# Stochastic Differential Equations in Hilbert Spaces - Approximation of the Mild Solution in the Sense of Euler Scheme

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# Introduction

The Euler scheme is a very powerful tool to solve differential equations approximately in the ordinary but also in the stochastic case. It is well known for ordinary one-dimensional differential equations with Lipschitz continuous coefficients. In this case we can prove, that the Euler approximation converges to the solution uniformly in time. A proof can be found in [KP92, chapter 8].

In the case of stochastic differential equations in d dimensions with a noise given by a m-dimensional Wiener process, $d, m \in \mathbb{N}$, we know by a fixed point argument under Lipschitz assumptions on the coefficients that there exists a pathwise unique strong solution. Even these equations can be solved approximately in the sense of Euler, namely by adding a time discrete realization of the Wiener process to the Euler scheme. A proof can be found in [KP92, chapter 10].

But the Euler scheme is not only used to construct approximations for existing solutions, but also to prove existence and uniqueness of solutions of equations with weaker conditions. For example in [Kry98, Chapter 1, p.1], [GK96] and [Gyö98] Gyöngy and Krylov proved convergence of the Euler scheme and established, that the limit is a solution under monotonicity assumptions on the coefficients.

In this thesis we will be concerned with stochastic differential equations on Hilbert spaces. We consider the following type of stochastic differential equations on a seperable Hilbert space $H$:

$$\begin{aligned} dX(t) &= [AX(t) + F(X(t))]dt + BdW(t),\ t \in [0,T] \\ X(0) &= x \in H \end{aligned} \tag{1}$$

where $W(t),\ t \in [0,T]$, is a cylindrical Wiener process on a probability space $(\Omega, \mathcal{F}, P)$ taking values in another Hilbert space $U$. $A$ is the generator of a $C_0$-semigroup of contractions $e^{tA},\ t \in [0,T]$.

If $F : H \to H$ is Lipschitz and $B \in L_2(U, H)$, then there exists a mild solution of problem (1) that is a predictable process $X(t)$, $t \in [0,T]$, such that

$$X(t) = e^{tA}x + \int_0^t e^{(t-s)A}F(X(s))ds + \int_0^t e^{(t-s)A}BdW(s) \quad \text{P-a.s.}$$

for all $t \in [0, T]$. A proof can be found in [KF01, theorem 3.2, p.68].

Applying the Euler scheme to this type of differential equations has not be done extensively up to now. This is owed to the fact that the mild solution is not a martingale and therefore we can not apply any of the well-known inequalities for martingales. One paper dealing with this type of equations is by Lord [Lor04], where he considered the Hilbert space of all continuous periodic functions and the Laplacian. He developed a modification of the classical Euler scheme, which gives us convergence to the mild solution.

Here we will show that there exists a numerical scheme approximating the mild solution of problem (1) in the sense of Euler.

In order to point out the specific properties of the Euler scheme, let us recall the Euler scheme for ordinary differential equations in one dimension: If we are concerned with the following type of differential equation on $\mathbb{R}$

$$\begin{aligned} y'(t) &= f(t, y(t)), \quad t \in [0, T] \\ y(0) &= c \in \mathbb{R} \end{aligned}$$

where $f : [0, T] \times \mathbb{R} \to \mathbb{R}$ is continuous and Lipschitz continuous in the second variable, we know by Picard-Lindelöf, that there exists a unique solution of the differential equation that is a differentiable function $\varphi : [0, T] \to \mathbb{R}$ such that

$$\varphi(t) = c + \int_0^t f(s, \varphi(s))ds, \quad t \in [0, T].$$

For the numerical approximation of $\varphi$ we consider a subdivision of $[0, T]$ $\Delta = \{u_0,\ u_1,\ ...,\ u_n\},\ 0 = u_0 < u_1 < ... < u_n = T$. Then the Euler scheme provides us a piecewise linear function $\varphi_\Delta : [0, T] \to \mathbb{R}$.

$\varphi_\Delta$ fulfills the following two properties, which are characteristical for the Euler scheme:

First, that there exists a function $g : \mathbb{R} \times [0, T] \times [0, T] \to \mathbb{R}$ depending on $c$ and $f$ with

$$\varphi_\Delta(u_i) = g(\varphi_\Delta(u_{i-1}),\ u_{i-1},\ u_i)$$

for $i \in \{0, ..., n\}$ and

$$\varphi_\Delta(t) = g(\varphi_\Delta(u_i),\ u_i,\ t)$$

for $u_i < t < u_{i+1},\ i \in \{0, ..., n-1\}$.

Secondly that the error $|\varphi_\Delta - \varphi|$ converges to 0 uniformly on $[0, T]$ as $\rho(\Delta)$ converges to 0, where $\rho(\Delta)$ is the maximum timestep of $\Delta$.

In the case of our stochastic differential equation on a Hilbert space $H$ we have to modify these two properties appropriately. We will show that we can define a process $X_\Delta(t),\ t \in [0, T]$, fulfilling the following two properties:

First that there exists a function $g : H \times [0,T] \times [0,T] \rightarrow H$ depending on $A,\ F,\ B,\ W$ with

$$X_\Delta(u_i) = g(X_\Delta(u_{i-1}),\ u_{i-1},\ u_i)$$

for $i \in \{0, ..., n\}$ and

$$X_\Delta(t) = g(X_\Delta(u_i),\ u_i,\ t)$$

for $u_i < t < u_{i+1},\ i \in \{0, ..., n-1\}$.

Secondly that

$$E(\sup_{0 \leq t \leq \tau} |X_\Delta(t) - X(t)|^p) \rightarrow 0$$

as $\rho(\Delta) \rightarrow 0$ for a stopping time $\tau$ and $p = 1, 2$.

The process, which fulfills these two properties, can be defined by

$$X_\Delta(t) := e^{tA}x + \sum_{\substack{1 \leq \nu \leq n \\ u_{\nu-1} \leq t}} e^{(t-u_{\nu-1})A} F(X_\Delta(u_{\nu-1}))\Delta u_{\nu-1} + e^{(t-u_{\nu-1})A} B \Delta W_{\nu-1},$$

where

- $\Delta u_{\nu-1} = u_\nu - u_{\nu-1}$,
- $\Delta W_{\nu-1} = W(u_\nu) - W(u_{\nu-1})$,
- in the last term differences $\Delta u_{\nu-1},\ \Delta W_{\nu-1}$ stand for $t - u_{\nu-1},\ W(t) - W(u_{\nu-1})$ respectively.

The first chapter is devoted to an introduction of terms and the notion of SDE in Hilbert spaces and mild solution.

In chapter 2 we develop the Euler scheme for the mild solution in detail. In section 2.1 we give an introduction to the Euler scheme for ordinary one-dimensional differential equations. We show that the two properties presented above are fulfilled in one dimension. In section 2.2 we state these two properties in the modified version and present the numerical scheme for our Hilbert space valued stochastic differential equation. Fulfilling the first property is proved in this section, fulfilling the second property is proved in section 2.3. In section 2.4 we consider the case of unbounded operator $A$. If we are obliged to compute with bounded generators only, we can approximate the Euler approximation by an Euler approximation relative to bounded generators, the Yosida approximation of $A$.

# Chapter 1

# SDE in Hilbert Spaces and Mild Solution

In this chapter we give a brief introduction to stochastic differential equations (SDEs) in Hilbert spaces and the notion of mild solution. Concerning the coefficients of the differential equation we already state the conditions being necessary for our later computations.

## 1.1 Preliminaries

Let $(H, \langle\ ,\ \rangle)$ and $(U, \langle\ ,\ \rangle_U)$ be separable Hilbert spaces and let $L(U, H)$ be the Banach space of all linear bounded operators from U into H endowed with the norm

$$\|T\|_{L(U,H)} := \sup\{\|Tx\| : x \in U, \|x\|_U = 1\},\ T \in L(U, H).$$

Define $L(H) := L(H, H)$.

Let $L_2 := L_2(U, H)$ be the Hilbert space of all operators $A$ from $L(U, H)$ with

$$\|A\|_{L_2}^2 := \sum_{k \in \mathbb{N}} \langle Ae_k, Ae_k\rangle < \infty$$

endowed with the inner product

$$\langle A, B\rangle_{L_2} := \sum_{k \in \mathbb{N}} \langle Ae_k, Be_k\rangle,$$

where $e_k,\ k \in \mathbb{N}$, is an arbitrary orthonormal basis of $U$. This space is called the *space of all Hilbert-Schmidt operators* from $U$ to $H$. For more details see [PR06, Chapter C, p.109]

A function $F : H \to H$ is called *Lipschitz continuous* if there exists a constant $K > 0$ such that

$$\|F(x) - F(y)\| \leq K\|x - y\|, \ x, y \in H.$$

A function $F : H \to H$ fulfills the *linear growth condition* if there exists a constant $L > 0$ such that

$$\|F(x)\| \leq L(1 + \|x\|), \ x \in H.$$

It comes out that the Lipschitz constant $K$ of a Lipschitz continuous function $F$ can always be chosen in such a way that $F$ fulfills also the linear growth condition with constant $K$. C.f. [KF01, Remark 3.1(iii), p.66].

For a function $f : [0, t] \to \mathbb{R}$, $t > 0$ and $p \geq 1$ we define $\|f\|_p := (\int_0^t |f(s)|^p ds)^{\frac{1}{p}}$ in the sense of Riemann if $\int_0^t |f(s)|^p ds < \infty$.

For $(X, \| \ \|)$ Banach space, $(\Omega, \mathcal{F}, \mu)$ measure space with finite measure $\mu$ and $f : \Omega \to X$ Bochner integrable we define $\int_\Omega f d\mu$ as the Bochner integral of $f$ with respect to $\mu$. (C.f. [PR06, Chapter A, p.99]) In our case we will fix a $T > 0$ and choose $\Omega = [0, T]$, $\mathcal{F}$ the Borel-$\sigma$-algebra restricted to $[0, T]$ and $\mu$ the Lebesgue measure restricted to $[0, T]$.

The following result is important for the theory of stochastic integrals.

**Proposition 1.1** *If $Q \in L(U)$ is nonnegative and symmetric then there exists exactly one element $Q^{\frac{1}{2}} \in L(U)$ nonnegativ and symmetric such that $Q^{\frac{1}{2}} \circ Q^{\frac{1}{2}} = Q$.*
*If, in addition, $Q$ is of finite trace we have $\|Q^{\frac{1}{2}}\|_{L_2}^2 = trQ$ and thus $Q^{\frac{1}{2}} \in L_2(U)$ and $L \circ Q^{\frac{1}{2}} \in L_2(U, H)$ for all $L \in L(U, H)$.*

**Proof**
[RS72, Theorem VI.9, p.196] $\square$

For $T > 0$, $(\Omega, \mathcal{F}, P)$ probability space and $Q \in L(U)$ nonnegative, symmetric and with finte trace we define $W(t)$, $t \in [0, T]$, as the $Q$-Wiener process on $(\Omega, \mathcal{F}, P)$ taking values in $U$. (C.f. [PR06, Definition 2.1.9, p.12])

For $T > 0$, $(\Omega, \mathcal{F}, P)$ probability space, $Q \in L(U)$ nonnegative and symmetric, $Q^{-\frac{1}{2}}$ the pseudo inverse of $Q^{\frac{1}{2}}$ in the case that $Q$ is not one to one, $\lambda_k \in (0, \infty)$, $k \in \mathbb{N}$, with $\sum_{k=1}^\infty \lambda_k^2 < \infty$, $e_k$, $k \in \mathbb{N}$, orthonormal basis of $Q^{\frac{1}{2}}(U)$, $J : Q^{\frac{1}{2}}(U) \to U$, $J(u) := \sum_{k=1}^\infty \lambda_k \langle Q^{-\frac{1}{2}} u, Q^{-\frac{1}{2}} e_k \rangle_U e_k$ and $Q_1 := JJ^* \in L(U)$ nonnegative, symmetric and with finite trace, we define $W(t)$, $t \in [0, T]$, as the $Q_1$-Wiener process on $(\Omega, \mathcal{F}, P)$ taking values in $U$.

This process is called *cylindrical $Q$-Wiener process.* (C.f. [PR06, Subsection 2.5.1, p.36])

For $T > 0$, $(\Omega, \mathcal{F}, P)$ probability space, $Q \in L(U)$ nonnegative, symmetric and with finite trace, $W(t),\ t \in [0,T]$, $U$-valued $Q$-Wiener process on $(\Omega, \mathcal{F}, P)$ and $\Phi : [0,T] \times \Omega \to L_2(Q^{\frac{1}{2}}(U), H)$ stochastically integrable with respect to $W$ we define $\int_0^T \Phi(s) dW(s)$ as the stochastic integral of $\Phi$ with respect to $W$. (C.f. [PR06, Section 2.3, p.20])

For $T > 0$, $(\Omega, \mathcal{F}, P)$ probability space, $Q \in L(U)$ nonnegative and symmetric, $W(t),\ t \in [0,T]$, $U$-valued cylindrical $Q$-Wiener process on $(\Omega, \mathcal{F}, P)$ and $\Phi : [0,T] \times \Omega \to L_2(Q^{\frac{1}{2}}(U), H)$ stochastically integrable with respect to $W$ we define $\int_0^T \Phi(s) dW(s) := \int_0^T \Phi(s) \circ (J^{-1}) dW(s)$. The right hand side is well-defined, because it holds $J$ is one-to-one,

$$\mathrm{Im}(J) = J(Q^{\frac{1}{2}}(U)) = Q_1^{\frac{1}{2}}(U)$$

and

$$\|\Phi \circ (J^{-1})\|_{L_2(Q_1^{\frac{1}{2}}(U),H)} = \|\Phi\|_{L_2(Q^{\frac{1}{2}}(U),H)}. \tag{1.1}$$

(C.f. [PR06, Subsection 2.5.2, p.38])

The following theorem deals with the Yosida approximation of the generator of a semigroup. Let $A : D(A) \subset H \to H$ be the generator of a $C_0-$semigroup of contractions $e^{tA},\ t \in [0,T]$, and $A_\alpha,\ \alpha > 0$, the Yosida approximation of $A$.

**Theorem 1.2** *Under these assumptions* $\|(e^{sA} - e^{sA_\alpha})h\|$ *converges to* $0$ *as* $\alpha \to \infty$ *for all* $h \in H$ *uniformly on bounded intervalls.*

**Proof**
[Paz83, Proof of theorem 3.1, p.10] □

## 1.2 SDE in Hilbert spaces

For our consideration of stochastic differential equations we fix a $T > 0$ and a cylindrical $Q$-Wiener process $W(t),\ t \in [0,T]$, on the probability space $(\Omega, \mathcal{F}, P)$ taking values in $U$. Therefore we choose $Q = I_U,\ \lambda_k \in (0,\infty),\ k \in \mathbb{N}$, with $\sum_{k=1}^\infty \lambda_k^2 < \infty$ and $e_k,\ k \in \mathbb{N}$, orthonormal basis of $Q^{\frac{1}{2}}(U) = U$.
We are here concerned with the following type of stochastic differential equations in $H$

$$\begin{aligned} dX(t) &= [AX(t) + F(X(t))]dt + BdW(t),\ t \in [0,T] \\ X(0) &= x \in H \end{aligned} \tag{1.2}$$

where

- $A : D(A) \subset H \to H$ is the generator of a $C_0-$semigroup of contractions $e^{tA},\ t \in [0,T]$,
- $F : H \to H$ is $\mathcal{B}(H)$-$\mathcal{B}(H)$-measurable, Lipschitz continuous,
- $B \in L_2(U, H)$.

**Definition 1.3** *A $H$-valued predictable process $X(t)$, $t \in [0,T]$, is called a mild solution of problem (1.2) if*

$$X(t) = e^{tA}x + \int_0^t e^{(t-s)A}F(X(s))ds + \int_0^t e^{(t-s)A}BdW(s) \quad P\text{-a.s.} \tag{1.3}$$

*for all $t \in [0,T]$.*

In particular, the integrals have to be well defined, i.e. that $e^{(t-s)A}F(X(s)),\ s \in [0,t]$, is $P$-a.s. Bochner integrable and that $e^{(t-s)A}B,\ s \in [0,t]$, is stochastically integrable.

**Theorem 1.4** *Under the given assumptions there exists a unique mild solution $X$ of problem (1.2) with $\sup\limits_{0\le t\le T} E(\|X(t)\|^2) < \infty$.*

**Proof**
[KF01, Theorem 3.2, p.68] □

**Proposition 1.5** *Under the given assumptions the mild solution $X$ of problem (1.2) has a continuous version.*

**Proof**
[KF01, Proposition 3.15, p.88] □

# Chapter 2

# Approximation in the Sense of Euler

Let us consider the stochastic differential equation given by (1.2) in section 1.2. By theorem 1.4 we know, that there exists a unique mild solution $X$ given by the implicit formula (1.3). If we are concerned with a nonlinear equation, i.e. $F \neq 0$ is nonlinear, it is not possible to state the solution explicitly.

In this chapter we show that we can approximate the mild solution in nearly in the same way we approximate the solutions of ordinary one-dimensional differential equations with the help of the Euler scheme.

In section 2.1 we give a brief introduction to the Euler scheme for ordinary one-dimensional differential equations. We present the two properties, *recursive calculability* and *uniform convergence of the error*, which we want to be fulfilled by the numerical scheme. Then we state the Euler scheme and proof the properties.

In section 2.2 we state these two properties in a modified version and present a numerical scheme for our Hilbert space valued stochastic differential equation, which fulfills these two properties. The modification is necessary, because we are concerned with stochastic instead of ordinary differential equations and Hilbert space valued instead of realvalued differential equations. Fulfilling the first property (recursive calculability) is proved here, fulfilling the second property (uniform convergence of the error) is proved in section 2.3.

In section 2.4 we consider the case of unbounded operator $A$. If we are obliged to compute with bounded generators only, we can approximate the Euler approximation by an Euler approximation relative to bounded generators, the Yosida approximation of $A$.

In section 2.5 we give some ideas for future research.

## 2.1 The Euler scheme

Let us consider the following one-dimensional ordinary differential equation on $\mathbb{R}$

$$
\begin{aligned}
y'(t) &= f(t, y(t)), \quad t \in [0,T] \\
y(0) &= c \in \mathbb{R}
\end{aligned}
$$

where $T > 0$ and $f : [0,T] \times \mathbb{R} \to \mathbb{R}$ is continuous in the first variable and fulfills the Lipschitz and linear growth condition concerning the second variable with constant $K > 0$.

By Picard-Lindelöf there exists a unique solution of the differential equation that is a differentiable function $\varphi : [0,T] \to \mathbb{R}$ such that

$$
\varphi(t) = c + \int_0^t f(s, \varphi(s))ds, \quad t \in [0,T].
$$

For the numerical approximation of $\varphi$ let us consider a subdivision of $[0,T]$ $\Delta = \{u_0,\ u_1,\ ...,\ u_n\}$, $0 = u_0 < u_1 < ... < u_n = T$. We want to have a numerical scheme, which provides us a function $\varphi_\Delta : [0,T] \to \mathbb{R}$ fulfilling the following two properties:

Fist that there exists a function $g : \mathbb{R} \times [0,T] \times [0,T] \to \mathbb{R}$ depending on $c$ and $f$ with

$$
\varphi_\Delta(u_i) = g(\varphi_\Delta(u_{i-1}),\ u_{i-1},\ u_i) \tag{2.1}
$$

for $i \in \{0, ..., n\}$ and

$$
\varphi_\Delta(t) = g(\varphi_\Delta(u_i),\ u_i,\ t) \tag{2.2}
$$

for $u_i < t < u_{i+1}$, $i \in \{0, ..., n-1\}$. Let us call this property *recursive calculability.*

Secondly that we have for the error

$$
\sup_{0 \le t \le T} |\varphi_\Delta(t) - \varphi(t)| \to 0 \text{ as } \rho(\Delta) \to 0 \tag{2.3}
$$

where $\rho(\Delta)$ is the maximal timestep of $\Delta$. Let us call this property *uniform convergence of the error.*

The *Euler approximation* of $\varphi$ works in the following way: Let us choose a timestep $h > 0$ and define $\varphi_\Delta(0) := c$, $t_k := kh, \quad k = 0, 1, ....$ Then compute recursively

$$
\varphi_\Delta(t_{k+1}) := \varphi_\Delta(t_k) + hf(t_k, \varphi_\Delta(t_k))
$$

for $k = 0, 1, ...,$ with $(k+1)h \le T$.

Then we have the following result concernig the error of the approximation:

$$
\sup_{\substack{0 \le k \\ kh \le T}} |\varphi_\Delta(t_k) - \varphi(t_k)| \le hTe^{TK}. \tag{2.4}
$$

This is proved e.g. in [KP92, Section 8.3, p.292].

In order to fulfill the two stated properties, we have to modify the Euler approximation, i.e. we have to define $\varphi_\Delta$ on the whole intervall $[0,T]$. Therefore consider (more generally) a subdivision of $[0,T]$ $\Delta = \{u_0,\ u_1,\ ...,\ u_n\},\ 0 = u_0 < u_1 < ... < u_n = T$ and define

$$\varphi_\Delta(t) := c + \int_0^t f(u(s), \varphi_\Delta(u(s)))ds, \quad t \in [0,T]$$

where $u(s),\ s \in [0,T]$, is defined by $u_\nu$ for $u_\nu \leq s < u_{\nu+1}$.

Claim: $\varphi_\Delta$ fulfills the two stated properties.

Proof: We have recursive calculability, because if we define $g(r,s,t) := r + (t-s)f(s,r)$ for $r \in \mathbb{R},\ s,t \in [0,T]$, we have

$$\begin{aligned}
\varphi_\Delta(u_i) &= c + \int_0^{u_i} f(u(s), \varphi_\Delta(u(s)))ds \\
&= c + \int_0^{u_{i-1}} f(u(s), \varphi_\Delta(u(s)))ds \\
&\qquad + \int_{u_{i-1}}^{u_i} f(u(s), \varphi_\Delta(u(s)))ds \\
&= \varphi_\Delta(u_{i-1}) \\
&\qquad + (u_i - u_{i-1}) f(u_{i-1}, \varphi_\Delta(u_{i-1})) \\
&= g(\varphi_\Delta(u_{i-1}),\ u_{i-1},\ u_i)
\end{aligned}$$

for $i \in \{0,...,n\}$ and

$$\begin{aligned}
\varphi_\Delta(t) &= c + \int_0^t f(u(s), \varphi_\Delta(u(s)))ds \\
&= c + \int_0^{u_i} f(u(s), \varphi_\Delta(u(s)))ds \\
&\qquad + \int_{u_i}^t f(u(s), \varphi_\Delta(u(s)))ds \\
&= \varphi_\Delta(u_i) \\
&\qquad + (t - u_i) f(u_i, \varphi_\Delta(u_i)) \\
&= g(\varphi_\Delta(u_i),\ u_i,\ t)
\end{aligned}$$

for $u_i < t < u_{i+1},\ i \in \{0,...,n-1\}$.

Let us consider the error of the approximation. By the linear growth condition we have for $t \in [0,T]$

$$\begin{aligned}|\varphi(t)| =&|c+\int_0^t f(s,\varphi(s))ds| \\ \leq&|c|+\int_0^t |f(s,\varphi(s))|ds \\ \leq&|c|+K\int_0^t (1+|\varphi(s)|)ds \\ \leq&|c|+KT+K\int_0^t |\varphi(s)|ds.\end{aligned}$$

With Gronwalls inequality (lemma 2.3) we get for $t \in [0,T]$

$$|\varphi(t)| \leq |c| + KT(1+KTe^{KT}).$$

Consequently $\varphi$ is bounded on $[0,T]$.

For the error we get by (2.4)

$$\begin{aligned}&\sup_{0\leq t\leq T} |\varphi_\Delta(t)-\varphi(t)| \\ \leq&\sup_{0\leq t\leq T}\int_0^t |f(u(s),\varphi_\Delta(u(s)))-f(s,\varphi(s))|ds \\ \leq&K\int_0^T |\varphi_\Delta(u(s))-\varphi(s)|ds \\ \leq&K\int_0^T |\varphi_\Delta(u(s))-\varphi(u(s))|ds + K\int_0^T |\varphi(u(s))-\varphi(s)|ds \\ \leq&K\int_0^T \rho(\Delta)Te^{TK}ds + K\int_0^T |\varphi(u(s))-\varphi(s)|ds \\ \leq&\rho(\Delta)KT^2e^{TK} + K\int_0^T |\varphi(u(s))-\varphi(s)|ds.\end{aligned}$$

The first term converges to 0 as $\rho(\Delta) \to 0$. $|\varphi(u(s))-\varphi(s)|$ converges to 0 as $\rho(\Delta) \to 0$ for all $s \in [0,T]$, because $\varphi$ is continuous and $u(s) \to s$ as $\rho(\Delta) \to 0$. Since $\varphi$ is bounded on $[0,T]$ we get by the dominated convergence theorem that the integral converges to 0 as $\rho(\Delta) \to 0$. □

## 2.2 Numerical scheme and basic properties

Let us consider the stochastic differential equation given by (1.2) in section 1.2. By theorem 1.4 we know, that there exists a unique mild solution $X$ given by the implicit formula (1.3). In order to approximate the mild solution in the sense of Euler let us consider a subdivision of $[0,T]$ $\Delta = \{u_0,\ u_1,\ ...,\ u_n\},\ 0 = u_0 < u_1 < ... < u_n = T$. As our Euler approximation of the mild solution we define for $t \in [0, T]$

$$X_\Delta(t) := e^{tA}x + \sum_{\substack{1 \le \nu \le n \\ u_{\nu-1} \le t}} e^{(t-u_{\nu-1})A} F(X_\Delta(u_{\nu-1}))\Delta u_{\nu-1} + e^{(t-u_{\nu-1})A} B \Delta W_{\nu-1} \tag{2.5}$$

where

- $\Delta u_{\nu-1} = u_\nu - u_{\nu-1}$,
- $\Delta W_{\nu-1} = W(u_\nu) - W(u_{\nu-1})$,
- in the last term differences $\Delta u_{\nu-1},\ \Delta W_{\nu-1}$ stand for $t - u_{\nu-1},\ W(t) - W(u_{\nu-1})$ respectively.

We show, that this process fulfills the following two properties, which are modifications of the properties (2.1), (2.2) and (2.3).

First we have *recursive calculability*, which means that there exists a function $g : H \times [0, T] \times [0, T] \to H$ depending on $A,\ F,\ B,\ W$ with

$$X_\Delta(u_i) = g(X_\Delta(u_{i-1}),\ u_{i-1},\ u_i)$$

for $i \in \{0, ..., n\}$ and

$$X_\Delta(t) = g(X_\Delta(u_i),\ u_i,\ t)$$

for $u_i < t < u_{i+1},\ i \in \{0, ..., n-1\}$.

Secondly we have *uniform convergence of the error*, which means that

$$E(\sup_{0 \le t \le \tau} |X_\Delta(t) - X(t)|^p) \to 0$$

as $\rho(\Delta) \to 0$ for a stopping time $\tau$ and $p = 1, 2$.

Indeed we show that

$$X_\Delta(u_i) = e^{(u_i - u_{i-1})A}\big[X_\Delta(u_{i-1}) + F(X_\Delta(u_{i-1}))\Delta u_{i-1} + B\Delta W_{i-1}\big] \tag{2.6}$$

for $i \in \{0, ..., n\}$,

$$X_\Delta(t) = e^{(t-u_i)A}\big[X_\Delta(u_i) + F(X_\Delta(u_i))(t - u_i) + B(W(t) - W(u_i))\big]. \tag{2.7}$$

for $u_i < t < u_{i+1},\ i \in \{0, ..., n-1\}$ and that

$$E(\sup_{0 \le t \le T} |X_\Delta(t) - X(t)|^p) \to 0 \tag{2.8}$$

as $\rho(\Delta) \to 0$ for a stopping time $\tau$ and $p = 1, 2$.

In section 2.3 we will prove (2.8).

If $A$ is not bounded we can split our approximation with the help of the Yosida approximation in order to deal with bounded generators only (c.f. section 2.4).

In the following we prove (2.6) and (2.7) and that the process is continuous:

For $t = 0$ the scheme gives us simply the initial value of the SDE:

$$\begin{aligned} X_\Delta(0) =& e^{0A}x + \sum_{\substack{1\le\nu\le n\\ u_{\nu-1}\le 0}} e^{(0-u_{\nu-1})A}F(X_\Delta(u_{\nu-1}))\Delta u_{\nu-1} + e^{(0-u_{\nu-1})A}B\Delta W_{\nu-1} \\ =& x. \end{aligned}$$

For $t = u_1$ we have

$$\begin{aligned} X_\Delta(u_1) =& e^{u_1 A}x \\ &+ \sum_{\substack{1\le\nu\le n\\ u_{\nu-1}\le u_1}} e^{(u_1-u_{\nu-1})A}F(X_\Delta(u_{\nu-1}))\Delta u_{\nu-1} \\ &\qquad + e^{(u_1-u_{\nu-1})A}B\Delta W_{\nu-1} \qquad (2.9) \\ =& e^{u_1 A}x + e^{(u_1-u_0)A}F(X_\Delta(u_0))\Delta u_0 + e^{(u_1-u_0)A}B\Delta W_0 \\ =& e^{u_1 A}x + e^{u_1 A}F(X_\Delta(0))\Delta u_0 + e^{u_1 A}B\Delta W_0 \\ =& e^{u_1 A}(x + F(x)\Delta u_0 + B\Delta W_0). \end{aligned}$$

For $t = u_2$ we have by equation (2.9)

$$\begin{aligned} X_\Delta(u_2) =& e^{u_2 A}x + \sum_{\substack{1\le\nu\le n\\ u_{\nu-1}\le u_2}} e^{(u_2-u_{\nu-1})A}F(X_\Delta(u_{\nu-1}))\Delta u_{\nu-1} + e^{(u_2-u_{\nu-1})A}B\Delta W_{\nu-1} \\ =& e^{u_2 A}x \\ &+ e^{(u_2-u_0)A}F(X_\Delta(u_0))\Delta u_0 + e^{(u_2-u_0)A}B\Delta W_0 \\ &+ e^{(u_2-u_1)A}F(X_\Delta(u_1))\Delta u_1 + e^{(u_2-u_1)A}B\Delta W_1 \\ =& e^{u_2 A}x \\ &+ e^{u_2 A}F(x)\Delta u_0 + e^{u_2 A}B\Delta W_0 \\ &+ e^{(u_2-u_1)A}F(X_\Delta(u_1))\Delta u_1 + e^{(u_2-u_1)A}B\Delta W_1 \\ =& e^{(u_2-u_1)A}(e^{u_1 A}x + e^{u_1 A}F(x)\Delta u_0 + e^{u_1 A}B\Delta W_0) \\ &+ e^{(u_2-u_1)A}F(X_\Delta(u_1))\Delta u_1 + e^{(u_2-u_1)A}B\Delta W_1) \\ =& e^{(u_2-u_1)A}X_\Delta(u_1) \\ &+ e^{(u_2-u_1)A}F(X_\Delta(u_1))\Delta u_1 + e^{(u_2-u_1)A}B\Delta W_1 \\ =& e^{(u_2-u_1)A}\big[X_\Delta(u_1) + F(X_\Delta(u_1))\Delta u_1 + B\Delta W_1\big]. \end{aligned}$$

In general we have for $t = u_i,\ i \in \{0, ..., n\}$

$$X_\Delta(u_i) = e^{u_i A}x + \sum_{\substack{1\le\nu\le n\\ u_{\nu-1}\le u_i}} e^{(u_i-u_{\nu-1})A}F(X_\Delta(u_{\nu-1}))\Delta u_{\nu-1} + e^{(u_i-u_{\nu-1})A}B\Delta W_{\nu-1}$$

$$
\begin{aligned}
=&e^{u_iA}x \\
&+e^{(u_i-u_0)A}F(X_\Delta(u_0))\Delta u_0+e^{(u_i-u_0)A}B\Delta W_0 \\
&\vdots \\
&+e^{(u_i-u_{i-1})A}F(X_\Delta(u_{i-1}))\Delta u_{i-1}+e^{(u_i-u_{i-1})A}B\Delta W_{i-1} \\
=&e^{(u_i-u_{i-1})A}\big[e^{u_{i-1}A}x \\
&+e^{(u_{i-1}-u_0)A}F(X_\Delta(u_0))\Delta u_0+e^{(u_{i-1}-u_0)A}B\Delta W_0 \\
&\vdots \\
&+e^{(u_{i-1}-u_{i-2})A}F(X_\Delta(u_{i-1}))\Delta u_{i-2}+e^{(u_{i-1}-u_{i-2})A}B\Delta W_{i-2}\big] \\
&+e^{(u_i-u_{i-1})A}F(X_\Delta(u_{i-1}))\Delta u_{i-1}+e^{(u_i-u_{i-1})A}B\Delta W_{i-1} \\
=&e^{(u_i-u_{i-1})A}X_\Delta(u_{i-1}) \\
&+e^{(u_i-u_{i-1})A}F(X_\Delta(u_{i-1}))\Delta u_{i-1}+e^{(u_i-u_{i-1})A}B\Delta W_{i-1} \\
=&e^{(u_i-u_{i-1})A}\big[X_\Delta(u_{i-1})+F(X_\Delta(u_{i-1}))\Delta u_{i-1}+B\Delta W_{i-1}\big]
\end{aligned}
$$

which is equation (2.6).

For $u_i < t < u_{i+1},\ i \in \{0,...,n-1\}$ we have

$$
\begin{aligned}
X_\Delta(t)=&e^{tA}x+\sum_{\substack{1\le\nu\le n\\ u_{\nu-1}\le t}} e^{(t-u_{\nu-1})A}F(X_\Delta(u_{\nu-1}))\Delta u_{\nu-1}+e^{(t-u_{\nu-1})A}B\Delta W_{\nu-1} \\
=&e^{tA}x \\
&+e^{(t-u_0)A}F(X_\Delta(u_0))\Delta u_0+e^{(t-u_0)A}B\Delta W_0 \\
&\vdots \\
&+e^{(t-u_i)A}F(X_\Delta(u_i))\Delta u_i+e^{(t-u_i)A}B\Delta W_i \\
=&e^{(t-u_i)A}\big[e^{u_iA}x \\
&+e^{(u_i-u_0)A}F(X_\Delta(u_0))\Delta u_0+e^{(u_i-u_0)A}B\Delta W_0 \\
&\vdots \\
&+e^{(u_i-u_{i-1})A}F(X_\Delta(u_{i-1}))\Delta u_{i-1}+e^{(u_i-u_{i-1})A}B\Delta W_{i-1}\big] \\
&+e^{(t-u_i)A}F(X_\Delta(u_i))(t-u_i)+e^{(t-u_i)A}B(W(t)-W(u_i)) \\
=&e^{(t-u_i)A}X_\Delta(u_i) \\
&+e^{(t-u_i)A}F(X_\Delta(u_i))(t-u_i)+e^{(t-u_i)A}B(W(t)-W(u_i)) \\
=&e^{(t-u_i)A}[X_\Delta(u_i)+F(X_\Delta(u_i))(t-u_i)+B(W(t)-W(u_i))]
\end{aligned}
$$

which is equation (2.7).

Why is $X_\Delta$ continuous?

Claim: For any $\omega \in \Omega$ $X_\Delta(\omega) : [0,T] \to H$ is continuous.

Proof: If $u_i < t < u_{i+1},\ i \in \{0, ..., n-1\}$ the continuity of $X_\Delta(\omega)(t)$ is obvious.
If $t = u_i,\ i \in \{0, ..., n-1\}$ we have for $s \searrow t$

$$\begin{aligned}\lim_{s \searrow u_i} X_\Delta(s) = &\lim_{s \searrow u_i} e^{(s-u_i)A}[X_\Delta(u_i) + F(X_\Delta(u_i))(s - u_i) + B(W(s) - W(u_i))]\\ =&X_\Delta(u_i).\end{aligned}$$

and for $s \nearrow t$

$$\begin{aligned}\lim_{s \nearrow u_i} X_\Delta(s) = &\lim_{s \nearrow u_i} e^{(s-u_{i-1})A}[X_\Delta(u_{i-1}) + F(X_\Delta(u_{i-1}))(s - u_{i-1}) + B(W(s) - W(u_{i-1}))]\\ =&e^{(u_i-u_{i-1})A}[X_\Delta(u_{i-1}) + F(X_\Delta(u_{i-1}))(u_i - u_{i-1}) + B(W(u_i) - W(u_{i-1}))]\\ =&X_\Delta(u_i). \quad \square\end{aligned}$$

## 2.3 Uniform convergence

In this section we show that (2.8) is fulfilled.

In the proofs we will see why the conditions on the coefficients given in section 1.2 are necessary.

We have:

**Theorem 2.1** *If $F = 0$ and $1 \leq p \leq 2$, then*

$$E(\sup_{0 \leq t \leq T} \|X_\Delta(t) - X(t)\|^p) \to 0 \text{ as } \rho(\Delta) \to \infty.$$

**Theorem 2.2** *If $1 \leq p \leq 2$, then*

$$E(\sup_{0 \leq t \leq T} \|X_\Delta(t) - X(t)\|^p) \to 0 \text{ as } \rho(\Delta) \to 0.$$

For the proofs we need some lemmas.

**Lemma 2.3** ***The Gronwall Inequality***
*Let $s, T \in \mathbb{R}_+$ and let $\alpha,\ \beta : [s,T] \to \mathbb{R}$ be integrable with*

$$0 \leq \alpha(t) \leq \beta(t) + L \int_s^t \alpha(s) ds$$

*for $t \in [s,T]$ where $L > 0$. Then*

$$\alpha(t) \leq \beta(t) + L \int_s^t e^{L(t-s)} \beta(s) ds$$

*for $t \in [s,T]$.*

**Proof**
[Pac06, Theorem 1.5.1., p.40] □

The next lemma is well-known in one dimension. If we consider a Gaussian random variable with state space $\mathbb{R}$, we can estimate the $p$-th moment by the second moment for even $p$. If we are concerned with a Gaussian random variable taking values in a Hilbert space (c.f. [KF01, Proposition 1.3, p.10]), we have the same result:

**Lemma 2.4** *Let $Y$ be a Gaussian random variable with state space $H$. If $m \in \mathbb{N}$ then there exists a constant $c_m > 0$ depending on $m$ with*

$$E\|Y\|^{2m} \leq c_m (E\|Y\|^2)^m.$$

**Proof**
We reduce it to the case of real valued Gaussian random variables. Let $e_k,\ k \in \mathbb{N}$, be an orthonormal basis of $H$. Since $Y$ is Gaussian, $\langle Y, e_k \rangle$ is Gaussian for all $k \in \mathbb{N}$ (c.f. [KF01, Proposition 1.5, p.10]). By the Hölder inequality we get

$$\begin{aligned}
&E\|Y\|^{2m} \\
=&E(\sum_{j=1}^{\infty} \langle Y, e_j \rangle^2)^m \\
=&\lim_{N\to\infty} E(\sum_{j=1}^{N} \langle Y, e_j \rangle^2)^m \\
=&\lim_{N\to\infty} E(\sum_{1\le j_1\le ...\le j_m\le N} \langle Y, e_{j_1} \rangle^2 ... \langle Y, e_{j_m} \rangle^2) \\
=&\lim_{N\to\infty} \sum_{1\le j_1\le ...\le j_m\le N} E(\langle Y, e_{j_1} \rangle^2 ... \langle Y, e_{j_m} \rangle^2) \\
\le&\lim_{N\to\infty} \sum_{1\le j_1\le ...\le j_m\le N} \left[(E\langle Y, e_{j_1} \rangle^{2m})^{\frac{1}{m}} ... (E\langle Y, e_{j_m} \rangle^{2m})^{\frac{1}{m}}\right].
\end{aligned}$$

Since $\langle Y, e_k \rangle$ is Gaussian for all $k \in \mathbb{N}$, we can apply the well-known equality for moments of real valued Gaussian random variables. For all $k, m \in \mathbb{N}$ there exists a constant $c_m > 0$ with

$$\begin{aligned}
&E\langle Y, e_k \rangle^{2m} \\
=&c_m(E\langle Y, e_k \rangle^2)^m.
\end{aligned}$$

So we get

$$\begin{aligned}
&\lim_{N\to\infty} \sum_{1\le j_1\le ...\le j_m\le N} \left[(E\langle Y, e_{j_1} \rangle^{2m})^{\frac{1}{m}} ... (E\langle Y, e_{j_m} \rangle^{2m})^{\frac{1}{m}}\right] \\
=&\lim_{N\to\infty} \sum_{1\le j_1\le ...\le j_m\le N} \left[c_m^{\frac{1}{m}} E\langle Y, e_{j_1} \rangle^2 ...\ c_m^{\frac{1}{m}} E\langle Y, e_{j_m} \rangle^2\right] \\
=&c_m \lim_{N\to\infty} \sum_{1\le j_1\le ...\le j_m\le N} \left[E\langle Y, e_{j_1} \rangle^2 ...\ E\langle Y, e_{j_m} \rangle^2\right] \\
=&c_m \lim_{N\to\infty} (\sum_{j=1}^{N} E\langle Y, e_j \rangle^2)^m \\
=&c_m \lim_{N\to\infty} (E\sum_{j=1}^{N} \langle Y, e_j \rangle^2)^m \\
=&c_m (E\sum_{j=1}^{\infty} \langle Y, e_j \rangle^2)^m \\
=&c_m (E\|Y\|^2)^m.
\end{aligned}$$

□

Definition: For $Q \in L(U)$ nonnegative, symmetric and with finite trace, $W(t),\ t \in [0,T]$, a $Q$-Wiener process on $(\Omega, \mathcal{F}, P)$, $\Phi : [0,T] \to L_2(Q^{\frac{1}{2}}(U), H)$ stochastically integrable with respect to $W$, $A : D(A) \subset H \to H$ the generator of a $C_0-$semigroup of contractions $e^{tA},\ t \in [0,T]$, we call

$$\int_0^t e^{(t-s)A}\Phi(s)dW(s) \tag{2.10}$$

the *stochastic convolution.*

Since B depends not on time, we will see in the next lemma that the difference $X_\Delta(t) - X(t)$ is a stochastic convolution, where $X$ is the mild solution of problem (1.2) and $X_\Delta$ is the Euler approxiation defined by (2.5).

Let us define $u(s)$ by $u(s) = 0$ for $s = 0$ and $u(s) = u_\nu$ for $u_\nu < s \leq u_{\nu+1},\ s \in (0,T]$ and

$$X_\Delta^\int(t) := e^{tA}x + \int_0^t e^{(t-u(s))A}F(X_\Delta(u(s)))ds + \int_0^t e^{(t-u(s))A}BdW(s)$$

for $t \in [0,T]$.

**Lemma 2.5** *It holds $X_\Delta = X_\Delta^\int$ P-a.s. Especially for $F = 0$ it holds*

$$X_\Delta(t) - X(t) = \int_0^t e^{(t-s)A}(e^{(s-u(s))A} - 1)BdW(s) \ P\text{-}a.s.$$

**Proof**
All equations are true P-a.s. It holds

$$\begin{aligned} &X_\Delta(t) - X(t) \\ =&X_\Delta(t) - X_\Delta^\int(t) + X_\Delta^\int(t) - X(t) \end{aligned}$$

where

$$\begin{aligned} &X_\Delta(t) - X_\Delta^\int(t) \\ =&e^{tA}x + \sum_{\substack{1\leq\nu\leq n \\ u_{\nu-1}\leq t}} e^{(t-u_{\nu-1})A}F(X_\Delta(u_{\nu-1}))\Delta u_{\nu-1} + e^{(t-u_{\nu-1})A}B\Delta W_{\nu-1} \\ &- e^{tA}x - \int_0^t e^{(t-u(s))A}F(X_\Delta(u(s)))ds - \int_0^t e^{(t-u(s))A}BdW(s) \\ =&\sum_{\substack{1\leq\nu\leq n \\ u_{\nu-1}\leq t}} e^{(t-u_{\nu-1})A}F(X_\Delta(u_{\nu-1}))\Delta u_{\nu-1} + e^{(t-u_{\nu-1})A}B\Delta W_{\nu-1} \\ &- \int_0^t e^{(t-u(s))A}F(X_\Delta(u(s)))ds - \int_0^t e^{(t-u(s))A}BdW(s) \end{aligned}$$

$$
\begin{aligned}
=&\sum_{\substack{1\le\nu\le n\\ u_{\nu-1}\le t}} e^{(t-u_{\nu-1})A}F(X_\Delta(u_{\nu-1}))\Delta u_{\nu-1}+e^{(t-u_{\nu-1})A}B\Delta W_{\nu-1}\\
&-\sum_{\substack{1\le\nu\le n\\ u_{\nu-1}\le t}}\Big[\int_{(u_{\nu-1},u_{\nu-1}+\Delta u_{\nu-1}]} e^{(t-u(s))A}F(X_\Delta(u(s)))ds-\int_{(u_{\nu-1},u_{\nu-1}+\Delta u_{\nu-1}]} e^{(t-u(s))A}BdW(s)\Big]\\
=&\sum_{\substack{1\le\nu\le n\\ u_{\nu-1}\le t}}\big[e^{(t-u_{\nu-1})A}F(X_\Delta(u_{\nu-1}))\Delta u_{\nu-1}-\int_{(u_{\nu-1},u_{\nu-1}+\Delta u_{\nu-1}]} e^{(t-u(s))A}F(X_\Delta(u(s)))ds\big]\\
&+\sum_{\substack{1\le\nu\le n\\ u_{\nu-1}\le t}}\big[e^{(t-u_{\nu-1})A}B\Delta W_{\nu-1}-\int_{(u_{\nu-1},u_{\nu-1}+\Delta u_{\nu-1}]} e^{(t-u(s))A}BdW(s)\big]\\
=&\sum_{\substack{1\le\nu\le n\\ u_{\nu-1}\le t}}\big[e^{(t-u_{\nu-1})A}F(X_\Delta(u_{\nu-1}))\Delta u_{\nu-1}-e^{(t-u_{\nu-1})A}F(X_\Delta(u_{\nu-1}))\Delta u_{\nu-1}\big]\\
&+\sum_{\substack{1\le\nu\le n\\ u_{\nu-1}\le t}}\big[e^{(t-u_{\nu-1})A}B\Delta W_{\nu-1}-e^{(t-u_{\nu-1})A}B\int_{(u_{\nu-1},u_{\nu-1}+\Delta u_{\nu-1}]} dW(s)\big]\\
=&\sum_{\substack{1\le\nu\le n\\ u_{\nu-1}\le t}}\big[e^{(t-u_{\nu-1})A}B\Delta W_{\nu-1}-e^{(t-u_{\nu-1})A}B\Delta W_{\nu-1}\big]\\
=&0.
\end{aligned}
$$

So we have for $F=0$

$$
\begin{aligned}
&X_\Delta(t)-X(t)\\
=&X_\Delta^{\int}(t)-X(t)\\
=&e^{tA}x+\int_0^t e^{(t-u(s))A}BdW(s)-e^{tA}x-\int_0^t e^{(t-s)A}BdW(s)\\
=&\int_0^t e^{(t-s)A}(e^{(s-u(s))A}-1)BdW(s).
\end{aligned}
$$

□

The next theorem gives us an estimate of the *stochastic convolution.*

**Theorem 2.6** *Let $Q\in L(U)$ nonnegative, symmetric and with finite trace and let $W(t),\ t\in[0,T]$, be the $Q$-Wiener process on $(\Omega,\mathcal{F},P)$. Assume that $A$ generates a contraction semigroup and $\Phi:[0,T]\to L_2(Q^{\frac{1}{2}}(U),H)$ is stochastically integrable. Then there exists a constant $c>0$ with*

$$
E(\sup_{0\le t\le T}\|\int_0^t e^{(t-s)A}\Phi(s)dW(s)\|^2)\le c\int_0^T\|\Phi(s)\|^2_{L_2(Q^{\frac{1}{2}}(U),H)}ds.
$$

**Proof**
[DPZ92, Theorem 6.10, p.160] □

Now we are prepared to prove theorem 2.1 and 2.2

**Proof of Theorem 2.1**

Since $1 \leq p \leq 2$ there exists by the Hölder inequality a constant $c_1 > 0$ depending on $p$ with

$$
\begin{aligned}
&E(\sup_{0\leq t\leq T} \|X_\Delta(t) - X(t)\|^p) \\
\leq &c_1(E(\sup_{0\leq t\leq T} \|X_\Delta(t) - X(t)\|^2))^{\frac{p}{2}}.
\end{aligned}
$$

For the expectation we get by lemma 2.5

$$
\begin{aligned}
&E(\sup_{0\leq t\leq T} \|X_\Delta(t) - X(t)\|^2) \\
=&E(\sup_{0\leq t\leq T} \| \int_0^t e^{(t-s)A}(e^{(s-u(s))A} - 1)BdW(s)\|^2).
\end{aligned}
$$

By theorem 2.6 there exists a constant $c_2 > 0$ with

$$
\begin{aligned}
&E(\sup_{0\leq t\leq T} \| \int_0^t e^{(t-s)A}(e^{(s-u(s))A} - 1)BdW(s)\|^2) \\
\leq &c_2 \int_0^T \|(e^{(s-u(s))A} - 1)B(J^{-1})\|^2_{L_2(Q_1^{\frac{1}{2}}(U),H)} ds \\
=&c_2 \int_0^T \|(e^{(s-u(s))A} - 1)B\|^2_{L_2(U,H)} ds \\
=&c_2 \int_0^T \sum_{k\in\mathbb{N}} \|(e^{(s-u(s))A} - 1)Be_k\|^2 ds,
\end{aligned}
$$

where $e_k,\ k \in \mathbb{N}$, is an arbitrary orthonormal basis of $U$. In order to show convergence to 0 we have to show for the integrand

$$
\lim_{\rho(\Delta)\to 0} \sum_{k\in\mathbb{N}} \|(e^{(s-u(s))A} - 1)Be_k\|^2 = 0
$$

for all $s \in [0, T]$ and that there exists an integrable dominating function.

In order to show the convergence of the integrand we have to show for fixed $s \in [0, T]$

$$
\lim_{\rho(\Delta)\to 0} \|(e^{(s-u(s))A} - 1)Be_k\|^2 = 0 \qquad \forall k \in \mathbb{N}
$$

and that there exists a sequence $y_k,\ k \in \mathbb{N}$, independent of $\rho(\Delta)$ such that

$$
\|(e^{(s-u(s))A} - 1)Be_k\|^2 \leq y_k
$$

for all $k \in \mathbb{N}$ and $\sum_{k\in\mathbb{N}} y_k < \infty$.

The latter is true because by the contraction property of the semigroup we get

$$
\|(e^{(s-u(s))A} - 1)Be_k\|^2 \leq 2^2\|Be_k\|^2 \qquad \forall k \in \mathbb{N}
$$

and $B \in L_2(U,H)$.

It is true that

$$\lim_{\rho(\Delta)\to 0} \|(e^{(s-u(s))A} - 1)Be_k\|^2 = 0 \qquad \forall k \in \mathbb{N}$$

because of the continuity of the semigroup. The integrable dominating function can be choosen by $2^2\|B\|^2_{L_2(U,H)}$. $\square$

**Proof of Theorem 2.2**

By the definition of $X_\Delta$ and $X$ and lemma 2.5 we get

$$\begin{aligned}
&X_\Delta(t) - X(t)\\
=&e^{tA}x + \int_0^t e^{(t-u(s))A}F(X_\Delta(u(s)))ds + \int_0^t e^{(t-u(s))A}BdW(s)\\
&- e^{tA}x - \int_0^t e^{(t-s)A}F(X(s))ds - \int_0^t e^{(t-s)A}BdW(s)\\
=&\int_0^t e^{(t-u(s))A}F(X_\Delta(u(s))) - e^{(t-s)A}F(X(s))ds\\
&+ \int_0^t e^{(t-s)A}(e^{(s-u(s))A} - 1)BdW(s) \quad \text{P-a.s.}
\end{aligned}$$

Let $S \in [0,T]$. Using the triangle inequality we get for $c := 2^p$

$$\begin{aligned}
&E(\sup_{0\le t\le S} \|X_\Delta(t) - X(t)\|^p)\\
\le& cE(\sup_{0\le t\le S} \|\int_0^t e^{(t-u(s))A}F(X_\Delta(u(s))) - e^{(t-s)A}F(X(s))ds\|^p)\\
&+ cE(\sup_{0\le t\le S} \|\int_0^t e^{(t-s)A}(e^{(s-u(s))A} - 1)BdW(s)\|^p).
\end{aligned}$$

Since $1 \le p \le 2$ the second term converges to 0 as $\rho(\Delta) \to 0$ by theorem 2.1. The first term can be divided into three parts, the constant $c$ changes, but depends still on $p$ :

$$\begin{aligned}
&cE(\sup_{0\le t\le S} \|\int_0^t e^{(t-u(s))A}F(X_\Delta(u(s))) - e^{(t-s)A}F(X(s))ds\|^p)\\
\le& cE(\sup_{0\le t\le S} \|\int_0^t e^{(t-u(s))A}F(X_\Delta(u(s))) - e^{(t-u(s))A}F(X(u(s)))ds\|^p)\\
+& cE(\sup_{0\le t\le S} \|\int_0^t e^{(t-u(s))A}F(X(u(s))) - e^{(t-u(s))A}F(X(s))ds\|^p)\\
+& cE(\sup_{0\le t\le S} \|\int_0^t e^{(t-u(s))A}F(X(s)) - e^{(t-s)A}F(X(s))ds\|^p) =: \mathcal{A} + \mathcal{B} + \mathcal{C}.
\end{aligned}$$

Consider $\mathcal{A}$:

By the Hölder inequality, the semigroup property, the Lipschitz condition and Fubini we get (constant $c$ changes but depends still on $p$)

$$
\begin{aligned}
&cE(\sup_{0\leq t\leq S}\|\int_0^t e^{(t-u(s))A}F(X_\Delta(u(s)))-e^{(t-u(s))A}F(X(u(s)))ds\|^p)\\
\leq&cE(\sup_{0\leq t\leq S}\int_0^t\|e^{(t-u(s))A}F(X_\Delta(u(s)))-e^{(t-u(s))A}F(X(u(s)))\|^p ds)\\
\leq&cE(\int_0^S\|F(X_\Delta(u(s)))-F(X(u(s)))\|^p ds)\\
\leq&cK^pE(\int_0^S\|X_\Delta(u(s))-X(u(s))\|^p\ ds)\\
=&cK^p\int_0^S E(\|X_\Delta(u(s))-X(u(s))\|^p\ )ds\\
\leq&cK^p\int_0^S E(\sup_{0\leq t\leq s}\|X_\Delta(u(t))-X(u(t))\|^p)ds\\
\leq&cK^p\int_0^S E(\sup_{0\leq t\leq s}\|X_\Delta(t)-X(t)\|^p)ds
\end{aligned}
$$

Consider $\mathcal{B}$:
By the Hölder inequality, the semigroup property, the Lipschitz condition and Fubini we get (constant $c$ changes but depends still on $p$)

$$
\begin{aligned}
&cE(\sup_{0\leq t\leq S}\|\int_0^t e^{(t-u(s))A}F(X(u(s)))-e^{(t-u(s))A}F(X(s))ds\|^p)\\
\leq&cE(\sup_{0\leq t\leq S}\int_0^t\|e^{(t-u(s))A}F(X(u(s)))-e^{(t-u(s))A}F(X(s))\|^p ds)\\
\leq&cE(\int_0^S\|F(X(u(s)))-F(X(s))\|^p ds)\\
\leq&cK^pE(\int_0^S\|X(u(s))-X(s)\|^p\ ds)\\
=&cK^p\int_0^S E\|X(u(s))-X(s)\|^p\ ds\\
\leq&cK^p\big(\int_0^S E\|X(u(s))-X(s)\|^2\ ds\big)^{\frac{1}{2}}.
\end{aligned}
$$

By Propostion 1.5 we know, that there exists a continuous version of $X$. Therefore we can assume that $\|X(\omega)(u(s))-X(\omega)(s)\|$ converges to 0 as $\rho(\Delta)\to 0$ for all $s\in[0,S],\omega\in\Omega$. So the integrand converges to 0 as $\rho(\Delta)\to 0$ for all $s\in[0,S],\omega\in\Omega$.
For the integrand we have

$$
\begin{aligned}
&\|X(u(s))-X(s)\|^2\\
\leq&2\|X(u(s))\|^2+2\|X(s)\|^2.
\end{aligned}
$$

Since $X$ is the mild solution it holds by definition that $\sup_{0 \le t \le T} E(\|X(t)\|^2) < \infty$ and therefore the integrand is integrable. Together with the dominated convergence theorem we obtain convergence to 0.

Consider $\mathcal{C}$:

By the Hölder inequality, the semigroup property and Fubini we get (constant $c$ changes but depends still on $p$)

$$\begin{aligned}
&cE(\sup_{0 \le t \le S} \| \int_0^t e^{(t-u(s))A} F(X(s)) - e^{(t-s)A} F(X(s)) ds \|^p) \\
=&cE(\sup_{0 \le t \le S} \| \int_0^t e^{(t-s)A} (e^{(s-u(s))A} - 1) F(X(s)) ds \|^p) \\
\le&cE(\sup_{0 \le t \le S} \int_0^t \| e^{(t-s)A} (e^{(s-u(s))A} - 1) F(X(s)) \|^p ds) \\
\le&cE(\sup_{0 \le t \le S} \int_0^t \| (e^{(s-u(s))A} - 1) F(X(s)) \|^p ds) \\
\le&cE(\int_0^S \| (e^{(s-u(s))A} - 1) F(X(s)) \|^p ds) \\
=&c \int_0^S E \| (e^{(s-u(s))A} - 1) F(X(s)) \|^p ds \\
\le&c( \int_0^S E \| (e^{(s-u(s))A} - 1) F(X(s)) \|^2 ds)^{\frac{p}{2}}.
\end{aligned}$$

The integrand converges to 0 as $\rho(\Delta) \to 0$ for each $s \in [0, S]$ and $\omega \in \Omega$.

By the linear growth condition we have for the integrand

$$\begin{aligned}
&\| (e^{(s-u(s))A} - 1) F(X(s)) \|^2 \\
\le&2^2 \| F(X(s)) \|^2 \\
\le&2^2 K^2 (1 + \|X(s)\|)^2 \\
=&2^2 K^2 (1 + 2\|X(s)\| + \|X(s)\|^2).
\end{aligned}$$

This term is integrable: By the Hölder inequality there exists a constant $d > 0$ with

$$\begin{aligned}
&\int_0^S E \|X(s)\| ds \\
\le&d( \int_0^S E \|X(s)\|^2 ds)^{\frac{1}{2}}.
\end{aligned}$$

Since $X$ is the mild solution it holds by definition that $\sup_{0 \le t \le T} E(\|X(t)\|^2) < \infty$ and therefore $2^2 K^2 (1 + 2\|X(s)\| + \|X(s)\|^2)$ is integrable. Together with the dominated convergence theorem we obtain convergence to 0.

If we put $\mathcal{A}$, $\mathcal{B}$ and $\mathcal{C}$ together we get for our error estimate

$$\begin{aligned}&E(\sup_{0\le t\le S}\|X_\Delta(t)-X(t)\|^p)\\ \le& cK^p\int_0^S E(\sup_{0\le t\le s}\|X_\Delta(t)-X(t)\|^p)ds+\mathcal{D}_\Delta(S)\end{aligned}$$

for $S\in[0,T]$, where $\mathcal{D}_\Delta:[0,T]\to\mathbb{R}_+$ is an increasing function for each approximation $\Delta$ and $\mathcal{D}_\Delta(T)$ converges to 0 as $\rho(\Delta)\to 0$. With Gronwalls inequality (lemma 2.3) we get

$$\begin{aligned}&E(\sup_{0\le t\le S}\|X_\Delta(t)-X(t)\|^p)\\ \le& cK^p\int_0^S e^{cK^p(S-s)}\mathcal{D}_\Delta(s)ds+\mathcal{D}_\Delta(S)\end{aligned}$$

for $S\in[0,T]$, especially for $S=T$ we get

$$\begin{aligned}&E(\sup_{0\le t\le T}\|X_\Delta(t)-X(t)\|^p)\\ \le& cK^p\int_0^T e^{cK^p(T-s)}\mathcal{D}_\Delta(s)ds+\mathcal{D}_\Delta(T)\\ \le& cK^p\int_0^T e^{cK^pT}\mathcal{D}_\Delta(T)ds+\mathcal{D}_\Delta(T)\\ \le&(cK^pTe^{cK^pT}+1)\mathcal{D}_\Delta(T)\end{aligned}$$

and the last term converges to 0 as $\rho(\Delta)\to 0$. $\square$

This proof is due to Kloeden, Platen [KP92, Theorem 10.2.2, p.342].

## 2.4 Uniform convergence for unbounded generator

In this section we show how we can modify our approximation $X_\Delta$ in the case of unbounded generator $A$ in order to deal with bounded generators only and to fulfill (2.8).

We make use of the Yosida approximation of $A$, called $A_\alpha$, $\alpha > 0$. The most important properties of the Yosida approximation are that it is bounded for every $\alpha > 0$ and that we have $\|(e^{sA} - e^{sA_\alpha})h\| \to 0$ as $\alpha \to \infty$ for all $h \in H$ uniformly on bounded intervalls (c.f. theorem 1.2).

Let us define

- $X_\alpha$ as the mild solution of problem (1.2) relative to $A_\alpha$
- $X_{\Delta,\alpha}$ as the Euler approximation of $X_\alpha$.

By theorem 2.1 and 2.2 we know that $X_{\Delta,\alpha}$ approximates $X_\alpha$ in the sense of (2.6) (2.7) and (2.8).

Here we will show, that we can approximate $X$ by $X_\alpha$ in the sense of (2.8).

Finally we have the possiblity to approximate $X$ by $X_{\Delta,\alpha}$ in the sense of (2.8). This kind of double approximation is proved in corollary 2.12 and 2.13.

For $\|x\| < \bar{c} < \infty$ we define the stopping time $\tau := \inf\{t \geq 0 | \|X(t)\| \geq \bar{c}\}$.

Then we have:

**Theorem 2.7** *If $F = 0$ and $p > 2$, then*

$$E(\sup_{0\leq t\leq T} \|X_\alpha(t) - X(t)\|^p) \to 0 \textit{ for } \alpha \to \infty.$$

**Theorem 2.8** *If $p > 2$, then*

$$E(\sup_{0\leq t\leq T\wedge\tau} \|X_\alpha(t) - X(t)\|^p) \to 0 \textit{ for } \alpha \to \infty.$$

Remark: We can show with the Hölder inequality, that both theorems are also true for $1 \leq p \leq 2$.

The proof of theorem 2.7 is given by [DPZ92, Theorem 5.12, p.129] but is also stated here in order to give more details. This would make it easier to compute the rate of convergence in practice. The proof of theorem 2.8 works similar to the proof of theorem 2.2.

For the proof of theorem 2.7 we need the following three lemmas, which are all due to Da Prato, Zabczyk [DPZ92, p.128].

Assume that $F = 0$ and $x = 0$. Then we have for the mild solution $X$ of problem (1.2)

$$X(t) = \int_0^t e^{(t-s)A} B dW(s) \quad \text{P-a.s.}$$

The next lemma gives us an alternative representation of the the right hand side, which is a stochastic convolution (c.f. definition (2.10)). With the help of this representation we obtain two helpfull estimates.

**Lemma 2.9** *Let $\beta \in (0,1)$. Then there is the following representation of the stochastic convolution.*

$$\int_0^t e^{(t-s)A} B dW(s) = \frac{\sin \pi\beta}{\pi} \int_0^t e^{(t-s)A}(t-s)^{\beta-1} Y(s) ds \quad P\text{-a.s.}$$

*where*

$$Y(s) := \int_0^s e^{(s-r)A}(s-r)^{-\beta} B dW(r).$$

**Proof**
The proof is based on the stochastic Fubini theorem and the formula

$$\int_r^t (t-s)^{\beta-1}(s-r)^{-\beta} ds = \frac{\pi}{\sin \pi\beta}$$

which holds for all $0 \leq r \leq t$, $0 < \beta < 1$. For a detailed proof see [KF01, Theorem 3.12, p.84]. $\square$

**Lemma 2.10** *Let $X(t) := \int_0^t e^{(t-s)A}(t-s)^{\beta-1} Y(s) ds$ with $Y : [0,T] \to H$, $\int_0^T \|Y(s)\|^{2m} ds < \infty$, $0 < \beta < 1$ and $m \in \mathbb{N}$, $m > \frac{1}{2\beta}$. Then there exists a constant $c > 0$ depending on $\beta$, $m$ and $T$ with*

$$\sup_{0 \leq t \leq T} \|X(t)\|^{2m} \leq c \int_0^T \|Y(s)\|^{2m} ds.$$

**Proof**
Define $Z(s) := e^{(t-s)A}(t-s)^{\beta-1}$ for $t \in [0,T]$, $s \in [0,t]$ and $q := \frac{1}{1-\frac{1}{2m}}$. Since $m > \frac{1}{2}$, $q$ is well defined. By the defintion of $X$ and the Hölder inequality we get

$$\begin{aligned}
&\|X(t)\|^{2m} \\
=&\|\int_0^t Z(s)Y(s)ds\|^{2m} \\
\leq&(\int_0^t \|Z(s)Y(s)\|ds)^{2m} \\
\leq&(\int_0^t \|Z(s)\|_{L(H)}\|Y(s)\|ds)^{2m} \\
\leq&(\int_0^t \|Z(s)\|_{L(H)}^q ds)^{\frac{2m}{q}} \int_0^t \|Y(s)\|^{2m} ds.
\end{aligned}$$

Consider $\int_0^t \|Z(s)\|_{L(H)}^q ds$:

By the defintion of $Z$ and the contraction property of the semigroup it holds

$$\begin{aligned}
&\int_0^t \|Z(s)\|_{L(H)}^q ds \\
=&\int_0^t \|e^{(t-s)A}(t-s)^{\beta-1}\|_{L(H)}^q ds \\
=&\int_0^t \|e^{(t-s)A}\|_{L(H)}^q (t-s)^{(\beta-1)q} ds \\
\leq&\int_0^t (t-s)^{(\beta-1)q} ds \\
=&\int_0^t s^{(\beta-1)q} ds \\
\leq&\int_0^T \frac{1}{s^b} ds
\end{aligned}$$

for $b := (1-\beta)q$. Since $m > \frac{1}{2\beta}$ we have $\beta > \frac{1}{2m} \Leftrightarrow 1-\beta < 1-\frac{1}{2m} \Leftrightarrow 1-\beta < \frac{1}{q} \Leftrightarrow b = (1-\beta)q < 1$ and thus $\int_0^T \frac{1}{s^b} ds \leq \infty$. So there exists a constant $c_1 > 0$ depending on $\beta,\ m,\ T$ with

$$\|X(t)\|^{2m} \leq c_2 \int_0^t \|Y(s)\|^{2m} ds$$

and finally

$$\sup_{0\leq t\leq T} \|X(t)\|^{2m} \leq c_2 \int_0^T \|Y(s)\|^{2m} ds.$$

$\square$

**Lemma 2.11** *Let $\beta \in (0, \frac{1}{2})$ and $Y(s)$ be as in lemma 2.9 and $m \in \mathbb{N}$. Then there exists a constant $c > 0$ depending on $m,\ \beta,\ T$ and $B$ with*

$$E \int_0^T \|Y(s)\|^{2m} ds \leq c.$$

**Proof**

For fixed $s \in [0,T]$ $Y(s) = \int_0^s e^{(s-r)A}(s-r)^{-\beta} B dW(r)$ is a Gaussian random variable with state space $H$. Since $m \in \mathbb{N}$ there exists by lemma 2.4 a constant $c_m > 0$ depending on $m$ with

$$\begin{aligned}
&E\|Y(s)\|^{2m} \\
\leq& c_m (E\|Y(s)\|^2)^m
\end{aligned}$$

By the definition of the stochastic integral, the Itô-isometry and the semigroup property we have

$$
\begin{aligned}
& E\|Y(s)\|^2 \\
=& E\|\int_0^s e^{(s-r)A}(s-r)^{-\beta} B dW(r)\|^2 \\
=& E\|\int_0^s e^{(s-r)A}(s-r)^{-\beta} B(J^{-1})\ dW(r)\|^2 \\
=& \int_0^s \|e^{(s-r)A}(s-r)^{-\beta} B(J^{-1})\|^2_{L_2(Q_1^{\frac{1}{2}}(U),H)} dr \\
=& \int_0^s (s-r)^{-2\beta}\|B\|^2_{L_2(U,H)} dr \\
=& \int_0^s r^{-2\beta}\|B\|^2_{L_2(U,H)} dr \\
\leq & \int_0^T r^{-2\beta}\|B\|^2_{L_2(U,H)} dr.
\end{aligned}
$$

Since $\beta \in (0, \frac{1}{2})$ and $B \in L_2$ there exists a constant $c > 0$ depending on $\beta,\ T,\ B$ with

$$\int_0^T r^{-2\beta}\|B\|^2_{L_2(U,H)} dr\ = c.$$

Consequently there exists a constant $d > 0$ depending on $m,\ \beta,\ T,\ B$ with

$$E\int_0^T \|Y(s)\|^{2m} ds \leq d.$$

□

Now we are prepared to prove theorem 2.7 and 2.8

**Proof of Theorem 2.7**

Since $p > 2$, $(\frac{1}{p}, \frac{1}{2}) \neq \emptyset$. Thus let $\frac{1}{p} < \beta < \frac{1}{2}$ and $m \in \mathbb{N}$, $m > \frac{p}{2}$. Since

$$X(t) = e^{tA}x + \int_0^t e^{(t-s)A} B dW(s)$$

and

$$X_\alpha(t) = e^{tA_\alpha}x + \int_0^t e^{(t-s)A_\alpha} B dW(s),$$

we have by the factorization method (c.f. lemma 2.9)

$$X(t) = e^{tA}x + \frac{\sin \pi\beta}{\pi}\int_0^t e^{(t-s)A}(t-s)^{\beta-1} Y(s) ds$$

where

$$Y(s) := \int_0^s e^{(s-r)A}(s-r)^{-\beta} B dW(r)$$

and

$$X_\alpha(t) = e^{tA_\alpha}x + \frac{\sin \pi\beta}{\pi}\int_0^t e^{(t-s)A_\alpha}(t-s)^{\beta-1} Y_\alpha(s) ds,$$

where

$$Y_\alpha(s) := \int_0^s e^{(s-r)A_\alpha}(s-r)^{-\beta} B dW(r)$$

respectively. Thus we can write

$$\begin{aligned}
&X_\alpha(t) - X(t) \\
=&e^{tA_\alpha}x - e^{tA}x \\
&+ \frac{\sin \pi\beta}{\pi} \int_0^t (e^{(t-s)A} - e^{(t-s)A_\alpha})(t-s)^{\beta-1} Y(s) ds \\
&+ \frac{\sin \pi\beta}{\pi} \int_0^t e^{(t-s)A_\alpha}(t-s)^{\beta-1}(Y(s) - Y_\alpha(s)) ds =: H_\alpha(t) + I_\alpha(t) + J_\alpha(t)
\end{aligned}$$

We have to show that $E\big(\sup_{0\le t\le T} \|H_\alpha(t)\|^p\big)$, $E\big(\sup_{0\le t\le T} \|I_\alpha(t)\|^p\big)$ and $E\big(\sup_{0\le t\le T} \|J_\alpha(t)\|^p\big)$ converge to 0 as $\alpha \to \infty$.

Consider $H_\alpha(t)$:

Since $A_\alpha$ is the Yosida approximation of $A$, we get by theorem 1.2 that $\|e^{sA_\alpha}h - e^{sA}h\|$ converges to 0 for $\alpha \to \infty$ for all $h \in H$ uniformly on bounded intervalls. Consequently we get

$$\lim_{\alpha\to\infty} \sup_{0\le t\le T} \|e^{tA_\alpha}x - e^{tA}x\|^p = 0.$$

Consider $I_\alpha(t)$:

There exists a constant $c_1 > 0$ depending on $p$ and $\beta$ with

$$\begin{aligned}
&\|I_\alpha(t)\|^p \\
=&\|\frac{\sin \pi\beta}{\pi} \int_0^t (e^{(t-s)A} - e^{(t-s)A_\alpha})(t-s)^{\beta-1} Y(s) ds\|^p \\
\le&c_1 (\int_0^t \|(e^{(t-s)A} - e^{(t-s)A_\alpha})(t-s)^{\beta-1} Y(s)\| ds)^p \\
\le&c_1 (\int_0^t |(t-s)^{\beta-1}| \|(e^{(t-s)A} - e^{(t-s)A_\alpha}) Y(s)\| ds)^p \\
=&c_1 \|fg\|_1^p,
\end{aligned}$$

where $f(s) := \|(e^{(t-s)A} - e^{(t-s)A_\alpha})Y(s)\|$ and $g(s) := |(t-s)^{\beta-1}|$. We now want to apply the Hölder inequality. Therefore define $q$ by $\frac{1}{p} + \frac{1}{q} = 1$. Consider $g$: It holds

$$\begin{aligned}
&\|g\|_q^p \\
=&(\int_0^t |(t-s)^{\beta-1}|^q ds)^{\frac{p}{q}} \\
=&(\int_0^t s^{(\beta-1)q} ds)^{\frac{p}{q}} \\
=&(\int_0^t \frac{1}{s^b} ds)^{\frac{p}{q}}
\end{aligned}$$

$$\leq (\int_0^T \frac{1}{s^b} ds)^{\frac{p}{q}} =: h$$

for $b := (1-\beta)q$. Since $\beta > \frac{1}{p} \Leftrightarrow 1-\beta < 1-\frac{1}{p} \Leftrightarrow 1-\beta < \frac{1}{q} \Leftrightarrow b = (1-\beta)q < 1$, we have $h < \infty$. Thus there exists a constant $c_2 > 0$ depending on $p,\ \beta,\ T$ with $\|g\|_q^p \leq c_2$. Hölder inequality gives us

$$\begin{aligned}
&E \sup_{0\leq t\leq T} \|fg\|_1^p \\
\leq &c_2 E \sup_{0\leq t\leq T} \|f\|_p^p \\
= &c_2 E \sup_{0\leq t\leq T} \int_0^t \|(e^{(t-s)A} - e^{(t-s)A_\alpha})Y(s)\|^p ds \\
\leq &c_2 E \sup_{0\leq t\leq T} \int_0^T \|(e^{(t\vee s-s)A} - e^{(t\vee s-s)A_\alpha})Y(s)\|^p ds \\
\leq &c_2 E \int_0^T \sup_{0\leq t\leq T} \|(e^{(t\vee s-s)A} - e^{(t\vee s-s)A_\alpha})Y(s)\|^p ds
\end{aligned}$$

Since $A_\alpha$ is the Yosida approximation of $A$, we get by theorem 1.2 that $\|(e^{sA} - e^{sA_\alpha})h\|$ converges to 0 as $\alpha \to \infty$ for all $h \in H$ uniformly on bounded intervalls.

By the semigroup property we have for the integrand

$$\|(e^{(t-s)A} - e^{(t-s)A_\alpha})Y(s)\|^p \leq 2^p \|Y(s)\|^p.$$

This term is integrable: Since $m > \frac{p}{2}$, there exists by the Hölder inequality a constant $c_3 > 0$ depending on $m,\ p$ with

$$E \int_0^T \|Y(s)\|^p ds \leq c_3 (E \int_0^T \|Y(s)\|^{2m} ds)^{\frac{p}{2m}}.$$

Since $\beta < \frac{1}{2}$ there exists by lemma 2.11 a constant $c_5 > 0$ depending on $m,\ \beta,\ T,\ B,\ p$ with

$$(E \int_0^T \|Y(s)\|^{2m} ds)^{\frac{p}{2m}} \leq c_5.$$

So we get an integrable dominating function and from the dominated convergence theorem follows

$$\lim_{\alpha\to\infty} E \sup_{0\leq t\leq T} \|I_\alpha(t)\|^p = 0.$$

Consider $J_\alpha(t)$:
Before we have a detailed look at $J_\alpha(t)$, note the following inequality concerning $E\|Y(s) - Y_\alpha(s)\|^2$: By the definition of the stochastic integral, the Itô-isometry

and equation (1.1) we get

$$
\begin{aligned}
&E\|Y(s)-Y_\alpha(s)\|^2\\
=&E\|\int_0^s (e^{(s-r)A}-e^{(s-r)A_\alpha})(s-r)^{-\beta}BdW(r)\|^2\\
=&\int_0^s \|(e^{(s-r)A}-e^{(s-r)A_\alpha})(s-r)^{-\beta}B\|^2_{L_2(U,H)}dr\\
=&\int_0^s (s-r)^{-2\beta}\|(e^{(s-r)A}-e^{(s-r)A_\alpha})B\|^2_{L_2(U,H)}dr\\
=&\int_0^s r^{-2\beta}\|(e^{rA}-e^{rA_\alpha})B\|^2_{L_2(U,H)}dr \qquad (2.11)\\
\leq&\int_0^T r^{-2\beta}\|(e^{rA}-e^{rA_\alpha})B\|^2_{L_2(U,H)}dr\\
=&\int_0^T r^{-2\beta}\sum_{k=1}^\infty \|(e^{rA}-e^{rA_\alpha})B(e_k)\|^2dr\\
=&\sum_{k=1}^\infty \int_0^T r^{-2\beta}\|(e^{rA}-e^{rA_\alpha})B(e_k)\|^2dr
\end{aligned}
$$

Since $m>\frac{p}{2}$, there exists by the Hölder inequality a constant $c_1>0$ depending on $m,\ p$ with

$$E\sup_{0\leq t\leq T}\|J_\alpha(t)\|^p\leq c_1(E\sup_{0\leq t\leq T}\|J_\alpha(t)\|^{2m})^{\frac{p}{2m}}.$$

Since $\beta>\frac{1}{p}\Leftrightarrow p>\frac{1}{\beta}\Leftrightarrow\frac{p}{2}>\frac{1}{2\beta},\ m>\frac{p}{2}$ and $p>2$, we have $m>\frac{1}{2\beta}$ and thus there exists by lemma 2.10 a constant $c_2>0$ depending on $\beta,\ m,\ T$ with

$$
\begin{aligned}
&E\sup_{0\leq t\leq T}\|J_\alpha(t)\|^{2m}\\
\leq&c_2E\int_0^T \|Y(s)-Y_\alpha(s)\|^{2m}ds.
\end{aligned}
$$

It remains to show, that the expectation converges to 0.

By Fubini we get

$$
\begin{aligned}
&E\int_0^T \|Y(s)-Y_\alpha(s)\|^{2m}ds\\
=&\int_0^T E\|Y(s)-Y_\alpha(s)\|^{2m}ds.
\end{aligned}
$$

Since $m\in\mathbb{N}$ and $Y(s)-Y_\alpha(s)$ is Gaussian there exists by lemma 2.4 a constant $c_3>0$ depending on $m$ with

$$
\begin{aligned}
&\int_0^T E\|Y(s)-Y_\alpha(s)\|^{2m}ds\\
\leq&c_4\int_0^T (E\|Y(s)-Y_\alpha(s)\|^2)^m ds.
\end{aligned}
$$

By inequality (2.11) we get

$$\begin{aligned}&\int_0^T (E\|Y(s)-Y_\alpha(s)\|^2)^m ds\\ \leq& T\big(\sum_{k=1}^{\infty}\int_0^T r^{-2\beta}\|(e^{rA}-e^{rA_\alpha})B(e_k)\|^2 dr\big)^m.\end{aligned}$$

It remains to show, that the sum converges to 0. For this reason we have to show, that

$$\lim_{\alpha\to\infty}\int_0^T r^{-2\beta}\|(e^{rA}-e^{rA_\alpha})B(e_k)\|^2 dr = 0 \text{ for all } k\in\mathbb{N}. \tag{2.12}$$

and that there exists a sequence $y_k,\ k\in\mathbb{N}$, such that

$$\int_0^T r^{-2\beta}\|(e^{rA}-e^{rA_\alpha})B(e_k)\|^2 dr \leq y_k \tag{2.13}$$

for all $k\in\mathbb{N},\ \alpha>0$ and $\sum_{k\in\mathbb{N}} y_k<\infty$.

Let us show (2.12). Since $A_\alpha$ is the Yosida approximation of $A$, we get by theorem 1.2 that $\|(e^{rA}-e^{rA_\alpha})h\|$ converges to 0 for $\alpha\to\infty$ for all $h\in H$ uniformly on bounded intervalls.

By the semigroup property we have for the integrand

$$\begin{aligned}&r^{-2\beta}\|(e^{rA}-e^{rA_\alpha})B(e_k)\|^2\\ \leq& r^{-2\beta}2^2\|B\|^2_{L_2(U,H)}.\end{aligned}$$

Since $\beta<\frac{1}{2}\Leftrightarrow 2\beta<1$, this term is Riemann integrable over $[0,T]$.

So we get an integrable dominating function and from the dominated convergence theorem follows (2.12).

Let us show (2.13). For $k\in\mathbb{N}$ define

$$y_k := 2^2\|B(e_k)\|^2\int_0^T r^{-2\beta}dr.$$

Since $\beta<\frac{1}{2}$ and $B\in L_2$ it holds

$$\sum_{k\in\mathbb{N}} y_k = 2^2\|B\|^2_{L_2(U,H)}\int_0^T r^{-2\beta}dr<\infty,$$

and thus the sequence $y_k,\ k\in\mathbb{N}$, fulfills (2.13). □

**Proof of Theorem 2.8**
By definition of $X_\alpha$ and $X$ we get

$$X_\alpha(t)-X(t)$$

$$
\begin{aligned}
=&e^{tA_\alpha}x + \int_0^t e^{(t-s)A_\alpha} F(X_\alpha(s))ds + \int_0^t e^{(t-s)A_\alpha} BdW(s) \\
&- e^{tA}x - \int_0^t e^{(t-s)A} F(X(s))ds - \int_0^t e^{(t-s)A} BdW(s) \\
=&e^{tA_\alpha}x - e^{tA}x \\
&+ \int_0^t e^{(t-s)A_\alpha} F(X_\alpha(s)) - e^{(t-s)A} F(X(s))ds \\
&+ \int_0^t (e^{(t-s)A_\alpha} - e^{(t-s))A}) BdW(s).
\end{aligned}
$$

Let $S \in [0, T]$. Using the triangle inequality we get for $c = 2^p$

$$
\begin{aligned}
&E(\sup_{0 \le t \le S \wedge \tau} \|X_\alpha(t) - X(t)\|^p) \\
\le & cE(\sup_{0 \le t \le S \wedge \tau} \|e^{tA_\alpha}x - e^{tA}x + \int_0^t (e^{(t-s)A_\alpha} - e^{(t-s)A}) BdW(s)\|^p) \\
+ & cE(\sup_{0 \le t \le S \wedge \tau} \| \int_0^t e^{(t-s)A_\alpha} F(X_\alpha(s)) - e^{(t-s)A} F(X(s))ds\|^p).
\end{aligned}
$$

Since $p > 2$, the first term converges to 0 as $\alpha \to \infty$ by theorem 2.7. The second term can be divided into two parts, the constant $c$ changes, but depends still on $p$ :

$$
\begin{aligned}
&cE(\sup_{0 \le t \le S \wedge \tau} \| \int_0^t e^{(t-s)A_\alpha} F(X_\alpha(s)) - e^{(t-s)A} F(X(s))ds\|^p) \\
\le & cE(\sup_{0 \le t \le S \wedge \tau} \| \int_0^t e^{(t-s)A_\alpha} F(X_\alpha(s)) - e^{(t-s)A_\alpha} F(X(s))ds\|^p) \\
+ & cE(\sup_{0 \le t \le S \wedge \tau} \| \int_0^t e^{(t-s)A_\alpha} F(X(s)) - e^{(t-s)A} F(X(s))ds\|^p) =: \mathcal{A} + \mathcal{B}.
\end{aligned}
$$

Consider $\mathcal{A}$:

By the Hölder inequality, the semigroup property, the Lipschitz condition and Fubini we get (constant $c$ changes, but depends still on $p$)

$$
\begin{aligned}
&cE(\sup_{0 \le t \le S \wedge \tau} \| \int_0^t e^{(t-s)A_\alpha} F(X_\alpha(s)) - e^{(t-s)A_\alpha} F(X(s))ds\|^p) \\
\le & cE(\sup_{0 \le t \le S \wedge \tau} \int_0^t \|e^{(t-s)A_\alpha} (F(X_\alpha(s)) - F(X(s)))\|^p ds) \\
\le & cE(\int_0^{S \wedge \tau} \|F(X_\alpha(s)) - F(X(s))\|^p ds) \\
\le & cK^p E(\int_0^S \|X_\alpha(s) - X(s)\|^p \ 1_{[0,\tau]}(s) ds) \\
= & cK^p \int_0^S E(\|X_\alpha(s) - X(s)\|^p \ 1_{[0,\tau]}(s)) ds
\end{aligned}
$$

$$\leq cK^p \int_0^S E(\sup_{0\leq t\leq s\wedge\tau} \|X_\alpha(t) - X(t)\|^p) ds.$$

Consider $\mathcal{B}$:

By the Hölder inequality we get (constant $c$ changes, but depends still on $p$)

$$\begin{aligned}
&cE(\sup_{0\leq t\leq S\wedge\tau} \| \int_0^t e^{(t-s)A_\alpha} F(X(s)) - e^{(t-s)A} F(X(s)) ds\|^p) \\
\leq &cE(\sup_{0\leq t\leq S\wedge\tau} \int_0^t \|(e^{(t-s)A_\alpha} - e^{(t-s)A}) F(X(s))\|^p ds) \\
\leq &cE(\sup_{0\leq t\leq S\wedge\tau} \int_0^T \|(e^{(t\vee s-s)A_\alpha} - e^{(t\vee s-s)A}) F(X(s))\|^p ds) \\
\leq &cE(\int_0^T \sup_{0\leq t\leq S\wedge\tau} \|(e^{(t\vee s-s)A_\alpha} - e^{(t\vee s-s)A}) F(X(s))\|^p ds).
\end{aligned}$$

Since $A_\alpha$ is the Yosida approximation of $A$, we know by theorem 1.2 that $\|(e^{sA} - e^{sA_\alpha})h\|$ converges to 0 as $\alpha \to \infty$ for all $h \in H$ uniformly on bounded intervalls. By the linear growth condition we get for $0 \leq t \leq S \wedge \tau$

$$\begin{aligned}
&\|(e^{(t\vee s-s)A_\alpha} - e^{(t\vee s-s)A}) F(X(s))\|^p \\
\leq &K^p \|e^{(t\vee s-s)A_\alpha} - e^{(t\vee s-s)A}\|^p_{L(H)} (1 + \|X(s)\|)^p \\
\leq &K^p 2^p (1 + \bar{c})^p.
\end{aligned}$$

So the dominated convergence theorem gives us convergence to 0 of $\mathcal{B}$.

If we put $\mathcal{A}$ and $\mathcal{B}$ together we get for our error estimate

$$\begin{aligned}
&E(\sup_{0\leq t\leq S\wedge\tau} \|X_\alpha(t) - X(t)\|^p) \\
\leq &cK^p \int_0^S E(\sup_{0\leq t\leq s\wedge\tau} \|X_\alpha(t) - X(t)\|^p) ds + \mathcal{D}_\alpha(S)
\end{aligned}$$

for $S \in [0, T]$, where $\mathcal{D}_\alpha : [0, T] \to \mathbb{R}_+$ is an increasing function for all $\alpha > 0$ and $\mathcal{D}_\alpha(T)$ converges to 0 as $\alpha \to \infty$. With Gronwalls inequality (lemma 2.3) we get

$$\begin{aligned}
&E(\sup_{0\leq t\leq S\wedge\tau} \|X_\alpha(t) - X(t)\|^p) \\
\leq &cK^p \int_0^S e^{cK^p(S-s)} \mathcal{D}_\alpha(s) ds + \mathcal{D}_\alpha(S)
\end{aligned}$$

for $S \in [0, T]$, especially for $S = T$ we get

$$\begin{aligned}
&E(\sup_{0\leq t\leq T\wedge\tau} \|X_\alpha(t) - X(t)\|^p) \\
\leq &cK^p \int_0^T e^{cK^p(T-s)} \mathcal{D}_\alpha(s) ds + \mathcal{D}_\alpha(T) \\
\leq &cK^p \int_0^T e^{cK^pT} \mathcal{D}_\alpha(T) ds + \mathcal{D}_\alpha(T) \\
\leq &(cK^pTe^{cK^pT} + 1)\mathcal{D}_\alpha(T)
\end{aligned}$$

and the last term converges to 0 as $\alpha \to \infty$. □

Now we can show how the approximation of the mild solution $X$ relative to a general operator $A$ works. Therefore define $X_{\Delta,\alpha}$ as the Euler approximation of the mild solution relative to the Yosida approximation $A_\alpha$. Then we have:

**Corollary 2.12** *If $F = 0$ and $1 \leq p \leq 2$ then*

$$E(\sup_{0\leq t\leq T} \|X_{\Delta,\alpha}(t) - X(t)\|^p) \to 0 \text{ as } \alpha \to \infty, \rho(\Delta) \to 0.$$

**Corollary 2.13** *If $1 \leq p \leq 2$ then*

$$E(\sup_{0\leq t\leq T\wedge\tau} \|X_{\Delta,\alpha}(t) - X(t)\|^p) \to 0 \text{ as } \alpha \to \infty, \rho(\Delta) \to 0.$$

**Proof of Corollary 2.12**
Using the triangle inequality we get for $c = 2^p$

$$\begin{aligned}
&E(\sup_{0\leq t\leq T} \|X_{\Delta,\alpha}(t) - X(t)\|^p) \\
\leq &cE(\sup_{0\leq t\leq T} \|X_{\Delta,\alpha}(t) - X_\alpha(t)\|^p) \\
+ &cE(\sup_{0\leq t\leq T} \|X_\alpha(t) - X(t)\|^p).
\end{aligned}$$

For a fixed $\alpha$ and because of $1 \leq p \leq 2$ the first term of the sum converges to 0 as $\rho(\Delta) \to 0$ by theorem 2.1. For the second term we have by the Hölder inequality for $q > 2$ (The constant $c$ changes and depends on $p$, $q$.)

$$\begin{aligned}
&cE(\sup_{0\leq t\leq T} \|X_\alpha(t) - X(t)\|^p) \\
\leq &c(E(\sup_{0\leq t\leq T} \|X_\alpha(t) - X(t)\|^q))^{\frac{p}{q}}.
\end{aligned}$$

This term converges to 0 as $\alpha \to \infty$ by theorem 2.7. □

**Proof of Corollary 2.13**
Using the triangle inequality we get for $c = 2^p$

$$\begin{aligned}
&E(\sup_{0\leq t\leq T\wedge\tau} \|X_{\Delta,\alpha}(t) - X(t)\|^p) \\
\leq &cE(\sup_{0\leq t\leq T\wedge\tau} \|X_{\Delta,\alpha}(t) - X_\alpha(t)\|^p) \\
+ &cE(\sup_{0\leq t\leq T\wedge\tau} \|X_\alpha(t) - X(t)\|^p).
\end{aligned}$$

For a fixed $\alpha$ and because of $1 \leq p \leq 2$ the first term of the sum converges to 0 as $\rho(\Delta) \to 0$ by theorem 2.2. For the second term we have by the Hölder inequality for $q > 2$ (The constant $c$ changes and depends on $p$, $q$.)

$$\begin{aligned}
&cE(\sup_{0\leq t\leq T\wedge\tau} \|X_\alpha(t) - X(t)\|^p) \\
\leq &c(E(\sup_{0\leq t\leq T\wedge\tau} \|X_\alpha(t) - X(t)\|^q))^{\frac{p}{q}}.
\end{aligned}$$

This term converges to 0 as $\alpha \to \infty$ by theorem 2.8. $\square$

As we see we have to choose a sufficient big $\alpha$ and depending on this a sufficient small $\rho(\Delta)$ in order to minimize the error.

## 2.5 Conclusion and future prospects

We have constructed a stochastic process, which approximates the mild solution of a Hilbert space valued stochastic differential equation. We have proved the existence of such a process for nonlinear equations with additive noise in the sense that we have recursive calculability and uniform convergence of the error. The latter means, that we have

$$E(\sup_{0 \le t \le T} |Y(t) - X(t)|^p) \to 0$$

for a certain approximating process $Y$ and $1 \le p \le 2$. Therefore we have also convergence in probability and convergence P-a.s. for a subsequence.

For future research we could think of the following questions, which are left open here:

- What rate of convergence do we have?
- What are the optimal values for $\beta$ and $m$ in the proof of theorem 2.7?
- Is it possible to approximate our Euler approximation by a finite dimensional process?
- Can we weaken the conditions concerning the coefficients, e.g. multiplicative noise instead of additive noise?
- Can we weaken the conditions concerning the coefficients, e.g. monotonicity condition instead of Lipschitz continuity?

In the latter case we do not know up to now, if there exists a mild solution. Therefore we would have to develop a numerical scheme, which has two properties: First, that it converges and secondly, that the limit is a mild solution. In [Kry98, Chapter 1, p.1], [GK96] and [Gyö98] Gyöngy and Krylov established these two properties for equations on $\mathbb{R}^d$ by using the Euler scheme.

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Ich versichere, dass ich die vorliegende Arbeit selbständig verfasst und keine anderen als die angegebenen Quellen und Hilfsmittel benutzt habe.

Bielefeld, den 21. Februar 2007