ by Proposition 4.57 is equivalent to their characterization as differentiable measures solving the *integration by parts* (for short, *IbP*) equations

$$\partial_{(\ell,k,i)}\mu(\mathrm{d}\omega) = b_{(\ell,k,i)}(\omega)\mu(\mathrm{d}\omega) \tag{4.305}$$

with the logarithmic derivatives $b_{(\ell,k,i)}$ defined by (4.303). An analogous characterization of the classical Gibbs states was given by Proposition 2.37.

Proposition 4.58 *(see [10]–[13]). Let $\mathcal{M}_b^{\mathrm{t}}$ denote the set of all tempered measures $\mu \in \mathcal{P}^{\mathrm{t}}$ which satisfy the (IbP)-formula*

$$\int_{\Omega} \partial_{(\ell,k,i)} f(\omega) \mathrm{d}\mu(\omega) = - \int_{\Omega} f(\omega) b_{(\ell,k,i)}(\omega) \mathrm{d}\mu(\omega) \tag{4.306}$$

for all test functions $f \in C^1_{\mathrm{dec}}(\Omega^{\mathrm{t}}; h_{(\ell,k,i)})$ and any direction $h_{(\ell,k,i)} \in \mathrm{bas}(\mathcal{H}_0)$. Then $\mathcal{G}^{\mathrm{t}} = \mathcal{M}_b^{\mathrm{t}}$.

Based on Proposition 4.58, instead of $\mu \in \mathcal{G}^{\mathrm{t}}$ defined as Markov fields on $\mathbb{L}$ we can study solutions to the (IbP)-formula (4.306), which in stochastic analysis are also called *symmetrizing measures.* For further connections to the reversible diffusion processes and Dirichlet operators in infinite dimensions we refer e.g. to [17, 18, 24, 56].

Remark 4.59 (i) The above $b_{(\ell,k,i)}$ depend only on the potentials $V_\ell, W_{\ell\ell'}$ and hence are the *same for all* $\mu \in \mathcal{G}^{\mathrm{t}}$ associated with the heuristic Hamiltonian (3.1). Actually, the flow and (*IbP*)-characterizations in Propositions 4.57 and 4.58 are true under *minimal assumptions* on the potentials, which guarantee just the continuity and local boundedness of the mappings (4.299), (4.303).

(ii) The *main difficulty* in dealing with the (*IbP*)-formula (4.306) is that we do not know in advance (until proving Theorem 3.19) *whether* $b_k \in L^1(\mu)$ for any $\mu \in \mathcal{G}^{\mathrm{t}}$. This problem may be overcome by the special choice (4.304) of test functions f, to which we can correctly apply both sides of the distributional equation (4.305).

The most progress achieved in the analytical approach is related with the problems of *existence* and *à-priori estimates* for the Gibbs measures. Until recently it remained the only *universal method* for studying the existence problem for general non-translation invariant interactions. So, the main statements of the joint papers [10]–[13] claim that, under the hypotheses more or less similar to Assumptions $(\mathbf{V}_8), (\mathbf{W}_8)$, the set of tempered Euclidean Gibbs measures is not empty at all temperature $\beta > 0$ and its elements obey the à-priori bounds (3.97). The *key point* of the proofs is that according to (4.306) each $\mu \in \mathcal{G}^{\mathrm{t}}$ might be viewed as a solution of the infinite system (4.305) of first order partial differential equations (*PDE*'s). Due to the above assumptions on the potentials $V_\ell, W_{\ell\ell'}$, the vector field $b := (b_\ell)_{\ell \in \mathbb{L}}$ possesses certain *coercivity properties* with respect to the tangent space $\mathcal{H}_0$. This enables us to employ an analog of the *Lyapunov function method,* well-known from finite dimensional *PDE*'s, to get uniform moment estimates (3.97) on $\mu \in \mathcal{G}^{\mathrm{t}}$. For any fixed boundary condition $\xi \in \Omega^{\mathrm{t}}$, the probability kernels $\pi_\Lambda(\mathrm{d}\omega|\xi)$ satisfy the same integration by parts formulas in directions $h_{(\ell,k,i)}$ with $\ell \in \Lambda$. Herefrom we can derive the moment estimates like (3.119) *uniformly* in volume, which by the compactness argument will ensure the existence of $\mu \in \mathcal{G}^{\mathrm{t}}$. In the extended review [13] we have also worked out an *abstract setting* for this approach and studied the measures on linear spaces satisfying integration by parts formulas with the given logarithmic gradient b. Other important and long-standing problem is to find sufficient conditions for the *uniqueness* of symmetrizing measures in infinite dimensions. Particular results on this topic were obtained in [56, 59, 60].

Furthermore, the (*IbP*)-description of $\mu \in \mathcal{G}^{\mathrm{t}}$ provides a background for the *stochastic dynamics method,* in which the Gibbs measures are treated as invariant distributions for certain infinite-dimensional stochastic evolution equations, see [24, 25] and Subsection 4.2.6.

From our viewpoint, the new method we developed in Subsections 3.2.2, 3.2.4 for proving Theorems 3.18, 3.19 seems to be *more elementary.* Furthermore, it allows to drop a number of technical conditions on the potentials $V_\ell, W_{\ell\ell'}$ and do not require their differentiability.

4.5.4 Essential self-adjointness of the Dirichlet operators

In this subsection we study essential self-adjointness of the Dirichlet operators $\mathbb{H}_\mu$ associated with the Gibbs measures $\mu \in \mathcal{G}^{\mathrm{t}}$ in different types of models dealt with in Chapters 2–4. As was already mentioned in Subsection 2.3.5 (ii), this is a fundamental property, which in particular implies the uniqueness of the corresponding equilibrium dynamics $\mathbb{T}_t := \exp(-t\mathbb{H}_\mu)$, $t \geq 0$, in $L^2(\mu)$. Depending on a concrete situation (classical or quantum systems, in infinite or finite volumes) we shall apply several modifications of the so-called *approximative self-adjointness criterion*, see Theorem 1 of [194], Theorem 1 of [16], or Theorem 3.1 of [191]. A common feature of such theorems is that they presume constructing a proper smooth approximation for coefficients of the considered operator; their proofs however could be technically quite disjoint.

For the classical lattice spin systems on $\mathbb{L} := \mathbb{Z}^d$, essential self-adjointness of the infinite dimensional Dirichlet operators $\mathbb{H}_\mu$ on natural domains like $\mathcal{F}C_{\mathrm{b}}^{\infty}(\Omega)$ was shown in [15, 16, 20, 166, 167, 210]. There are also few results related to the quantum lattice systems, see respectively [191] for $\beta < \infty$ and [155] for $\beta = \infty$, and to the Euclidean quantum fields in finite volume, see [82, 194]. Concerning the models of our interest, the techniques developed so far are principally limited to the pair potentials $W_{\ell\ell'}$ having at most quadratic growth (i.e., $R = 2$) and the one-particle potentials V_ℓ obeying certain *coercivity* and *semi-monotonicity* properties. Our self-adjointness criteria, Theorems 4.61, 4.63 and 4.64 below, impose the most *general* assumptions on $V_\ell, W_{\ell\ell'}$ of such type and are stated for the *whole* class of Gibbs measures $\mu \in \mathcal{G}^{\mathrm{t}}$. In particular, this covers the result of [191] which concerned only with the "superstable" Gibbs states, cf. Remark 3.34. A possible extension to arbitrary $\mu \in \mathcal{G}^{\mathrm{t}}$ relies on the regularity properties of those measures established in Subsections 3.2.1, 3.2.4 and respectively 2.2.3, 2.2.4. To construct the required approximations of the operators $\mathbb{H}_\mu$ we shall use smoothing by convolutions in the spin spaces $\mathbb{R}^\nu$ and Cesàro partial sums in C_β. For introductory material on the Dirichlet operators and forms associated with the Gibbs measures $\mu \in \mathcal{G}^{\mathrm{t}}$ see Subsection 2.3.5(ii).

(i) Classical case

We first place ourselves in the situation of Chapter 2 and consider the spin system (2.1), with possibly infinite range of the interaction, which lives on some indexing set $\mathbb{L}$. To this end, additionally to the main Assumptions $(\mathbf{L}_d)$, $(\mathbf{W})$, $(\mathbf{J})$, and either $(\mathbf{V})$ or $(\mathbf{V}_1)$

holding with $P \geq R = 2$, we suppose that $V_\ell \in C^1(\mathbb{R}^\nu)$ and $W_{\ell\ell'} \in C^1(\mathbb{R}^{2\nu})$ satisfy the following conditions (for convenience, we here continue the previous numbering):

Assumption $(\mathbf{W}_9)$ *There exists* $C_W > 0$ *such that for all* $\ell, \ell' \in \mathbb{L}$ *and* $x_\ell, x_{\ell'}, \tilde{x}_\ell, \tilde{x}_{\ell'} \in \mathbb{R}^\nu$,

$$|W_{\ell\ell'}(x_\ell, x_{\ell'})| + |\partial_{x_\ell} W_{\ell\ell'}(x_\ell, x_{\ell'})| \leq \frac{1}{2} J_{\ell\ell'}(|x_\ell| + |x_{\ell'}| + C_W), \tag{4.307}$$

$$|(\partial_{x_\ell} W_{\ell,\ell'}(x_\ell, x_{\ell'}) - \partial_{x_\ell} W_{\ell,\ell'}(\tilde{x}_\ell, x_{\ell'}) \, , \, x_\ell - \tilde{x}_\ell)| \leq \frac{1}{2} J_{\ell\ell'}|x_\ell - \tilde{x}_\ell|^2, \tag{4.308}$$

$$|(\partial_{x_\ell} W_{\ell,\ell'}(x_\ell, x_{\ell'}) - \partial_{x_\ell} W_{\ell,\ell'}(x_\ell, \tilde{x}_{\ell'}) \, , \, x_{\ell'} - \tilde{x}_{\ell'})| \leq \frac{1}{2} J_{\ell\ell'}|x_{\ell'} - \tilde{x}_{\ell'}|^2. \tag{4.309}$$

Assumption $(\mathbf{V}_9)$ *There exist* $P \geq 2$, $C_V > 0$, *and* $a_V \in \mathbb{R}$ *such that for all* $\ell \in \mathbb{L}$ *and* $x_\ell, \tilde{x}_\ell \in \mathbb{R}^\nu$,

$$|V_\ell^{(k)}(x_\ell)| \leq C_V \exp\left(C_V |x_\ell|\right), \quad k = 0, 1, \tag{4.310}$$

$$(V_\ell'(x_\ell) - V_\ell'(\tilde{x}_\ell) \, , \, x_\ell - \tilde{x}_\ell)_{\mathbb{R}^\nu} \geq a_V |x_\ell - \tilde{x}_\ell|^2. \tag{4.311}$$

If $\nu \geq 2$, *we additionally claim that each of* V_ℓ *allows the representation*

$$V_\ell(x_\ell) := (h_\ell, x_\ell) + \frac{1}{2} b_\ell |x_\ell|^2 + u_\ell(|x_\ell|) + Q_\ell(x_\ell), \tag{4.312}$$

with $h_\ell \in \mathbb{R}^\nu, b_\ell \in \mathbb{R}$, *such that* $\sup_\ell \{|h_\ell|, |b_\ell|\} =: B < \infty$, *and*

$$u_\ell \in C^1(\mathbb{R}), \ u_\ell(0) = 0, \ u_\ell'(s) \geq u_\ell'(\tilde{s}) \ \text{ if } \ s \geq \tilde{s} \geq 0, \tag{4.313}$$

$$Q_\ell \in C^1_{\mathrm{b}}(\mathbb{R}^\nu), \ \sup_\ell ||Q_\ell||_{C^1_{\mathrm{b}}} =: C_Q < \infty. \tag{4.314}$$

Remark 4.60 Instead of monotonicity of $u_\ell'(s)$ we can assume that

$$(u_\ell'(s) - u_\ell'(\tilde{s}))(s - \tilde{s}) \geq c_\ell (s - \tilde{s})^2, \quad \text{with} \quad c_\ell \in \mathbb{R}, \quad \sup_\ell |c_\ell| < \infty,$$

and then extract the quadratic terms $\frac{1}{2} c_\ell s^2$ from $u_\ell(s)$.

Theorem 4.61 *Let the above hypotheses be fulfilled. Then, for any tempered Gibbs measure* $\mu \in \mathcal{G}^{\mathrm{t}}$, *the associated Dirichlet operator* $\mathbb{H}_\mu \restriction \mathcal{F}C_{\mathrm{b}}^\infty(\Omega)$ *(which is given by the differential expression (2.206) or (4.52)) is essentially self-adjoint in* $L^2(\mu)$.

Proof. We shall apply the general self-adjointness criterion for Dirichlet operators due to V. Liskevich and M. Röckner, see Theorem 1 of [194]. To this end, we approximate the logarithmic derivative $b(x)$ by *smooth cylinder* mappings in the following way. Pick some $0 \leq \rho \in C_0^\infty(\mathbb{R}^\nu)$ such that $|\rho|_{L^1(\mathbb{R}^\nu)} = 1$ and $\mathrm{supp}\rho \subseteq \{s \in \mathbb{R}^\nu \mid |s| \leq 1\}$, and construct a sequence of mollifiers $0 \leq \rho_M \in C_0^\infty(\mathbb{R}^\nu)$, $M \in \mathbb{N}$, by

$$\rho_M(s) := M^\nu \rho(sM), \;\; s \in \mathbb{R}^\nu. \tag{4.315}$$

Define the convolutions (see e.g. Subsection 1.3.2 in [272])

$$\begin{aligned} V_{M,\ell}(x_\ell) &:= (\rho_M * V_\ell)(x_\ell) = \int_{\mathbb{R}^\nu} \rho_M(x_\ell - y_\ell) V_\ell(y_\ell) \mathrm{d}y_\ell = \int_{\mathbb{R}^\nu} \rho_M(y_\ell) V_\ell(x_\ell - y_\ell) \mathrm{d}y_\ell, \\ W_{M,\ell\ell'}(x_\ell, x_{\ell'}) &:= \int_{\mathbb{R}^\nu} \int_{\mathbb{R}^\nu} \rho_M(x_\ell - y_\ell) \rho_M(x_{\ell'} - y_{\ell'}) W_{\ell\ell'}(y_\ell, y_{\ell'}) \mathrm{d}y_\ell \mathrm{d}y_{\ell'}. \end{aligned} \tag{4.316}$$

It is obvious that $V_{M,\ell} \in C^\infty(\mathbb{R}^\nu)$, $W_{M,\ell,\ell'} \in C^\infty(\mathbb{R}^{2\nu})$ pointwise converge to V_ℓ, $W_{\ell,\ell'}$ as $M \to \infty$. Since $V'_{M,\ell} = \rho_M * V'_\ell = \rho'_M * V_\ell$, the same convergence holds for the derivatives $V'_{M,\ell}$, $\partial_{x_\ell} W_{M,\ell,\ell'}$. An advantage of this construction is that the growth and dissipativity conditions in Assumptions $(\mathbf{V}_9)$**,** $(\mathbf{W}_9)$ do not change, which in particular means

$$|\partial_{x_\ell} W_{M,\ell\ell'}(x_\ell, x_{\ell'})| \leq \frac{1}{2} J_{\ell\ell'}(|x_\ell| + |x_{\ell'}| + C_W + 1/M), \tag{4.317}$$

$$\sup \left\{ |\partial^2_{x_\ell} W_{M,\ell\ell'}|, .|\partial^2_{x_\ell, x_{\ell'}} W_{M,\ell\ell'}| \right\} \leq \frac{1}{2} J_{\ell\ell'}, \tag{4.318}$$

$$|V'_{M,\ell}(x_\ell)| \leq C_V \exp\left\{C_V(|x_\ell| + 1/N)\right\}, \quad V''_{M,\ell}(x_\ell) \geq a_V \mathbf{Id}_\nu. \tag{4.319}$$

Take a cofinal sequence $\Lambda^{(N)} \nearrow \mathbb{L}$ as $N \to \infty$. For a fixed $p > d$, consider the Hilbert space

$$\Omega_p := \left\{ x \in \Omega \;\middle|\; \|x\|_p := \left[\sum_\ell (1 + |\ell|)^{-p} |x_\ell|^2 \right]^{1/2} < \infty \right\} \tag{4.320}$$

(cf. the definition (2.15) with $R = 2$). By Chebyshev's inequality and Theorem 2.15, any $\mu \in \mathcal{G}^{\mathrm{t}}$ is surely a probability measure on Ω_p. Furthermore, (2.81), (4.307), and (4.310) together guarantee that

$$\begin{aligned} \int_\Omega ||b(x)||_p^2 \mathrm{d}\mu &\leq 2 \sum_\ell (1 + |\ell|)^{-p} \int_\Omega |V'_\ell(x_\ell)|^2 \mathrm{d}\mu \\ &+ 2||\mathbf{J}||_0 ||\mathbf{J}||_p \sum_\ell (1 + |\ell|)^{-p} \int_\Omega \left(C_9^2 + 2|x_\ell|^2\right) \mathrm{d}\mu < \infty. \end{aligned} \tag{4.321}$$

In the case of $\nu = 1$ we next proceed as follows. Let $\psi \in C_{\mathrm{b}}^\infty(\mathbb{R})$ be a cut-off function such that $\psi(s) = s$ for $s \in (-1, 1)$, $\psi(s) = 2$ for $|s| \geq 3$, and $\psi(s) = -\psi(-s)$, $0 \leq \psi'(s) \leq 1$ for all $s \in \mathbb{R}$. For each $L \in \mathbb{N}$, we put

$$\psi_L(s) := L\psi_N(L^{-1}s), \quad s \in \mathbb{R}. \tag{4.322}$$

Define cylinder mappings $b_I \in C_{\mathrm{b}}^2(\Omega \to \Omega)$ indexed by $I := (L, M, N) \in \mathbb{N}^3$

$$\begin{aligned} b_{I,\ell}(x) &= 0, \quad \ell \notin \Lambda^{(N)}, \\ b_{I,\ell}(x) &:= -V'_{M,\ell}\left[\psi_L(x_\ell|)\right] - \sum_{\ell' \in \Lambda^{(N)}} \partial_{x_\ell} W_{M,\ell\ell'}\left[\psi_L(x_\ell), \psi_L(x_{\ell'})\right], \quad \ell \in \Lambda^{(N)}. \end{aligned} \tag{4.323}$$

If $\nu \geq 2$, set respectively

$$b_{I,\ell}(x) := -h_\ell - b_\ell \psi_L(|x_\ell|)\frac{x_\ell}{|x_\ell|} - u'_{M,\ell}\left[\psi_L(|x_\ell|)\right]\frac{x_\ell}{|x_\ell|} - Q'_{M,\ell}(x_\ell) - \frac{\sum_{\ell'\in\Lambda^{(N)}} \partial_{x_\ell} W_{M,\ell\ell'}(x_\ell, x_{\ell'})}{\left(1+L^{-1}||x_{\Lambda^{(N)}}||_p^2\right)^{1/2}}, \quad \ell \in \Lambda^{(N)}. \tag{4.324}$$

By the Liskevich-Röckner criterion, it suffices to check the following:

(i) $\lim_{n\to\infty} \mathbf{E}_\mu ||b - b_{I_n}||_p^2 = 0$ along some subsequence $I_n := (L_n, M_n, N_n)$, $n \in \mathbb{N}$;

(ii) There exists $c \in \mathbb{R}$ such that for all $I = (L, M, N)$ and $x, y \in \Omega_{\Lambda^{(N)}}$

$$\sum_{\ell,\ell'\in\Lambda^{(N)}} \left(\partial_{x_\ell} b_{I,\ell'}(x) y_{\ell'}, y_\ell\right)(1+|\ell|)^p \leq c \sum_{\ell\in\Lambda^{(N)}} |y_\ell|^2 (1+|\ell|)^p. \tag{4.325}$$

Since $\lim_{n\to\infty} b_{I_n,\ell}(x) = b_\ell(x)$ for all $x \in \Omega_p$, the first condition is obvious by (4.317), (4.319), (4.321) and Lebesgue's dominated convergence theorem. For $\nu = 1$, the second condition follows with $c := |a_V| + ||\mathbf{J}||_p$ from the estimate

$$\begin{aligned}
&\sum_{\ell,\ell'\in\Lambda^{(N)}} \left(\partial_{x_\ell} b_{I,\ell'}(x) y_{\ell'}, y_\ell\right)(1+|\ell|)^p \\
&= -\sum_{\ell\in\Lambda^{(N)}} \left(V''_{M,\ell}(\psi_L(x_\ell))\,\psi'_L(x_\ell)\right) y_\ell, y_\ell)(1+|\ell|)^p \\
&\quad - \sum_{\ell,\ell'\in\Lambda^{(N)}} \left(\partial^2_{x_\ell} W_{M,\ell\ell'}(\psi_L(x_\ell), \psi_L(x_{\ell'}))\,\psi'_L(x_\ell) y_\ell, y_\ell\right)(1+|\ell|)^p \\
&\quad - \sum_{\ell,\ell'\in\Lambda^{(N)}} \left(\partial^2_{x_\ell, x_{\ell'}} W_{M,\ell\ell'}(\psi_L(x_\ell), \psi_L(x_{\ell'}))\,\psi'_L(x_{\ell'}) y_\ell, y_{\ell'}\right)(1+|\ell|)^p \\
&\leq \left(|a_V| + \frac{1}{2}||\mathbf{J}||_0 + \frac{1}{2}||\mathbf{J}||_0^{1/2}||\mathbf{J}||_p^{1/2}\right) \sum_{\ell\in\Lambda^{(N)}} |y_\ell|^2 (1+|\ell|)^p,
\end{aligned} \tag{4.326}$$

where we used the Cauchy inequality and (4.318), (4.319). In the case of $\nu \geq 1$ we observe that for all $x \in \Omega_{\Lambda^{(N)}}$, $y_\ell \in \mathbb{R}^\nu$,

$$\begin{aligned}
&\left(\partial_{x_\ell}\left\{u'_{M,\ell}\left[\psi_L(|x_\ell|)\right]\frac{x_\ell}{|x_\ell|}\right\} y_\ell, y_\ell\right) \\
&= u''_{M,\ell}\left[\psi_L(|x_\ell|)\right]\psi'_L(|x_\ell|)\frac{(x_\ell, y_\ell)^2}{|x_\ell|^2} + \frac{u'_{M,\ell}\left[\psi_L(|x_\ell|)\right]}{|x_\ell|}\left(1 - \frac{(x_\ell, y_\ell)^2}{|x_\ell|^2|y_\ell|^2}\right)|y_\ell|^2 \geq 0
\end{aligned} \tag{4.327}$$

and

$$\left(\partial_{x_\ell}\left\{\psi_L(|x_\ell|)\frac{x_\ell}{|x_\ell|}\right\} y_\ell, y_\ell\right) \geq -2|y_\ell|^2, \tag{4.328}$$

since $0 \le \psi'_L(s) \le 1$ and $u'_{M,\ell}(s)$, $u''_{M,\ell}(s) \ge 0$ for $s \ge 0$. Using this together with (4.314), (4.318), and (4.324) gives us that

$$\sum_{\ell.\ell' \in \Lambda^{(N)}} (\partial_{x_\ell} b_{I,\ell'}(x) y_{\ell'}, y_\ell)\,(1+|\ell|)^p \le \tag{4.329}$$
$$\sum_{\ell \in \Lambda^{(N)}} \left[\frac{||\mathbf{J}||_p + 2(1+C_W^2)}{\left(1 + L^{-1}||x_{\Lambda^{(N)}}||_p^2\right)^{1/2}} + 2B + C_Q \right] |y_\ell|^2 (1+|\ell|)^p,$$

which implies (4.325) with

$$c := 2B + ||\mathbf{J}||_p + C_Q + 2(1 + C_W^2). \tag{4.330}$$

and completes the proof. ■

Note that in finite dimensions, i.e., $\mathbb{R}^n$, essential self-adjointness of the Dirichlet operators $\mathbb{H}_\mu$ takes place under a much weaker sufficient condition

$$|b|_{\mathbb{R}^n} \in L^4(\mathbb{R}^n, \mu), \tag{4.331}$$

see the discussion related to Theorem 1 in [194]. In our situation, such assertion can be checked similarly to (4.321) by employing the moment estimate (2.27). The corresponding result for the local Gibbs distributions $\mu_{\Lambda,y}$ now reads as follows.

Theorem 4.62 *Let $V_\ell \in C^1(\mathbb{R}^\nu)$ and $W_{\ell\ell'} \in C^1(\mathbb{R}^{2\nu})$ satisfy Assumptions* (**W**), (**J**), *and* (**V**) *with some $P > R \ge 2$ and, in addition, the exponential bound (4.310) from Assumption* ($\mathbf{V}_9$)*. Then, for all $\Lambda \Subset \mathbb{L}$ and $y \in \Omega^{\mathrm{t}}$, the finite volume Dirichlet operators $\mathbb{H}_{\Lambda,y} \upharpoonright \mathcal{F}C_{\mathrm{b}}^\infty(\Omega_\Lambda)$ are essentially self-adjoint in $L^2(\Omega_\Lambda, \mu_{\Lambda,y})$.*

Let us stress that, unlike the preceding Theorem 4.61, here we need to claim that the pair potentials have at most quadratic growth. As is seen from (4.326), such assumption was crucial to check the uniform coercivity property (4.325). Clearly, the above results can be extended to more general Hamiltonians with the N-particle interactions $W_{\{\ell_1,\dots,\ell_N\}}$, which were described in Subsection 2.2.5.

Next, we present a modification of Theorem 4.61 related to the spin system (4.10) living on an infinite graph $\mathbb{G}(\mathbb{V}, \mathbb{E})$. We suppose that this graph satisfies the regularity Assumption ($\mathbf{G}_\delta$) with some $\delta_{\mathbb{G}} \ge 0$, cf. Subsection 4.1.1. The interaction matrix $\mathbf{J} = (J_{vv'})_{\mathbb{V}\times\mathbb{V}}$ is now given by $J_{vv'} := J$ if $v \sim v'$ and $J_{vv'} = 0$ otherwise. Among the hypotheses on $V_\ell := V_v$, $W_{\ell\ell'} := W_{vv'}$ listed in Subsection 4.1.2 we need only (4.11) and (4.14). They should be supplemented by the former Assumptions $(\mathbf{W}_9), (\mathbf{V}_9)$. Adapting the arguments used in proving Theorem 4.61 to the Hilbert spaces $\mathcal{H}_\delta := \Omega_\delta$ with $\delta > \delta_{\mathbb{G}}$, we show the following:

Theorem 4.63 *In the situation described above, the statement of Theorem 4.61 holds true for the graph spin system (4.10).*

(iii) Quantum case

In the rest of this subsection, we discuss self-adjointness of the Dirichlet operators corresponding to the Euclidean Gibbs measures. First we prepare the corresponding set-up on loop spaces.

Consider the system of quantum oscillators (3.1), (3.2) indexed by an infinite set $\mathbb{L}$, and let the interaction potentials $V_\ell, W_{\ell\ell'}$ be the same as Theorem 4.61. By the definition (3.71) and the à-priori bound (3.97), each of $\mu \in \mathcal{G}^{\mathrm{t}}$ is now supported by the tempered configurations from $\Omega^{\mathrm{t}} := \bigcap_{p>d} \Omega_p$, where according to (3.63), (3.69) we set $\Omega_p := \Omega \cap \left[l_p^2(\mathbb{L}) \otimes L_\beta^2\right]$. Recall that the tangent Hilbert space $\mathcal{H}_0 := l^2(\mathbb{L}) \otimes L_\beta^2$, with inner product $(\cdot,\cdot)_0$ and orthobasis $\mathrm{bas}(\mathcal{H}_0) := \{h_{(\ell,k,i)}\}$, was introduced in (4.297), (4.298). For a fixed $p > d$, consider the rigging of $\mathcal{H}_0$

$$\mathcal{W}_+ \subset \mathcal{H}_+ \ \subset\ \mathcal{H}_0 \ \subset\ \mathcal{H}_- \ \subset\ \mathcal{W}_- \tag{4.332}$$

by the Hilbert spaces $\mathcal{H}_- := l_p^2(\mathbb{L}) \otimes L_\beta^2$, $\mathcal{W}_- := l_p^2(\mathbb{L}) \otimes W_\beta^{-2}$ and their dual $\mathcal{H}_+$, $\mathcal{W}_+$. Denote by $\mathcal{F}C_{\mathrm{b}}^\infty(\Omega) := \bigcup_{\Lambda \Subset \mathbb{L}} \mathcal{F}C_{\mathrm{b}}^\infty(\Omega_\Lambda)$ the set of all smooth cylinder functions $f : \Omega \to \mathbb{C}$ constructed by means of $\mathrm{bas}(\mathcal{H}_0)$. In particular,

$$\|\omega\|_{\mathcal{H}_-}^2 := \sum_\ell (1+|\ell|)^{-p} |\omega_\ell|_{L_\beta^2}^2, \quad \|\omega\|_{\mathcal{W}_-}^2 := \sum_\ell (1+|\ell|)^{-p} |A^{-1}\omega_\ell|_{L_\beta^2}^2.$$

By the integration by parts formula (4.306) one can straightforwardly check that the *Dirichlet operator* associated with the symmetric form

$$\mathcal{E}_\mu(f,g) := \int_\Omega (\nabla f(\omega), \nabla g(\omega))_0 \mathrm{d}\mu(\omega) = (\mathbb{H}_\mu f, g)_{L^2(\mu)}, \quad f, g \in \mathcal{F}C_{\mathrm{b}}^\infty(\Omega), \tag{4.333}$$

is given by the differential expression

$$\mathbb{H}_\mu f(\omega) = -\Delta f(\omega) - (b(\omega), \nabla f(\omega))_0, \quad f \in \mathcal{F}C_{\mathrm{b}}^\infty(\Omega). \tag{4.334}$$

We denote here

$$\Delta f(\omega) := \mathrm{trace}_{\mathcal{H}_0}(f''(\omega)) = \sum_{\ell,k,i} \partial_{(\ell,k,i)}^2 f(\omega), \tag{4.335}$$

$$\nabla f(\omega) := \sum_{\ell,k,i} \partial_{(\ell,k,i)} f(\omega) \cdot h_{(\ell,k,i)} \in \mathcal{H}_0.$$

According to (4.301)–(4.303), the *logarithmic gradient* of every $\mu \in \mathcal{G}^{\mathrm{t}}$ is a measurable vector field

$$b : \Omega_p \to \mathcal{W}_-, \quad b := \alpha + F,$$

with the components

$$\alpha_\ell(\omega) := -A\omega_\ell, \quad F_\ell(\omega) := -V_\ell'(\omega_\ell) - \sum\nolimits_{\ell'(\neq \ell)} \partial_{q_\ell} W_{\ell\ell'}(\omega_\ell, \omega_{\ell'}),$$
$$\alpha := (\alpha_\ell)_{\ell \in \mathbb{L}} : \Omega_p \to \mathcal{W}_-, \quad F := (F_\ell)_{\ell \in \mathbb{L}} : \Omega_p \to \mathcal{H}_- \tag{4.336}$$

Taking regard of (3.97), (4.307), and (4.310), we check similarly to (4.321) that

$$\int_{\Omega} ||b(\omega)||^2_{\mathcal{W}_-} \mathrm{d}\mu(\omega) < \infty. \tag{4.337}$$

Since the embedding $\mathcal{H}_0 \hookrightarrow \mathcal{W}_-$ belongs to the Hilbert-Schmidt class, the definitions (4.334), (4.335) extend by continuity to all $f \in C^2_{\mathrm{b}}(\mathcal{W}_-)$. Recall that here $W^{-2}_\beta \supset L^2_\beta$ is the dual Sobolev space with the norm $|\omega_\ell|^2_{W^{-2}_\beta} := |A^{-1}\omega_\ell|^2_{L^2_\beta}$, where $A := (-m\mathrm{d}^2/\mathrm{d}\tau^2 + a) \otimes \mathbf{Id}_\nu$ is the positive self-adjoint operator in L^2_β with the maximal domain $\mathcal{D}(A) := W^2_\beta$.

Theorem 4.64 *Let the interaction potentials $V_\ell, W_{\ell\ell'}$ satisfy the hypotheses of Theorem 4.61. Then, for each tempered Euclidean Gibbs measure $\mu \in \mathcal{G}^{\mathrm{t}}$, the associated Dirichlet operator $\mathbb{H}_\mu \restriction \mathcal{F}C^\infty_{\mathrm{b}}(\Omega)$ is essentially self-adjoint in $L^2(\mu)$.*

Our prior observation is that Theorem 1 of [194], which we successfully employed before, does not apply in the quantum case (as well as the alternative approach of [82]). The reason is that we are not able to control the dissipativity properties of $\gamma(\omega)$ in the Hilbert space $\mathcal{W}_-$. Instead, we shall use a *modified* version of the general self-adjointness criterion established by S. Albeverio, Yu. Kondratiev, and M. Röckner in Theorem 1 of [16] (see also Theorem 3.1 of [191]). Note that by the standard approximation argument (cf. Lemma 6 in [16]), the statement of the theorem is equivalent to the essential self-adjointness of the operator $\mathbb{H}_\mu \restriction C^2_{\mathrm{b}}(\mathcal{W}_-)$.

Proof. To prove the theorem it is enough (see page 116 of [16] or Equation (3.9) in [191]) to construct mappings $\alpha_n, F_n \in C^2(\mathcal{W}_- \to \mathcal{H}_-)$, which are twice Fréchet differentiable and have globally bounded continuous derivatives

$$\alpha'_n, F'_n : \mathcal{W}_- \to \mathcal{L}(\mathcal{W}_-, \mathcal{H}_-), \quad \alpha''_n, F''_n : \mathcal{W}_- \to \mathcal{L}(\mathcal{W}_-, \mathcal{L}(\mathcal{W}_-, \mathcal{H}_-))$$

with the following properties:

(i) There exist $a_1, c_1 > 0$ such that for all $n \in \mathbb{N}$ and $\omega \in \mathcal{W}_-$, $\xi \in \mathcal{H}_-$

$$\text{(a)} \quad (\alpha'_n(\omega)\xi, \xi)_{\mathcal{H}_-} \le a_1 ||\xi||^2_{\mathcal{H}_-}; \quad \text{(b)} \quad (F'_n(\omega)\xi, \xi)_{\mathcal{H}_-} \le c_1 ||\xi||^2_{\mathcal{H}_-}; \tag{4.338}$$

(ii) There exist $a_2, c_2(n) > 0$ such that for $n \in \mathbb{N}$ and $\omega, \zeta \in \mathcal{W}_-$

$$\text{(a)} \quad (\alpha'_n(\omega)\zeta, \zeta)_{\mathcal{W}_-} \le a_2 ||\zeta||^2_{\mathcal{W}_-}; \quad \text{(b)} \quad (F'_n(\omega)\zeta, \zeta)_{\mathcal{W}_-} \le c_2(n) ||\zeta||^2_{\mathcal{W}_-}; \tag{4.339}$$

(iii) The sequences $(\alpha_n)_{n\in\mathbb{N}}$, $(\gamma_n)_{n\in\mathbb{N}}$ approximate α, γ in the L^2-sense

$$\text{(a)} \lim_{n\to\infty} e^{b_2(n)} \int_{\Omega} ||\alpha - \alpha_n||^2_{\mathcal{W}_-} \mathrm{d}\mu = 0; \quad \text{(b)} \lim_{n\to\infty} \int_{\Omega} ||F - F_n||^2_{\mathcal{H}_-} \mathrm{d}\mu = 0. \tag{4.340}$$

To this end we shall combine the averaging via the Cesàro partial sums (4.276) with the regularization of the potentials $V_\ell, W_{\ell\ell'}$ by the convolutions (4.316) and cutoffs (4.322) already used in proving Theorem 4.61. Recall that in L^2_β we fix the orthobasis

$\{h_{(k,i)} \mid k \in \mathbb{Z},\ 1 \le i \le \nu\}$ such that $Ah_{(k,i)} = \lambda_k h_{(k,i)}$. For each multi-index $I := (J, K, L, M, N) \in \mathbb{N}^5$ and $\omega \in \mathcal{W}_-$, we define

$$\begin{gathered} \alpha_I := (\alpha_{I,\ell})_{\ell\in\mathbb{L}}, \quad F_I := (F_{I,\ell})_{\ell\in\mathbb{L}}, \quad \alpha_{I,\ell} = \gamma_{I,\ell} = 0 \ \text{ for } \ell \notin \Lambda^{(N)}, \\ \alpha_{I,\ell}(\omega) := -\sum_{|k|\le J,\ 1\le i\le\nu} \lambda_k (\omega_\ell, h_{(k,i)})_{L^2_\beta} h_{(k,i)}, \ \ \ell \in \Lambda^{(N)}. \end{gathered} \tag{4.341}$$

If $\ell \in \Lambda^{(N)}$, we further set for $\nu = 1$

$$F_{I,\ell}(\omega) := \mathbb{M}_K V'_{M,\ell}[\psi_L(\mathbb{M}_K\omega_\ell)] - \sum_{\ell'\in\Lambda^{(N)}} \partial_{q_\ell} W_{K,L,M,\ell\ell'}(\omega_\ell, \omega_{\ell'}), \tag{4.342}$$

and, respectively, for $\nu \ge 2$

$$\begin{aligned} F_{I,\ell}(\omega) &:= -h_\ell - b_\ell \psi_L(|\mathbb{M}_K\omega_\ell|)\frac{\mathbb{M}_K\omega_\ell}{|\mathbb{M}_N\omega_\ell|} - u'_{M,\ell}\left[\psi_L(|\mathbb{M}_K\omega_\ell|)\right]\frac{\mathbb{M}_K\omega_\ell}{|\mathbb{M}_K\omega_\ell|} \\ &- Q'_{M,\ell}(\omega_\ell) - \frac{\sum_{\ell'\in\Lambda^{(N)}} \partial_{q_\ell} W_{K,M,\ell\ell'}(\omega_\ell, \omega_{\ell'})}{\left(1 + L^{-1}||\omega_{\Lambda^{(N)}}||_p^2\right)^{(P-1)/2}}, \end{aligned} \tag{4.343}$$

where for shorthand we denote

$$\begin{gathered} U_{K,M,\ell\ell'}(\omega_\ell, \omega_{\ell'}) := \mathbb{M}_K \partial_{q_\ell} W_{M,\ell\ell'}(\mathbb{M}_K\omega_\ell, \mathbb{M}_K\omega_{\ell'}), \\ U_{K,L,M,\ell\ell'}(\omega_\ell, \omega_{\ell'}) := \mathbb{M}_K \partial_{q_\ell} W_{M,\ell\ell'}[\psi_L(\mathbb{M}_K\omega_\ell), \psi_L(\mathbb{M}_K\omega_{\ell'})]. \end{gathered} \tag{4.344}$$

It is clear that these α_I satisfy the conditions (i) (a) and (ii) (a) with $a_1 = a_2 = 0$. Condition (ii) (b) for γ_I is also obvious, whereas (iii) (b) is implied by Lebesgue's dominated convergence theorem and (4.277), (4.317), (4.319), (4.321). For $\nu = 1$ the property (i) (b) is confirmed by the following computations based on (4.325), (4.326), (4.342), and (4.344)

$$\begin{aligned} (F'_I(\omega)\xi, \xi)_{\mathcal{H}_-} &= -\sum_{\ell\in\Lambda^{(N)}} (V''_{M,\ell}[\psi_L(\mathbb{M}_K\omega_\ell)]\psi'_L(\mathbb{M}_K\omega_\ell)\mathbb{M}_K\xi_\ell, \mathbb{M}_K\xi_\ell)_{L^2_\beta}(1+|\ell|)^{-p} \\ &- \sum_{\ell,\ell'\in\Lambda^{(N)}} (\partial_{q_\ell}[U_{K,L,M,\ell\ell'}(\omega_\ell,\omega_{\ell'})]\psi'_L(\mathbb{M}_K\omega_\ell)\mathbb{M}_K\xi_\ell, \mathbb{M}_K\xi_\ell)_{L^2_\beta}(1+|\ell|)^{-p} \\ &- \sum_{\ell,\ell'\in\Lambda^{(N)}} (\partial_{q_{\ell'}}[U_{K,L,M,\ell\ell'}(\omega_\ell,\omega_{\ell'})]\psi'_L(\mathbb{M}_K\omega_\ell)\mathbb{M}_N\xi_\ell, \mathbb{M}_N\xi_{\ell'})_{L^2_\beta}(1+|\ell|)^{-p} \\ &\le (|a_V| + ||\mathbf{J}||_p)||\xi_\Lambda||^2_{\mathcal{H}_-}, \quad \omega \in \mathcal{W}_-,\ \xi \in \mathcal{H}_-. \end{aligned}$$

Analogously, taking into account (4.327)–(4.329) and (4.343), we get for $\nu \ge 2$

$$\begin{aligned} &\left(\partial_{\xi_\ell}\left[u'_{M,\ell}\left[\psi_L(|\mathbb{M}_K\omega_\ell|)\right]\frac{\mathbb{M}_K\omega_\ell}{|\mathbb{M}_K\omega_\ell|}\right], \xi_\ell\right)_{L^2_\beta} \\ &= \int_0^\beta u''_{M,\ell}\left[\psi_L(|\mathbb{M}_K\omega_\ell|)\right]\psi'_L(|\mathbb{M}_K\omega_\ell|)\frac{(\mathbb{M}_K\omega_\ell, \mathbb{M}_K\xi_\ell)^2}{|\mathbb{M}_K\omega_\ell|^2}\mathrm{d}\tau \\ &+ \frac{u'_{M,\ell}\left[\psi_L(|\mathbb{M}_K\omega_\ell|)\right]}{|\mathbb{M}_K\omega_\ell|}\left(1 - \frac{(\mathbb{M}_K\omega_\ell, \mathbb{M}_K\xi_\ell)^2}{|\mathbb{M}_K\omega_\ell|^2|\mathbb{M}_K\xi_\ell|^2}\right)|\mathbb{M}_K\xi_\ell|^2 \ge 0 \end{aligned} \tag{4.345}$$

and thus

$$(F_I'(\omega)\xi,\xi)_{\mathcal{H}_-} \le \left[2B + ||\mathbf{J}||_p + C_Q + 2(1+C_W^2)\right] \cdot ||\xi||^2_{\mathcal{H}_-} \tag{4.346}$$

which implies (i) (b) with $c_1 := c_{4.330}$. Finally, by (3.38)–(3.41) and Proposition 3.20 we conclude that

$$\begin{aligned}\int_\Omega ||\alpha - \alpha_J||^2_{\mathcal{W}_-}\mathrm{d}\mu &= \sum_\ell (1+|\ell|)^{-p} \sum_{|k|>J,\ 1\le i\le\nu} \int_\Omega \left(\omega_\ell, h_{(k,i)}\right)^2_{L^2_\beta} \mathrm{d}\mu(\omega) \\ &\le \Xi_p C_{4.347} \sum_{|k|>J,\ 1\le i\le\nu} \int_{C_\beta} \left(\omega_\ell, h_{(k,i)}\right)^2_{L^2_\beta} \mathrm{d}\chi(\omega_\ell) \\ &= \nu \Xi_p C_{4.347} \sum_{|k|>J} \lambda_k^{-1} \to 0, \quad J\to\infty, \end{aligned} \tag{4.347}$$

with some universal constant $C_{4.347} > 0$. Thus, we get the desired approximation $b_{I_n} := \alpha_{I_n} + F_{I_n}$ by choosing large enough $J = J(K,L,M,N)$ in accordance with (ii) (b) and (4.347). ■

Remark 4.65 To prove essential self-adjointness of the Dirichlet operators (4.334) one can use an alternative scheme (similar to that realized in [155] for $|\Lambda| < \infty$ and $\beta = \infty$), which employs the *stochastic quantization* dynamics discussed in Subsection 4.2.6. In this case, we need to approximate only the nonlinear terms F in the logarithmic derivative $b(\omega) := -A\omega + F(\omega)$. The result is obtained by mimicking the proof of the self-adjointness criterion, Theorem 1 of [16], and using the properties of the solutions to the Cauchy problem (4.125) with smooth F_n tending to F as $n \to \infty$.

Similarly to the classical case, the essential self-adjointness of the Dirichlet operators corresponding to the local Gibbs distributions $\mu_{\Lambda,\xi}(\mathrm{d}\omega_\Lambda)$ holds under rather general assumptions on the interaction.

Theorem 4.66 *Let the potentials $V_\ell \in C^1(\mathbb{R}^\nu)$ and $W_{\ell\ell'} \in C^1(\mathbb{R}^{2\nu})$ be the same as in Theorem 4.62. Then, for all $\Lambda \Subset \mathbb{L}$ and $\xi \in \Omega^{\mathrm{t}}$, the finite volume Dirichlet operators $\mathbb{H}_{\Lambda,\xi} \restriction \mathcal{F}C_{\mathrm{b}}^\infty(\Omega_\Lambda)$ are essentially self-adjoint in $L^2(\Omega_\Lambda, \mu_{\Lambda,\xi})$.*

Proof. Analogously to (4.336), the corresponding logarithmic derivative $b = (b_\ell)_{\ell\in\Lambda} : \Omega_\Lambda \to [W_\beta^{-2}]^\Lambda$ can be represented as $b_\ell(\omega) := -A\omega_\ell + \gamma_\ell(\omega)$ with

$$\gamma_\ell(\omega) := -V_\ell'(\omega_\ell) - \sum\nolimits_{\ell'\in\Lambda} \partial_{q_\ell} W_{\ell\ell'}(\omega_\ell,\omega_{\ell'}) - \sum\nolimits_{\ell'\in\Lambda^c} \partial_{q_\ell} W_{\ell\ell'}(\omega_\ell,\xi_{\ell'}). \tag{4.348}$$

We again apply the self-adjointness criterion, Theorem 1 of [194]. Define the cylinder approximation $\alpha_{K,L}(\omega_\ell)$ of $A\omega_\ell$ by

$$\alpha_{K,L}(\omega_\ell) := \frac{A\mathbb{P}_K\omega_\ell}{\sqrt{1 + L^{-1}|\mathbb{P}_K\omega_\ell|^2_{W_\beta^{-1}}}}, \quad K,L\in\mathbb{N}, \tag{4.349}$$

where

$$\mathbb{P}_K\omega_\ell := \sum_{|k|\le K,\ 1\le i\le\nu} (\omega_\ell, h_{(k,i)})_{L^2_\beta} h_{(k,i)}, \quad |\mathbb{P}_K\omega_\ell|^2_{W^{-1}_\beta} := (A^{-1}\mathbb{P}_K\omega_\ell, \mathbb{P}_K\omega_\ell,)_{L^2_\beta}.$$

We claim that they satisfy the required assumptions:

$$\text{(i)}\quad \left(\alpha'_{K,L}(\omega_\ell)\zeta,\zeta\right)_{W^{-2}_\beta} \ge a_1|\zeta|^2_{W^{-2}_\beta};\quad \text{(ii)}\quad \lim_{K,L\to\infty}\int_\Omega |A\omega_\ell - \alpha_{K,L,\ell}|^2_{W^{-2}_\beta}\mathrm{d}\mu_{\Lambda,\xi} = 0. \tag{4.350}$$

The validity of (4.350) (i) with $a_1 = 0$ is implied by the estimate

$$\begin{aligned}
&\left(\alpha'_{K,L}(\omega_\ell)\zeta,\zeta\right)_{W^{-2}_\beta} \\
&= \frac{|\mathbb{P}_K\zeta|^2_{W^{-1}_\beta}}{\left(1+L^{-1}|\mathbb{P}_K\omega_\ell|^2_{W^{-1}_\beta}\right)^{1/2}} - \frac{L^{-1}\left(\mathbb{P}_K\omega_\ell,\mathbb{P}_K\zeta\right)^2_{W^{-1}_\beta}}{\left(1+L^{-1}|\mathbb{P}_K\omega_\ell|^2_{W^{-1}_\beta}\right)^{3/2}} \\
&\ge \frac{L^{-1}|\mathbb{P}_K\zeta|^2_{W^{-1}_\beta}|\mathbb{P}_K\omega_\ell|^2_{W^{-1}_\beta} - \left(\mathbb{P}_K\omega_\ell,\mathbb{P}_K\zeta\right)^2_{W^{-1}_\beta}}{\left(1+L^{-1}|\mathbb{P}_K\omega_\ell|^2_{W^{-1}_\beta}\right)^{3/2}} \ge 0.
\end{aligned}$$

On the other hand,

$$\begin{aligned}
&|A\omega_\ell - \alpha_{K,L,\ell}|_{W^{-2}_\beta} \\
&\le |A\omega_\ell - A\mathbb{P}_K\omega_\ell|_{W^{-2}_\beta} + |A\mathbb{P}_K\omega_\ell|_{W^{-2}_\beta}\left[1-\left(1+L^{-1}|\mathbb{P}_K\omega_\ell|^2_{W^{-1}_\beta}\right)^{-1/2}\right] \\
&\le |\omega_\ell - \mathbb{P}_K\omega_\ell|_{L^2_\beta} + L^{-1}|\mathbb{P}_K\omega_\ell|_{W^{-1}_\beta}|\mathbb{P}_K\omega_\ell|_{L^2_\beta} \le |\omega_\ell - \mathbb{P}_K\omega_\ell|_{L^2_\beta} + (aL)^{-1}|\omega_\ell|^2_{L^2_\beta},
\end{aligned}$$

where $a > 0$ is the parameter related to the operator A, cf. (3.38). Proceeding similarly to (4.347), we can check (4.350) (ii)

$$\begin{aligned}
&\int_\Omega |A\omega_\ell - \alpha_{K,L,\ell}|^2_{W^{-2}_\beta}\mathrm{d}\mu_{\Lambda,\xi} \le 2\nu C_{4.347}(\Lambda,\xi)\sum_{|k|>K}\lambda_k^{-1} \\
&+2(aL)^{-2}\int_{\Omega_\Lambda}|\omega_\ell|^2_{L^2_\beta}\mathrm{d}\mu_{\Lambda,\xi}(\omega_\Lambda) \to 0, \quad \text{as}\ \ K,\ L\to\infty.
\end{aligned} \tag{4.351}$$

Finally, we note that

$$\sup_{\ell\in\Lambda}\int_{\Omega_\Lambda} ||\gamma_\ell(\omega_\Lambda)||^4_{L^2_\beta}\mathrm{d}\mu_{\Lambda,\xi}(\omega_\Lambda) < \infty.$$

The criterion we apply says that such L^4-integrable "*perturbation*" terms $\gamma_\ell(\omega)$ cannot destroy self-adjointness, which completes the proof. ■

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