DIPLOMARBEIT

A Transformation Rule for Measures via Differentiation of Measures

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Dezember 2008

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Introduction

Using methods of functional analysis, we present a transformation rule for signed measures.

Let $\nu \in M(\mathbb{E}, \mathbb{B}(\mathbb{E}))$, I be an interval, $\mathbb{E}$ be a Banach space and $\mathbb{H} \subset \mathbb{E}$ be a continuously embedded Hilbert space. For certain maps $F : I \times \mathbb{E} \to \mathbb{E}$, which are e.g. differentiable w.r.t. $\mathbb{H}$ and I and respect null sets, we gain a kind of Maruyama-Girsanov-Cameron-Martin or Ramer formula for different measures (cf. Theorem 5.4.1). We define the measure valued map

$$\begin{array}{rcl} f_F^\nu \colon \mathbb{R} & \to & M(\mathbb{E}, \mathbb{B}(\mathbb{E})) \\ s & \mapsto & \nu^s := F^\diamondsuit(s, \nu) := F^\diamondsuit(s, \cdot)^*(\nu), \end{array}$$

where $F^\diamondsuit(s, F(s, \cdot)) = \mathrm{id}_\mathbb{E}$, and obtain the following formula

$$\frac{d\nu^t}{d\nu^0} = \det F_2'(t,x) \exp\{\int_0^t {}_{\tilde{C}_b^1}\beta_\mathbb{H}^\nu(F_1'(s,x))(F(s,x))ds\}, \tag{0.1}$$

where ${}_{\tilde{C}_b^1}\beta_\mathbb{H}^\nu$ denotes the logarithmic gradient. This result is obtained using the differentiation of signed measures. Namely (cf. Theorem 5.3.1), we prove and use in the sequel that f_F^ν is (τ_{tv}-)differentiable at t in $\mathbb{H}$ with logarithmic derivative

$$\frac{d\nu^{t'(\tau_{tv})}}{d\nu^t}(x) = {}_{\tilde{C}_b^1}\beta_\mathbb{H}^\nu(F_1'(t,x))(F(t,x)) + \mathrm{tr}({F''}_{12}(t,x) \circ (F_2'(t,x))^{-1})$$

At the very end we notice that (0.1) is a generalization of the Maruyama-Girsanov-Cameron-Martin formula, if $F(t, \cdot) = id + th$, where $h \in \mathbb{H}$.

The main aim of the presented thesis is to illustrate and motivate the idea of the differentiation of a measure and to prove the theorems of [SvW95] rigorously, where we weaken some of the assumptions. We try to do this in such a detailed way, that it is suitable for everyone with a good knowledge in measure theory, probability theory, basic stochastic analysis and Malliavin calculus, where the latter two are only used for the examples. Beside that a profound functional analytic background is quite helpful. We try to keep the presentation as self-contained as possible.

Historic overview

Transformation rules for measures

A fundamental role plays the theorem of Girsanov and Cameron-Martin (cf. [CM49]). For an absolutely continuous function h, with $h(0) = 0$, we define $T : C([0,1])_0 \to C([0,1])_0, T(x) = x + h$. Let ν be the classical Wiener measure, then

$$\frac{d(\nu(id + h(\cdot))}{d\nu} \quad = \quad \exp\left(\int_0^1 \dot{h}(t)dW_t - \frac{1}{2}\int_0^1 (\dot{h}(s))^2 ds\right).$$

If B is a Brownian motion under ν, then (by Girsanov) $B_t - \int_0^1 h(s)ds$ is a Brownian motion under $\nu(id + h)$.

Many generalizations of this formula have been studied for the Gaussian measure, e.g. [Kue68, Gro60, Kus82, Ram74, Kus03]. [Ram74] is an essential result for the generalization of the transformation of abstract Gaussian measures. A condition, when the Ramer formula looks like the Girsanov formula is given by [ZZ92]. Until the paper of [Bel90] most transformation rules for measure on infinite dimensional Banach spaces $\mathbb{E}$ were stated for Gaussian measures. In [Bel90] an arbitrary Borel measure and the translation $T_t(x) := I(x) + tK(x),\ t \in [0,1],\ x \in \mathbb{E}$, where K is e.g. a contraction, are considered. For $\nu_t := \nu(T_t)$ this is (cf. [Bel90, P.20, (10)])

$$\frac{d\nu_t}{d\nu} = \exp\left(\int_0^1 \mathcal{L}[K \circ T_s^{-1}](T_s \circ T^{-1}(x))ds\right),$$

where $\mathcal{L}$ denotes the integration by parts operator.

For a detailed overview we refer to e.g. [UZ00]. The modern theory of Gaussian measures is presented in e.g. [Bog98].

Differentiation of measures

In 1966 the theory of differentiable measures on infinite dimensional spaces was started by Fomin. In 1971 this theory was extended in [ASF71]. Its key idea is similar to the concept of Gâuteaux-differentiation. The idea is to evaluate the measure for every set permitted and varying it in one direction. In 1993 in [SvW93] the differentiation of signed measures of finite total variation was stated in a more general context.

Structure and results

In Chapter 1 we repeat basic definitions and outline general assertions needed for the theory. After introducing the basic concepts of differentiation in Chapter 2, we generalize them in Chapter 3 and define the general derivative of a signed measure. In Chapter 4 we formulate some conditions, which insure that we can work with the theory presented so far. Chapter 5 is reserved for the key results, which include a transformation rule for signed measures and finally we give in Chapter 6 the Gaussian and Wiener measure as examples and derive the Maruyama-Girsanov-Cameron-Martin formula. For the readers convenience an index of the introduced notation is included.

After stating the general framework for the presented thesis at the beginning of Chapter 2 we introduce the concept of Fomin-differentiability. We deduce a few properties of the Fomin-derivative. These include that the Fomin-derivative of a signed measure ν is absolutely continuous w.r.t. to this signed measure. Thus we define the logarithmic derivative and motivate its name. Then the β-differentiability is introduced. Using its transformation rule we show that each Fomin-differentiable signed measure on $\mathbb{R}^n$ is absolutely continuous w.r.t. the Lebesgue measure. The linearity and continuity of the Fomin-derivative are presented as well. At the end we introduce the logarithmic gradient and find out that it is (ν-quasi) linear and continuous. Moreover, we give a definition for the logarithmic gradient along a vector field and develop a condition such that it exists.

Motivated by the formula of integrations by part we introduce the concept of C-differentiability in Chapter 3. After having seen different concepts of differentiation, we establish the general concept of differentiation of signed measures w.r.t. a Hausdorff topology, give three examples of differentiation in this general context and outline how the Fomin-, β- and C-derivative fit in the general picture. A summary of their connections, which include a kind of main theorem of calculus, is illustrated by a graphic. We remark that Sections 3.3 to 3.5 are independent of the other parts (excluding Chapter 1). Though these parts can be understood without reading the other parts, it is helpful to read the preceding parts carefully as a motivation and for a better understanding of the general concept.

In Chapter 4 we present reasonable examples and conditions for the so far introduced concepts. Thus we derive that Chapter 3 is applicable to an interesting set of functions. Furthermore we prepare the proof of the key results. We prove the existence of a norm-defining set, for which the key results will be demonstrated (Theorems 5.3.1 and 5.4.1). We give explicit and sufficient conditions for the existence of a local flow, which is used to define the differentiability along a vector field. For this end we adapt methods of the theory of evolutionary equations to our needs. A further preparation it to adjust the well known Lebesgue Theorem. At the end we prove a unique correspondence between C-differentiability and τ_C-differentiability for a special set $\tilde{C}_b^1$, which allows us to exploit the results of Section 3.5 for the C-differentiability. Generally speaking Chapter 4 is independent of the other parts.

In Chapter 5 we present the key results. The Key Proposition divulges a formula for the logarithmic gradient along a vector field in terms of the logarithmic derivative. These results contribute heavily to the proof of the two main results, namely the Main Theorem and the transformation rule for signed measures. The results of Chapter 5 heavily depend on the results of Section 3.5 and Chapter 4.

Finally in Chapter 6 we expound as an example the Gaussian and Wiener measure. In the case of a Gaussian measure we deduct a Ramer type formula and in the adapted Wiener case we rediscover the well known Maruyama-Girsanov-Cameron-Martin formula. For this end we give explicit conditions and use the theory about Carleman operators to deduce these assertions.

New aspects of the Thesis

We note that most of the results of Chapter 1 are essentially known. Chapter 2 is inspired by [ASF71] and most of the ideas of Chapter 3 can be found in [SvW93]. The calculation and postulation of Chapter 4 are not done in the mentioned papers. In Chapter 5 the assertions and part of their proofs are indicated in [SvW95]. For technical reasons we assume additional conditions. The postulated smooth property is removed. If the smoothness should be kept, stronger assumption would have to hold for the norm-defining set and the local flows. Though the main assertions of Chapter 6 can be found in [SvW95], there is given short shrift about how to obtain them.

In Chapter 2 we explicitly prove that the Fomin-derivative is a signed measure, which is of finite total variation (cf. Theorem 2.2.6). We point out that in general $\nu_h'^{F+}$ and ${\nu^+}_h'^{F}$ do not coincide (cf. Example 2.2.11). The definition of $\| \, \|_\beta$ and M_β (Definition 2.3.1) are not mentioned in [ASF71]. They are used to show the continuity and linearity part in the definition of the β-differentiability (Definition 2.3.3). We introduce the definition of being semi-β-differentiable to gain a deeper insight of the transformation rule (Proposition 2.3.8). Furthermore we altered the assumption to close a small gap in the prove of the transformation rule. Introducing the (new) concept of being uniformly Fomin-differentiable (cf. Definition 2.2.2) we close a little gap in the prove of [ASF71, Proposition 4.1.1] (cf. Proposition 2.5.1). Although by the results of Chapter 3 we obtain that it coincides with the concept of being Fomin-differentiable (cf. Remark 2.2.3), it is helpful to understand the connection with the β-differentiability and to keep Chapter 2 independent of Chapter 3. The definition of the logarithmic gradient (cf. [SvW95]) is altered. The other definitions and assertions of Section 2.6 are new and allow us to expose a condition, such that the logarithmic gradient exists. Namely, we prove that it exists for finitely based vector fields (cf. Lemma 2.6.8).

In Chapter 3 the motivation of the C-differentiability with the formula of integration by parts is not revealed in [SvW95]. We altered the definition of C-differentiability (cf. Definition 3.2.3 vs. [SvW95, P.105]). This allows us to change the assumptions for the key results in Chapter 5. We changed the definition of being τ-differentiable along a vector field (cf. Definition 3.3.7 vs. [SvW93, P.471]), because the original seems to be too harsh. Furthermore this allows us to formulate a condition for the existence of a local flow in Theorem 4.2.11. The connection of the different notations has not been visualized (cf. Section 3.4). In Section 3.5 we explicitly write down the considered topologies in the proofs. That τ_S- implies τ_C-differentiability (Lemma 3.5.1) is omitted in the mentioned papers.

The results of Sections 4.1 and 4.2 are new. The adaption of the Lebesgue dominated convergence theorem (cf. Section 4.3) has not been mentioned before. Furthermore only one direction of the assertion of Proposition 4.4.1 is proposed in [SvW95, Propostion 1]. We extend and prove it, because the extension is needed to apply the results of Chapter 3 for C-differentiability. Thus a gap in the proof of 5.2.1 is closed.

In Chapter 5 we weaken the conditions for the key results (cf. Remark 5.1.2 and [SvW95]). Furthermore we explicitly reveal the dependence of the parameter in [SvW95, Lemma 1]. Writing down the details of the proof of [SvW93, Proposition 8.2], we have to assume stronger conditions to gain the existence of Bochner integrals and that we may apply Fubini. The same is true for the proofs of Theorem 5.3.1 and Theorem 5.4.1. The last two assumptions of the latter are used to check that Theorem 3.5.4 is applicable, which was not spelled out in [SvW95]. The rest of the additional assumptions are used to explicitly write down the details of rewriting the formula.

In [SvW95] there is no clue given how to prove the results of Chapter 6. For the proofs the theory of Carleman operators is applied. The assumptions needed are explicitly stated. We like to point out that we develop a new condition that all eigenvalues of an integral operator are 0 (cf. Theorem 6.2.7).

Acknowledgment

I am grateful that Prof. M. Röckner presented to me within his lecture series this fascinating topic. I dearly thank him for the inspiring meetings and new insights. I appreciate that Dr. O. Kutovyi took the time to review this thesis and made me rethink some of the notations and formulations. Furthermore I am grateful for the fruitful discussions with my fellow students. Last but not least, I thank my family and friends for their constant encouragement and enduring support.

Chapter 1

Elemental concepts

Throughout the following chapters we will work with a lot of different derivatives of signed measures. Thus we start with a few familiar definitions to bring to mind the basic properties of taking derivative. We use this opportunity to establish some notations. For the proofs or further remarks we refer to the lecture series of Professor Röckner, [DS57, chapter II and III], [Wer05], [Bau01] and [Zei98].

In this paper $\mathbb{R}$ denotes the set of real numbers and $\mathbb{N} = \{1, 2, \dots\}$ is the set of natural numbers. Furthermore the exclamation mark (!) indicate that a fact is to be shown.

By a function we mean a map, which maps to $\mathbb{R}$, and by a numerical function we mean a map mapping to $\overline{\mathbb{R}} := [-\infty, \infty]$. We use [for the left limit of the interval to indicate, that the point is included in the interval and], if it is not. Viceversa this is meant for the right limit of the interval.

For any space E the map id_E indicates the identity mapping on E, i.e $\mathrm{id}\colon E \to E$, $x \mapsto x$.

D_1, respectively D_2 denotes the derivative in the first, respectively second component.

Moreover the plain symbol $\triangle$ points out that the triangle inequality is used. The symbol $\clubsuit$ indicates a sequence of signed measures.

Definition 1.0.1 (vector field).
Let $(\mathbb{E}, \| \ \|_{\mathbb{E}})$, $(\mathbb{H}, \| \ \|_{\mathbb{H}})$ be normed vector spaces and $U \subset \mathbb{E}$. A vector field is a mapping $h : U \to \mathbb{H}$. $vect(U, \mathbb{H})$ denotes the vector space of all vector fields from U to $\mathbb{H}$. If $\mathbb{E} = \mathbb{H}$, then $vect(U) := vect(U, \mathbb{E})$.

Definition 1.0.2 (Borel σ-algebra).
For every topological space (E, τ) we define the Borel σ-algebra $\mathbb{B}(E)$ as the σ-algebra generated by all the open sets in E. If $E \subset \mathbb{R}$, we assume that $\mathbb{B}(E)$ is complete w.r.t. Lebesgue measure, i.e. $\forall N' \subset \mathbb{R} : \exists N \in \mathbb{B}(E) : N' \subset N,\ \nu(N) = 0$ it follows that $N' \in \mathbb{B}(E)$.

Definition 1.0.3 (invariant w.r.t. translations).
Let $(\mathbb{E}, \| \, \|_{\mathbb{E}})$ *be a Banach space and* $(\mathbb{H}, \| \, \|_{\mathbb{H}})$ *a subspace of* $\mathbb{E}$*. A Borel-*σ*-algebra* $\mathbb{B}(\mathbb{E})$ *is said to be invariant w.r.t. translations of elements of* $\mathbb{H}$*, iff for all sets* $A \in \mathbb{B}(\mathbb{E}), h \in \mathbb{H} : A + th \in \mathbb{B}(\mathbb{E})$*.*

Definition 1.0.4 (signed measure).
Let $(\mathbb{E}, \| \, \|_{\mathbb{E}})$ *be a Banach space.* $\nu : \mathbb{E} \to \mathbb{R}$ *is a signed measure if it is the difference of two* σ*-additive measures on* $(\mathbb{E}, \mathbb{B}(\mathbb{E}))$*. Let* $\mathfrak{M}(\mathbb{E})$ *denote the vector space of all signed measures.*

Definition 1.0.5 (measure space).
By a measure space we mean a triple $(\mathbb{E}, \mathbb{B}(\mathbb{E}), \nu)$*, where* $\mathbb{E}$ *is a Banach space,* $\mathbb{B}(\mathbb{E})$ *its Borel* σ*-algebra and* $\nu \in \mathfrak{M}(\mathbb{E})$*.*

Definition 1.0.6 (ν-a.e., null-set).
Let $(\mathbb{E}, \| \, \|_{\mathbb{E}})$ *be a Banach space and* $\nu \in \mathfrak{M}(\mathbb{E})$ *a signed measure. A property is said to hold* ν*-a.e. if it holds* $|\nu|$*-a.e., i.e. there exists a null-set* $N \in \mathbb{B}(\mathbb{E})$ *:* $|\nu|(N) = 0$ *and the property holds for all* $x \in N^C := \mathbb{E} \backslash N$*.*

Theorem 1.0.7 (Hahn decomposition, ν^+, ν^-,$\mathbb{E}^+(\nu)$).
Let $(\mathbb{E}, \| \, \|_{\mathbb{E}})$ *be a Banach space.*
Every σ*-additive function* $\nu : \mathbb{B}(\mathbb{E}) \to \mathbb{R}$ *is a signed measure and there exists a measurable set* $\mathbb{E}^+ =: \mathbb{E}^+(\nu) \in \mathbb{B}(\mathbb{E})$*:* $\nu^+ := \mathbb{1}_{\mathbb{E}^+}\nu$ *and* $\nu^- := -(1 - \mathbb{1}_{\mathbb{E}^+})\nu$ *are nonnegative and* $\nu = \nu^+ - \nu^-$*.*

Definition 1.0.8 ($L^p(\nu)$,$L^\infty(\nu)$).
Let $(\mathbb{E}, \mathbb{B}(\mathbb{E}), \nu)$ *be a measure space.*
By $L^p(\nu)$ *we denote the set of all measurable function* $f : \mathbb{E} \to \mathbb{R}$*, for which*

$$\|f\|_{L^p(\nu)} := \|f\|_{L^p(|\nu|)} := \int_{\mathbb{E}} |f(x)|^p |\nu|(dx) < \infty$$

By $L^\infty(\nu)$ *we denote the set of all measurable function* $f : \mathbb{E} \to \mathbb{R}$*, that are* ν*-a.e. bounded.*

Definition 1.0.9 (total variation norm, $\| \, \|^\nu_{tv}$).
Let $(\mathbb{E}, \mathbb{B}(\mathbb{E}), \nu)$ *be a measure space and* $S \in \mathbb{B}(\mathbb{E})$*. Then the total variation norm of* S *w.r.t.* ν *is defined as*

$$\|S\|^\nu_{tv} := \sup \left\{ \sum_{S' \in \pi} |\nu(S')| \,\middle|\, \pi\text{-partition of } S \text{ into a finite number of disjoint borel subsets} \right\}$$

A set S *is said to be of bounded variation w.r.t.* ν*, if* $\|S\|^\nu_{tv} < \infty$*. The measure* ν *is of bounded variation (or finite total variation), if* $\|\nu\|_{tv} := \|\mathbb{E}\|^\nu_{tv} < \infty$*.*

Remark 1.0.10.
We notice using the Hahn decomposition that $\|S\|_{tv}^{\nu} = \|S \cap \mathbb{E}^{+}\|_{tv}^{\nu} + \|S \backslash \mathbb{E}^{+}\|_{tv}^{\nu}$.

Definition 1.0.11 ($M(\mathbb{E}, \mathbb{B}(\mathbb{E}))$).
Let $\mathbb{E}$ be a Banach space. By $M(\mathbb{E}, \mathbb{B}(\mathbb{E}))$ we define the vector space of all signed measures on $\mathbb{E}$ with finite total variation.

Definition 1.0.12 (absolutely continuous w.r.t.).
Let $\mathbb{E}$ be a Banach space and $\nu, \nu' \in \mathfrak{M}(\mathbb{E})$. ν' is said to be absolutely continuous w.r.t. ν (or said that ν dominates ν'), if the following property holds for all sets $A \in \mathbb{B}(\mathbb{E})$

$$\|A\|_{tv}^{\nu} = 0 \Rightarrow \|A\|_{tv}^{\nu'} = 0.$$

If ν' is absolutely continuous w.r.t. ν, we denote this by $\nu' \ll \nu$. If ν' is absolutely continuous w.r.t. to the Lebesgue measure, that is $\nu' \ll \lambda$, we say that it is absolute continuous.

Theorem 1.0.13 (Radon-Nikodym).
Let ν and μ be measures on a σ-algebra $\mathcal{A}$ of a set Ω. If μ is σ-finite, the following two assertions are equivalent:

1. *ν has a density with respect to ν, i.e. there exists a non-negative, $\mathcal{A}$-measurable, numerical function f on Ω such that*

$$\nu(A) = \int_A f d\mu \quad \forall A \in \mathcal{A}.$$

2. *ν is absolute continuous w.r.t. μ.*

Theorem 1.0.14 (Radon-Nikodym(for signed measure)).
Let ν and μ be signed measures on a Borel σ-algebra $\mathbb{B}(\mathbb{E})$ in a Banach space $\mathbb{E}$ with μ having finite total variation. The following two assertions are equivalent:

1. *ν is absolute continuous w.r.t. μ.*

2. *There exists a non-negative, $\mathbb{B}(\mathbb{E})$-measurable $f : \mathbb{E} \to \overline{\mathbb{R}}$ such that*

$$\nu(A) = \int_A f d\mu \quad \forall A \in \mathbb{B}(\mathbb{E}). \tag{1.1}$$

Definition 1.0.15 (Radon-Nikodym density).
The numerical function f in (1.1) is called Radon-Nikodym density.

Remark 1.0.16.
If in Theorem 1.0.14 $\|\nu\|_{tv} < \infty$, then $f \in L^1(\nu)$.

The following lemma tells us how to obtain new signed measures.

Lemma 1.0.17.
Let $(\mathbb{E}, \mathbb{B}(\mathbb{E}), \nu)$ *be a measure space. If* $f \in L^1(\nu)$ *then* $f\nu$, *i.e.* $(f\nu)(A) := \int_A f(x)\nu(dx)$, *is a signed measure.*

Proof.
Let $f = f^+ - f^-$ be the decomposition of f into the positive and negative part and let $\nu = \nu^+ - \nu^-$ be the Hahn decomposition of ν (c.f. Theorem 1.0.7). Then we define

$$(f\nu)^+ := f^+\nu^+ + f^-\nu^- \text{ and } (f\nu)^- := f^+\nu^- + f^-\nu^+,$$

which are positive finite measures (for $(f\nu)^-$ it follows similarly). Namely,

$$(f\nu)^+(\mathbb{E}) = \int_{\mathbb{E}} \underbrace{f^+(x)}_{\leq |f(x)|} \nu^+(dx) + \int_{\mathbb{E}} \underbrace{f^-(x)}_{\leq |f(x)|} \nu^-(dx)$$

$$\leq \int_{\mathbb{E}^+} |f(x)|\nu^+(dx) + \int_{\mathbb{E}^-} |f(x)|\nu^-(dx) = \int_{\mathbb{E}^+ \cup \mathbb{E}^-} |f(x)||\nu|(dx) \overset{f \in L^1(\nu)}{<} \infty$$

It remains to show the σ-additivity of $(f\nu)^+$. Since taking integral is additive, w.l.o.g. f and ν are positive: Choosing disjoint sets $A_n \in \mathbb{B}(\mathbb{E})$ we observe

$$(f\nu)(\bigcup_{n=1}^{\infty} A_n) = \int \mathbb{1}_{\bigcup_{n=1}^{\infty} A_n}(x) f(x)\nu(dx)$$

$$= \int \lim_{N\to\infty} \mathbb{1}_{\bigcup_{n=1}^{N} A_n}(x) f(x)\nu(dx) \overset{f \in L^1(\nu),\ \text{Lebesgue}}{=} \lim_{N\to\infty} \int \mathbb{1}_{\bigcup_{n=1}^{N} A_n}(x) f(x)\nu(dx)$$

$$= \sum_{n=1}^{\infty} \int \mathbb{1}_{A_n}(x) f(x)\nu(dx) = \sum_{n=1}^{\infty} (f\nu)(A_n)$$

□

We obtain an useful connection of $\| \ \|_{tv}$ and $\| \ \|_{L^1}$, namely

Lemma 1.0.18.
Let $(\mathbb{E}, \mathbb{B}(\mathbb{E}), \nu)$ *be a measure space. If* $f \in L^1(\nu)$, *then* $\|f\nu\|_{tv} = \|f\|_{L^1(\nu)}$.

Proof.
Since $f\nu$ is a signed measure (cf. Lemma 1.0.17), we can apply Theorem 1.0.7. We set $\mathbb{E}^+ := \mathbb{E}^+(\nu)$, $\mathbb{E}^- := \mathbb{E}\backslash\mathbb{E}^+$, $A^+ := \mathbb{E}^+(f\nu)$, $A^- := \mathbb{E}\backslash A^+$ (for notation cf. also Section 2.1), $f^+ = \max(f, 0)$, $f^- = -\min(f, 0)$, $\nu^+ = \nu(\cdot \cap \mathbb{E}^+)$ and $\nu^- = -\nu(\cdot \cap \mathbb{E}^-)$. Since ν is a signed measure we note $\forall A_i \in \mathbb{B}(\mathbb{E})$

$$\left|\int_{A_i} f d\nu\right| = |\int_{A_i \cap A^+} f d\nu + \int_{A_i \cap A^-} f d\nu| = \int_{A_i \cap A^+} f d\nu - \int_{A_i \cap A^-} f d\nu$$

$$= \int_{A_i \cap A^+ \cap \mathbb{E}^+} f d\nu + \int_{A_i \cap A^+ \cap \mathbb{E}^-} f d\nu - \int_{A_i \cap A^- \cap \mathbb{E}^+} f d\nu - \int_{A_i \cap A^- \cap \mathbb{E}^-} f d\nu$$

$$
\begin{aligned}
&= \int_{A_i\cap A^+\cap\mathbb{E}^+} f d\nu^+ - \int_{A_i\cap A^+\cap\mathbb{E}^-} f d\nu^- - \int_{A_i\cap A^-\cap\mathbb{E}^+} f d\nu^+ + \int_{A_i\cap A^-\cap\mathbb{E}^-} f d\nu^- \\
&= \int_{A_i\cap A^+} f^+ d\nu^+ + \int_{A_i\cap A^+} f^- d\nu^- + \int_{A_i\cap A^-} f^- d\nu^+ + \int_{A_i\cap A^-} f^+ d\nu^- \\
&= \int_{A_i} |f| d\nu^+ + \int_{A_i} |f| d\nu^- = \int_{A_i} |f| d|\nu| \qquad (1.2)
\end{aligned}
$$

$|f||\nu|$ is a measure on $\mathbb{E}$ and thus we have

$$
\|f\nu\|_{tv} = \sup\left\{ \sum_{i=1}^{n} \left| \int_{A_i} f d\nu \right| \middle| A_i \in \mathbb{B}(\mathbb{E}) \text{ disjoint subsets} \right\}
$$

$$
\overset{(1.2)}{=} \sup\left\{ \sum_{i=1}^{n} \int_{A_i} |f| d|\nu| \middle| A_i \in \mathbb{B}(\mathbb{E}) \text{ disjoint subsets} \right\} = \int_{\mathbb{E}} |f| d|\nu| = \|f\|_{L^1(\nu)}
$$

□

Definition 1.0.19 ($L(\mathbb{E}, F)$).
Let $(\mathbb{E}, \| \|_{\mathbb{E}}), (F, \| \|_F)$ be normed vector spaces, then

$$
L(\mathbb{E}, F) := \left\{ f : \mathbb{E} \to F \middle| f \text{ linear and continuous} \right\}. \qquad (1.3)
$$

If $F = \mathbb{R}$, we have $\mathbb{E}' := L(\mathbb{E}, \mathbb{R})$.

Definition 1.0.20 (Gâuteaux-differentiable).
Let $(\mathbb{E}, \| \|_{\mathbb{E}}), (F, \| \|_F)$ be normed vector spaces, $U \subset \mathbb{E}$ open and $f : U \mapsto F$ be a mapping. f is Gâuteaux-differentiable at $x_0 \in U$, if there exists a continuous linear operator $T_{x_0} \in L(\mathbb{E}, F)$ such that

$$
\| \|_F - \lim_{\varepsilon\to 0} \frac{f(x_0 + \varepsilon h) - f(x_0)}{\varepsilon} = T_{x_0} h \qquad \forall h \in \mathbb{E}
$$

is satisfied. If f is Gâuteaux-differentiable at every point $x \in U$, it is said to be Gâuteaux-differentiable. Its derivate is denoted by f'_h.

Theorem 1.0.21 (Mean Value Theorem).
Let $(\mathbb{E}, \| \|_{\mathbb{E}}), (F, \| \|_F)$ be normed vector spaces, $U \subset \mathbb{E}$ open and $f : U \mapsto F$ Gâuteaux-differentiable. Define the 'interval' $I = \{x_0 + \lambda h : 0 \leq \lambda \leq 1\} \subset U$. Then

$$
\|f(x_0) + f(x_0 + h)\|_F \leq \sup_{x\in I} \|f'(x)\| \ \ \|h\|_{\mathbb{E}}
$$

Definition 1.0.22 (Frèchet-differentiable).
Let $(\mathbb{E}, \| \|_{\mathbb{E}})$ and $(F, \| \|_F)$ be normed vector spaces. Let U be an open set in $\mathbb{E}$, and $f : U \to F$. Then f is Fréchet differentiable at the point $x_0 \in U$, if there exists a linear operator $f'_.(x_0) \in L(\mathbb{E}, F)$, such that:

$$
\lim_{\|h\|_{\mathbb{E}}\to 0} \frac{\|f(x_0 + h) - f(x_0) - f'_h(x_0)\|_F}{\|h\|_{\mathbb{E}}} = 0
$$

Remark 1.0.23.
If a mapping is Gâuteaux-differentiable and if $f' : U \to L(\mathbb{E}, F)$, $x_0 \mapsto f'_\cdot(x_0)$ *is continuous with respect to* x_0, *then it is Frèchet differentiable.*

Definition 1.0.24 ($C^\alpha(U, F)$)**.**
Let $(\mathbb{E}, \| \ \|_{\mathbb{E}})$, $(F, \| \ \|_F)$ *be normed vector spaces,* $U \subset \mathbb{E}$ *open and* $\alpha \in \mathbb{N}$. *Then*

$$\begin{aligned}
C^\alpha(U, F) &:= \left\{ f : U \to F \middle| f \text{ is } \alpha \text{ times continuously Gâuteaux-differentiable} \right\} \\
C^\alpha_b(U, F) &:= \left\{ \phi \in C^\alpha(U, F) \middle| \text{ each derivative is bounded in its operator norm} \right\} \\
C^\infty(U, F) &:= \bigcap_{\alpha \in \mathbb{N}} C^\alpha(U, F) \\
C^\infty_b(U, F) &:= \bigcap_{\alpha \in \mathbb{N}} C^\alpha_b(U, F)
\end{aligned}$$

If U, F *are obvious, we may omit them, i.e.* C^α, C^α_b, C^∞ *and* C^∞_b.

Definition 1.0.25.
Let $(F, \| \ \|_F)$ *be a normed vector space and* C *denote any set of functions from* F *to* $\mathbb{R}$. *Then we define*

$$\begin{aligned}
\mathfrak{F}C := \{ f \in C \mid f(u) = f(l_1(u), \ldots, l_n(u)), \\
n \in \mathbb{N}, f : \mathbb{R}^n \to \mathbb{R}, l_i \in F' \ \forall 1 \leq i \leq n, u \in F \},
\end{aligned}$$

The elements of $\mathfrak{F}C$ *are called finitely based functions.*

Definition 1.0.26 (ν_h)**.**
Let $(\mathbb{E}, \| \ \|_{\mathbb{E}})$ *be a Banach space. For any signed measure* $\nu \in M(\mathbb{E}, \mathbb{B}(\mathbb{E}))$ *and any* $h \in \mathbb{E}$ *we define* $\nu_h := \nu(\cdot + h)$.

Remark 1.0.27.
Of course, the above notation can occur for elements h *or* th' *of a normed vector space* $(\mathbb{E}, \| \ \|_{\mathbb{E}})$ *and a measure denoted* $\nu'^F_h \in M(\mathbb{E}, \mathbb{B}(\mathbb{E}))$ *(in 2.2 we define* ν'^F_h *as the Fomin-derivative), i.e.*

$$\nu'^F_{h\ th'}, \ \nu'^F_{h\ h\ th'} = \nu'^F_{h\ h+th'}, \ldots,$$

which means

$$\begin{aligned}
&\nu'^F_{h\ th'} \overset{\text{Def. 1.0.26}}{=} \nu'^F_h(\cdot + th'), \\
&\nu'^F_{h\ h\ th'} \overset{\text{Def. 1.0.26}}{=} \nu'^F_h(\cdot + h)_{th'} = \nu'^F_h(\cdot + h + th'), \ldots
\end{aligned}$$

Sometimes we may write for clarification $\big((\nu'^F_h)_h\big)_{th'}$.

Chapter 2

Differentiation of measures

In chapter 1 we have repeated known concepts to motivate the idea of Fomin-differentiability. In this chapter we introduce the idea of differentiation of a measure on a Banach space, namely the Fomin-differentiation. Later on we examine conditions under which two signed measures are absolutely continuous to each other. In Chapter 3 this concept is generalized.

In Section 2.1 we outline the general framework, in which we will work. Furthermore we introduce some notations which will be used in the proceeding chapters. Throughout the following sections and chapters a lot of new notations will be introduced. For the convenience of the reader these can be found in the index.

In Section 2.2 the (uniform) Fomin-derivative and its basic properties are introduced and illustrated within an example. By results of Chapter 3 the uniform Fomin- and Fomin-differentiability are the same. Furthermore we state a Mean-Value-Theorem. By showing that the derivative of a measure can be a signed measure, we see that defining the concept of differentiation without signed measures would be a wild goose chase. Thus the use of signed measures in the general framework is motivated.

In Section 2.3 we present the β-differentiation, which is similar to the idea of Fomin-differentiation. We give details about the connections of these concepts of differentiation. For the β-differentiation we obtain a transformation rule.

In Section 2.4 it is used to show that each Fomin-differentiable signed measure on $\mathbb{R}^n$ is absolutely continuous. Furthermore we show that a linear combination of directions, in which a signed measure is Fomin-differentiable, gives a new direction, in which it is differentiable and that being Fomin-differentiable implies that the $\| \|_{tv} - \lim$ exists on bounded sets (cf. Theorem 2.4.2).

In Section 2.5 we observe, using the transformation rule for β-differentiation, that taking (uniform) Formin-derivative is linear and continuous.

Finally in Section 2.6 we derive that the logarithmic gradient ${}_F\beta^{\nu}_{\mathbb{H}}$ is ν-quasi-linear, continuous and even ν-a.e linear and define the logarithmic gradient along a vector field. For finitely based vector fields we present a condition, such that the logarithmic gradient exists and is well defined, i.e. independent of the chosen orthonormal base (cf. Lemma 2.6.8).

We basically follow [ASF71] for Sections 2.2 to 2.5 and explicitly write down the omitted details, i.e. that the Fomin derivative of a finite signed measure is finite. The examples of Section 2.2 and the details of the dependence on null sets (e.g. Lemma 2.6.4 and Corollary **??**) in Section 2.6 are new. Using the basic idea of [ASF71, Theorem 3.2.1] we formulate in Theorem 2.4.2 a few more connections than there were stated in [ASF71, Theorem 3.2.1]. The definition of the logarithmic gradient for functions, i.e. Definition 2.6.6, was stated in [SvW95], but there were no comments made or references given about the existence.

2.1 General Framework

From now on, we will work on a fixed measure space $(\mathbb{E}, \mathbb{B}(\mathbb{E}), \nu)$.

We assume that $(\mathbb{E}, \| \ \|_{\mathbb{E}})$ is a separable Banach space.

Furthermore we fix a Hilbert subspace $\mathbb{H} \subset \mathbb{E}$, i.e. $\mathbb{H}$ is a vector subspace equipped with the structure of a Hilbert space such that the canonical embedding

$$\begin{array}{rcl} i\colon (\mathbb{H}, \| \ \|_{\mathbb{H}}) & \to & (\mathbb{E}, \| \ \|_{\mathbb{E}}) \\ h & \mapsto & h \end{array}$$

is continuous and $\| \ \|_{\mathbb{E}} \leq \| \ \|_{\mathbb{H}} \ \forall h \in \mathbb{H}$.

Let $\mathbb{B}(\mathbb{E})$ be the Borel-σ-algebra of $\mathbb{E}$. We suppose that $\mathbb{B}(\mathbb{E})$ is complete w.r.t. ν and that $\mathbb{B}(\mathbb{E})$ is invariant w.r.t. translations of elements of $\mathbb{H}$.

The signed measure $\nu \in M(\mathbb{E}, \mathbb{B}(\mathbb{E}))$ has in particular a finite total variation, i.e. $\|\nu\|_{tv} < \infty$. Furthermore ν^+ and ν^- denote the positive and negative part of the decomposition of ν, i.e. $\nu = \nu^+ - \nu^-$ and $\nu^- = |\nu^-|, \nu^+ = |\nu^+|$ (cf. Theorem 1.0.7). In the sequel the signed measure ν will be fixed.

For any signed measure $\tilde{\nu} \in M(\mathbb{E}, \mathbb{B}(\mathbb{E}))$ we denote by $\mathbb{E}^+_{\tilde{\nu}} =: \mathbb{E}^+(\tilde{\nu}) \in \mathbb{B}(\mathbb{E})$ the set of the Hahn-Banach decomposition, i.e. $\tilde{\nu}^+(\mathbb{E}^+_{\tilde{\nu}}) \geq 0$. Of course, this set is unique except for a $\tilde{\nu}$ null-set. Moreover we define $\mathbb{E}^-(\tilde{\nu}) := \mathbb{E}^-_{\tilde{\nu}} := \mathbb{E} \backslash \mathbb{E}^+_{\tilde{\nu}} \in \mathbb{B}(\mathbb{E})$ and obtain

$$\tilde{\nu}^+(A) = \tilde{\nu}(A \cap \mathbb{E}^+_{\tilde{\nu}}) \text{ and } \tilde{\nu}^-(A) = -\tilde{\nu}(A \backslash \mathbb{E}^+_{\tilde{\nu}}) \ \forall A \in \mathbb{B}(\mathbb{E}).$$

If $\tilde{\nu} = \nu$, we write $\mathbb{E}^+ := \mathbb{E}^+_{\nu}$ and $\mathbb{E}^- := \mathbb{E}^-_{\nu}$.

2.2 Fomin-Differentiable

We now treat the Fomin-derivative, which was first introduced by Fomin in 1966. Its key idea is similar to the concept of Gâuteaux-differentiation. The idea is to evaluate the measure for every set permitted and varying it in one direction. Furthermore we state a few properties of the Fomin-derivative and prove them in detail to get familiar to the concept. At the end of this section we give a basic example to see what the derivative looks like and to understand why it is natural to work with signed measures in this context.

In [ASF71, P.142] or [SvW95, P.105] we find:

Definition 2.2.1 (Fomin-differentiable).
The signed measure $\nu \in M(\mathbb{E}, \mathbb{B}(\mathbb{E}))$ is said to be Fomin-differentiable along a vector $h \in \mathbb{H} \subset \mathbb{E}$ if for every Borel set $A \in \mathbb{B}(\mathbb{E})$ the function

$$\mathbb{R} \ni t \mapsto \nu(A + th) \in \mathbb{R}$$

is differentiable at t=0. This expression is well-defined, because $\mathbb{B}(\mathbb{E})$ is invariant w.r.t. translations of $h \in \mathbb{H}$.
If ν is Fomin-differentiable, its derivative $\nu_h'^F$ is called the Fomin-derivative, i.e

$$\nu_h'^F(A) = \lim_{t \to 0} \frac{\nu(A + th) - \nu(A)}{t} \tag{2.1}$$

Definition 2.2.2 (uniformly Fomin-differentiable).
The signed measure ν is called uniformly Fomin-differentiable along a vector $h \in \mathbb{H}$, if the limit in (2.1) exists uniformly for all $A \in \mathbb{B}(\mathbb{E})$.

Remark 2.2.3.
By the results of chapter 3 we see that ν being Fomin-differentiable is equivalent with ν being uniformly Fomin-differentiable. In detail we use Example 3.3.9 and Theorem 3.5.5 in connection with Lemma 3.5.1 (and Remark 3.3.10) for $C = \{\mathbb{1}_B | B \in \mathbb{B}(\mathbb{E})\}$, Theorem 2.2.6 ($\gamma_t := \cdot + th,\ t \in I,\ h \in \mathbb{H}$, $\mathbb{B}(\mathbb{E})$ invariant w.r.t. translations of $\mathbb{H}$ (cf. Section 2.1)) and Example 3.3.9. We notice that the used results of 3 are independent of the results about the Fomin-differentiability.

Remark 2.2.4.

1. *The Fomin-differentiability relies on the differentiation of a function in $\mathbb{R}$. Later on we will see that taking Fomin-differentiability is continuous and linear (cf. Section 2.5).*

2. *The symbol $'^F$ indicates that the Fomin derivative is taken. Later we introduce different notions of taking derivative of a measure. Thus in order to avoid confusion, the letter "F" in the notation of the derivative should remind us that we consider the Fomin-derivative. Getting to know further concepts of differentiation we will extend this notation.*

   Naturally the signed measure has not to be denoted by ν. Assume $h, h' \in \mathbb{H}, \phi : \mathbb{H} \to \mathbb{H}$ and $t \in \mathbb{R}$. If the derivative $\nu_{th}'^F$ is again Fomin differentiable *along $\phi(h')$, we see that ${\nu'^F_{th}}'^F_{\phi(h')}$ is as well a suitable use of this notation. Other notations could be $\psi(\nu)'^F_h$ or ${\nu_{B_2}}'^F_h$, where $\nu_{B_2}, \psi(\nu) \in M(\mathbb{E}, \mathbb{B}(\mathbb{E}))$ and will be defined later (cf. Proposition 2.3.8 and Proposition 2.5.1).*

Example 2.2.5.
Suppose that a measure $\mu \in M(\mathbb{R}^d, \mathbb{B}(\mathbb{R}^d))$ is absolute continuous w.r.t. to Lebesgue measure λ. Thus, by Radon-Nikodym (cf. Theorem 1.0.14), it is of the form $\rho\lambda$, where $\rho \in L^1(\lambda)$ is a density. We suppose furthermore that ρ is positive, $\rho'_h \in L^1(\lambda)$ exists and that there exists $g \in L^1(\lambda)$ and $\delta > 0 : \ \forall t : |t| < \delta : |\frac{\rho(x+th)-\rho(x)}{t}| \leq g(x)$. We calculate the Fomin-derivative of $\rho\lambda$ in direction $h \in \mathbb{R}^d$ for each $A \in \mathbb{B}(\mathbb{R}^d)$:

$$
\begin{aligned}
(\rho\lambda)'^F_h(A) &= \lim_{t\to 0} \frac{(\rho\lambda)(A+th) - (\rho\lambda)(A)}{t} \\
&= \lim_{t\to 0} \frac{\int_{A+th} \rho(x)dx - \int_A \rho(x)dx}{t} \\
&= \lim_{t\to 0} \int_A \frac{\rho(x+th) - \rho(x)}{t} dx \\
&\overset{Lebesgue}{=} \int_A \rho'_h(x)dx = (\rho'_h\lambda)(A).
\end{aligned}
$$

Theorem 2.2.6 (Properties of the Fomin-derivative)**.**
Let ν'^F_h be the Fomin derivative of $\nu = \nu^+ - \nu^-$ in the direction $h \in \mathbb{H}$, that is

$$\nu'^F_h : A \mapsto \frac{d}{dt}\nu(A+th)\Big|_{t=0}$$

Then

1. *ν'^F_h defines a signed measure,*
2. *$\|\nu'^F_h\|_{tv} < \infty$,*
3. *the positive and negative part of ν are Fomin-differentiable,*
4. *the positive and negative part of ν'^F_h are absolutely continuous w.r.t. to the positive and negative part of ν, i.e.*
$$\nu^{+'F}_h \ll \nu^+ \text{ and } \nu^{-'F}_h \ll \nu^-.$$
5. *$\nu'^F_h \ll \nu$.*

Definition 2.2.7 (logarithmic derivative)**.**
By Radon-Nikodym (cf. Theorem 1.0.14) there exists a density, denoted by ${}_F\beta^\nu(h,\cdot)$, such that $\nu'^F_h = {}_F\beta^\nu(h,\cdot)\,\nu$. In our context we will call ${}_F\beta^\nu(h,\cdot)$ the logarithmic derivative of ν.

Remark 2.2.8.

1. *The name "logarithmic" derivative is motivated by the following: We consider a measure $\mu = \rho\lambda$ as in Example 2.2.5. Then the Fomin-derivative of μ in direction $h \in \mathbb{R}^d$ can be written as*

$$\mu_h'^F = \rho_h'\lambda = (\ln\rho)_h'\rho\lambda = (\ln\rho)_h'\mu$$

and thus the logarithmic derivative is of the form $(\ln\rho)_h'$, which motivates the name logarithmic derivative.

2. *The subindex "F" should remind us that we talk about the logarithmic derivative in the context of the Fomin-differentiability.*

3. *The notation of the logarithmic derivative will be extended later, when we get to know different notions of differentiability. Whenever we see a function named β with indices throughout this paper, it will denote a logarithmic derivative (or gradient). However, the letter F might be replaced by other symbols, who indicate different contexts of differentiation. For now this motivation should be sufficient.*

Proof of Theorem 2.2.6.

1. We define

$$\nu_n(A) := \frac{\nu(A + \frac{1}{n}h) - \nu(A)}{\frac{1}{n}} \quad \forall A \in \mathbb{B}(\mathbb{E}).$$

Then each ν_n is a signed measure on the Borel σ-algebra $\mathbb{B}(\mathbb{E})$ and by the Fomin-differentiability of ν its limit exists pointwise for all sets A. Thus we gain by a Nikodym corollary of Vitali-Hahn-Saks (cf. [DS57, Corollary III.7.4, p.160]) that the limit of the ν_n is σ-additive. The limit of the ν_n is by definition the derivative $\nu_h'^F$ of ν. Thus $\nu_h'^F$ is a signed measure.
The Hahn decomposition Theorem (cf. Theorem 1.0.7) tells us that for $\nu_h'^F$ there exists a set $\mathbb{E}^+(\nu_h'^F) \in \mathbb{B}(\mathbb{E})$:

$$\nu_h'^{F^+} := \mathbb{1}_{\mathbb{E}^+(\nu_h'^F)}\nu_h'^F$$

2. By the Fomin-differentiability we know that for any set $A \in \mathbb{B}(\mathbb{E})$ $\exists\, t_0 \in \mathbb{R}$:

$$\begin{aligned} |\nu_h'^F(A)| &\le 1 + \left|\frac{\nu(A + t_0 h) - \nu(A)}{|t_0|}\right| \\ &\le 1 + \frac{2\,\|\nu\|_{tv}}{|t_0|} =: N(A) < \infty, \end{aligned} \tag{2.2}$$

where we used that $\|\nu\|_{tv}$ is finite. Then the Theorem of Nikodýn ([DS57, Theorem IV.9.8, p.309f]) states that it is even uniformly bounded, and hence we obtain that $\exists N < \infty$:

$$|\nu_h'^F(A)| < N \;\forall A \in \mathbb{B}(\mathbb{E}).$$

Thus we conclude

$$\|\nu_h'^F\|_{tv} = \left|\nu_h'^F\left(\mathbb{E}^+(\nu_h'^F)\right)\right| + \left|\nu_h'^F\left(\mathbb{E}^-(\nu_h'^F)\right)\right| \leq 2N.$$

3. We conduct the proof following [ASF71, Theorem 2.6.1]. We know that the function $\mathbb{R} \ni t \mapsto \nu(\mathbb{E}^+ + th) \in \mathbb{R}$ is differentiable at 0 and that it has a local maximum at 0, because $\forall A \in \mathbb{B}(\mathbb{E})$:

$$\nu(A) \leq \nu^+(A) = \nu(A \cap \mathbb{E}^+) \leq \nu(\mathbb{E}^+).$$

Using this fact we will prove the claim:

$$\begin{aligned}
& \frac{\nu((A+th)\cap\mathbb{E}^+) - \nu(A\cap\mathbb{E}^+)}{t} \\
\stackrel{!}{=} \; & \frac{\nu((A+th)\cap(\mathbb{E}^+ + th)) - \nu(A\cap\mathbb{E}^+)}{t} \\
+ \; & \frac{\nu((A+th)\cap(\mathbb{E}^+\backslash(\mathbb{E}^+ + th))) - \nu((A+th)\cap((\mathbb{E}^+ + th)\backslash(\mathbb{E}^+)))}{t} \\
= \; & \frac{\nu((A\cap\mathbb{E}^+) + th) - \nu(A\cap\mathbb{E}^+)}{t} \\
+ \; & \frac{\nu((A+th)\cap(\mathbb{E}^+\backslash(\mathbb{E}^+ + th))) - \nu((A+th)\cap((\mathbb{E}^+ + th)\backslash(\mathbb{E}^+)))}{t}
\end{aligned} \tag{2.3}$$

where we applied

$$\begin{aligned}
\mathbb{E}^+ &= ((\mathbb{E}^+ + th)\cup\mathbb{E}^+)\cap((\mathbb{E}^+ + th)^C\cup\mathbb{E}^+) \\
&= ((\mathbb{E}^+ + th)\cup(\mathbb{E}^+\cap(\mathbb{E}^+ + th)^C))\cap((\mathbb{E}^+ + th)\cap\mathbb{E}^{+C})^C \\
&= ((\mathbb{E}^+ + th)\,\dot\cup\,(\mathbb{E}^+\backslash(\mathbb{E}^+ + th)))\backslash((\mathbb{E}^+ + th)\backslash\mathbb{E}^+).
\end{aligned}$$

We notice that $(A+th)\cap(\mathbb{E}^+\backslash(\mathbb{E}^+ + th)) \subset \mathbb{E}^+$ and that $(A+th)\cap((\mathbb{E}^+ + th)\backslash(\mathbb{E}^+)) \subset \mathbb{E}^-$. Thus, since ν^+ and ν^- are measures, we would possibly only enlarge the fraction by considering it without the $A + th$ intersection. Thus we replace the fraction in (2.3) by

$$\frac{\nu(\mathbb{E}^+\backslash(\mathbb{E}^+ + th))}{t} - \frac{\nu((\mathbb{E}^+ + th)\backslash(\mathbb{E}^+))}{t}$$

We prove that the limit of both summands is zero, thus the estimate without the $A+th$ intersection was not to harsh and the limit of the original fraction is 0 as well. We justify our estimate by proving the following claim:

$$\lim_{t\to 0}\frac{\nu((\mathbb{E}^+ + th)\backslash\mathbb{E}^+)}{t} = 0, \; \lim_{t\to 0}\frac{\nu(\mathbb{E}^+\backslash(\mathbb{E}^+ + th))}{t} = 0 \tag{2.4}$$

Using the above fact about the maximum, i.e. $\nu_h^{\prime F}(\mathbb{E}^+) = 0$, we obtain

$$\begin{aligned} 0 &= -\nu_h^{\prime F}(\mathbb{E}^+) = -\lim_{t\to 0} \frac{\nu(\mathbb{E}^+ + th) - \nu(\mathbb{E}^+)}{t} \\ &= -\lim_{t\to 0} \frac{\nu((\mathbb{E}^+ + th)\backslash \mathbb{E}^+)}{t} + \lim_{t\to 0} \frac{\nu(\mathbb{E}^+ \backslash (\mathbb{E}^+ + th))}{t} \\ &= \lim_{t\to 0} \frac{\nu^-(\mathbb{E}^+ + th)}{t} + \lim_{t\to 0} \frac{\nu^+(\mathbb{E}^+ \backslash (\mathbb{E}^+ + th))}{t} \end{aligned}$$

Since each of the summands is positive, we gain that each summand converges to zero and thus the claim is proved. Therefore, envisioning that ν is Fomin differentiable for $A \cap \mathbb{E}^+$, we get

$$\nu^{+\prime F}_{\ h}(A) = \lim_{t\to 0} \frac{\nu^+(A + th) - \nu^+(A)}{t} \overset{(2.3),(2.4)}{=} \nu_h^{\prime F}(A \cap \mathbb{E}^+) \quad (2.5)$$

By similar arguments (0 is a local minimum for $\mathbb{R} \ni t \mapsto \nu(\mathbb{E}^- + th) \in \mathbb{R}$) we have

$$\nu^{-\prime F}_{\ h}(A) = \lim_{t\to 0} \frac{\nu^-(A + th) - \nu^-(A)}{t} = \nu_h^{\prime F}(A \backslash \mathbb{E}^+) \quad (2.6)$$

By applying the first part of this theorem to ν^+ and ν^- we conclude that $\nu^{+\prime F}_{\ h}$ and $\nu^{-\prime F}_{\ h}$ are signed σ-additive measure. An alternative proof is to use Lemma 1.0.17 for $\mathbb{1}_{\mathbb{E}^+}$ and $\mathbb{1}_{\mathbb{E}^-}$.

4. It is enough to show it for ν^+(ν^- follows analogously). Let $A \in \mathbb{B}(\mathbb{E})$ such that $\|A\|_{tv}^{\nu^+} = 0$. Using the idea of the proof of [SvW93, Proposition 3.1, p.461] we obtain

$$0 = \nu^+(A) \leq \nu^+(A + th) \geq 0.$$

Moreover $t = 0$ is a local minimum and the Fomin derivative of ν^+ at A is 0. Thus $\nu^{+\prime F}_{\ h}(A) = 0$. Repeating this for $A \cap \mathbb{E}^+ (\nu^{+\prime F}_{\ h})$ and $A \cap \mathbb{E}^- (\nu^{+\prime F}_{\ h})$ we have $\|A\|_{tv}^{\nu^{+\prime F}_{\ h}} = 0$. Hence (cf. Definition 1.0.12), $\nu^{+\prime F}_{\ h} \ll \nu^+$.

5. By the last two assertions the claim follows:

$$\nu_h^{\prime F} = \nu_h^{\prime F}(\cdot \cap \mathbb{E}^+) + \nu_h^{\prime F}(\cdot \backslash \mathbb{E}^+) \overset{(2.5),(2.6)}{=} \nu^{+\prime F}_{\ h} + \nu^{-\prime F}_{\ h} \ll \nu^+ + \nu^- = \nu$$

□

Remark 2.2.9.

1. *The key point of the third statement of the Theorem is that the function* $t \mapsto \nu(\mathbb{E}^+ + th)$ *has a local maximum at* $t = 0$. *The key idea of the fourth statement of the Theorem is to use that for all null-sets* A *the function* $t \mapsto \nu^+(A + th)$ *has a local minimum.*

2. We have proved that ${\nu^+}_h'^F(\cdot) = \nu_h'^F(\cdot \cap \mathbb{E}^+)$.

In Example 2.2.10 we show that in general ${\nu^+}_h'^F$ and ${\nu_h'^F}^+$ do not coincide and in Example 2.2.11 that the derivative of a measure is not in general a measure.

Example 2.2.10.
Define for $a \leq b \in \mathbb{R}$

$$\nu([a,b]) := \int_a^b x^3 \mathbb{1}_{]-10,10[}(x)dx.$$

Then ν *can be extended uniquely to* $\mathbb{B}(\mathbb{R})$. *This extension is a signed measure, which is Fomin-differentiable along each* $h \in \mathbb{R}$. *We will see that its derivative is absolutely continuous.*

Proof.
Let $a, b, c \in \mathbb{R}$. We define $a \vee b := min(a,b), a \wedge b := max(a,b)$.

$$\nu^+([a,b]) := \int_{a\wedge 0}^{b\wedge 0} x^3 \mathbb{1}_{]-10,10[}(x)dx \text{ and } \nu^-([a,b]) := -\int_{a\vee 0}^{b\vee 0} x^3 \mathbb{1}_{]-10,10[}(x)dx$$

This definition is independent of the representation of the interval, i.e.

$$\nu^+([a,b]\dot\cup[b,c]) := \int_{a\wedge 0}^{b\wedge 0} x^3 \mathbb{1}_{]-10,10[}(x)dx + \int_{b\wedge 0}^{c\wedge 0} x^3 \mathbb{1}_{]-10,10[}(x)dx$$
$$= \int_{a\wedge 0}^{c\wedge 0} x^3 \mathbb{1}_{]-10,10[}(x)dx = \nu^+([a,c])$$

ν^+ is a measure on $\mathfrak{R} = \left\{ \bigcup_{i=1}^n [a_i, b_i[\middle| a_i, b_i \in \mathbb{R} \right\}$, because
$\nu^+(\emptyset) = 0$, $\nu^+([a,b[) = \left[\frac{1}{4}x^4\right]_{a\wedge 0}^{b\wedge 0} \geq 0$ and by definition

$$\nu^+\left(\dot{\bigcup_n}[a_n, b_n[\right) \stackrel{\text{by def.}}{=} \sum_n \int_{a_n\wedge 0}^{b_n\wedge 0} x^3 \mathbb{1}_{]-10,10[}(x)dx \stackrel{\text{by def}}{=} \sum_n \nu^+([a_n, b_n[).$$

Since ν^+ is σ-finite on $\mathbb{R}$ we gain by Carathéodory (cp. [Röc05a, Theorem 3.3, p.21] or [Bau01, Theorem 5.3]) applied to $\mathfrak{R}$ a unique extension $\overline{\nu^+}$of the measure ν^+on $\sigma(\mathbb{R})$. The same is true for $\nu^-$and thus we obtain $\overline{\nu^-}$ as the unique extension of ν^-.
We know

$$\nu_h'^F([-1,1]) = \frac{d}{dt}\int_{-1+th}^{1+th} x^3 \mathbb{1}_{]-10,10[}(x)dx\Big|_{t=0} = 2h \tag{2.7}$$

$$\nu([-1,1]) = \int_{-1}^{1} x^3 \mathbb{1}_{]-10,10[}(x)dx = \left[\frac{1}{4}x^4\right]_{-1}^{1} = 0$$

Thus we see that we have found an example, where $\nu([-1,1]) = 0$ and for $h \in \mathbb{R}\backslash\{0\}$ $\nu_h'^F([-1,1]) \neq 0$. Here we see the importance of using the $\| \|_{tv}$ in the definition of absolute continuity of a measure w.r.t. another measure (cf. Definition 1.0.12). We will prove that $\overline{\nu}$ is Fomin-differentiable.

Let $P \in \sigma(\mathcal{R})$, then $\overline{\nu}(P) := \overline{\nu^+}(P) - \overline{\nu^-}(P)$ is unique by Carathéodory. We calculate the derivative of the uniquely extended measure $\overline{\nu}$ along $h \in \mathbb{R}$. To this end we have to show that the following limit exists.

$$\begin{aligned}
\exists\ \overline{\nu}_h'^F(P) &\overset{!}{=} \lim_{t\to 0} \frac{\overline{\nu}(P+th) - \overline{\nu}(P)}{t} \\
= \lim_{t\to 0} &\frac{\overline{\nu}^+(P+th) - \overline{\nu}^-(P+th) - (\overline{\nu}^+(P) - \overline{\nu}^-(P))}{t} \\
= \lim_{t\to 0} &\frac{\overline{\nu}^+(P+th) - \overline{\nu}^+(P) - (\overline{\nu}^-(P+th) - \overline{\nu}^-(P))}{t}
\end{aligned}$$

We calculate the value of the associated outer measure of $\overline{\nu^+}$

$$\overline{\nu^+}(P+th) = \inf\left\{\sum_{n=1}^{\infty} \nu^+(A_n) \middle| A_n \in \mathcal{R} \text{ and } P+th \subset \dot{\bigcup_n} A_n\right\}$$

We may assume w.l.o.g. $A_n = [a_n, b_n[$, where $a_n, b_n \in \mathbb{R}$, and define $\tilde{a_n} := a_n + th, \tilde{b_n} := b_n + th$.

$$\begin{aligned}
&= \inf\left\{\sum_{n=1}^{\infty} \nu^+([\tilde{a_n}, \tilde{b_n}[) \middle| [\tilde{a_n}, \tilde{b_n}[\in \mathcal{R} \text{ and } P+th \subset \dot{\bigcup_n} [\tilde{a_n}, \tilde{b_n}[\right\} \\
&\overset{\text{def}}{=} \inf\left\{\sum_{n=1}^{\infty} \int_{10\vee(b_n+th)\wedge 0}^{10\vee(a_n+th)\wedge 0} x^3 dx \middle| [a_n, b_n[\in \mathcal{R} \text{ and } P+th \subset \dot{\bigcup_n} [a_n+th, b_n+th[\right\}
\end{aligned}$$

Using the definition of Lebesgue measure (cp. [Röc05a] or [Bau01]) we obtain that the latter expression is equal to

$$\int \mathbb{1}_P(x)(x+th)^3 \mathbb{1}_{]-th,10-th[}(x)dx$$

Analogously we gain $\overline{\nu^-}(P+th) = -\int \mathbb{1}_P(x)(x+th)^3 \mathbb{1}_{]-10-th,-th[}(x)dx$. If we plug this in, we receive

$$\overline{\nu}_h'^F(P) = \lim_{t\to 0} \frac{\overline{\nu}(P+th) - \overline{\nu}(P)}{t}$$
$$= \lim_{t\to 0} \frac{\overline{\nu^+}(P+th) - \overline{\nu^-}(P+th) - (\overline{\nu^+}(P) - \overline{\nu^-}(P))}{t}$$
$$= \lim_{t\to 0} \frac{\int \mathbb{1}_P(x)(x+th)^3 \mathbb{1}_{]-10-th,10-th[}(x)dx - \int \mathbb{1}_P(x)x^3\mathbb{1}_{]-10,10[}(x)dx}{t}$$
$$\overset{\substack{\text{dom. conv.}\\ \text{Lebesgue}}}{=} \int \mathbb{1}_P(x)3x^2h\mathbb{1}_{]-10,10[}(x)dx$$

Therefore $\overline{\nu}$ is Fomin-differentiable. Using Remark 2.2.9 we see

$$\overline{\nu^+}_h'^F(P) = \overline{\nu}_h'^F(P\cap\mathbb{R}^+) = \int_0^{10} \mathbb{1}_P(x)3x^2h\ dx$$
$$\overline{\nu^-}_h'^F(P) = \overline{\nu}_h'^F(P\cap\mathbb{R}^-) = \int_{-10}^{0} \mathbb{1}_P(x)3x^2h\ dx$$

□

Example 2.2.11.
If we define $\nu([a,b]) = \int_a^b x^2\mathbb{1}_{]-10,10[}(x)dx$ for $a, b \in \mathbb{R}$, we obtain by the same construction via Carathéodory a measure on $\mathbb{B}(\mathbb{R})$.
For this measure $\nu^+ = \nu$ and

$$\nu_1'^F([a,b]) = \int_b^a 2x\mathbb{1}_{]-10,10[}(x)dx.$$

Here we see that the derivative of the measure ν along 1 is a signed measure. Moreover

$$\nu_1'^{F^+} = \nu_1'^F(\cdot\cap\mathbb{R}^+) \neq \nu^+{}_1'^F = \nu_1'^F.$$

By [ASF71, Theorem 1.3.2] we have

Theorem 2.2.12 (Mean-Value-Theorem)**.**
If ν is Fomin-differentiable in the direction $h \in \mathbb{H}$, then we obtain $t \in \mathbb{R}$:

$$\|\nu_{th} - \nu - t\nu_h'^F\|_{tv} \leq |t| \sup_{0<\tau<t} \|\nu_h'^F{}_{\tau h} - \nu_h'^F\|_{tv}$$
$$= |t| \sup_{0<\tau<t} \|\nu_h'^F(\cdot + \tau h) - \nu_h'^F(\cdot)\|_{tv} \tag{2.8}$$

2.3 β-differentiable

In this section we introduce a new type of differentiability, which is analogue to the Fomin-differentiability and state a condition such that they are equal. For this β-differentiability we obtain a transformation rule, which we will use later to prove the linearity and continuity of taking Fomin-derivative. In [ASF71] not all of the details were shown, at heart only the first assertion of the transformation rule (Proposition 2.3.8) was demonstrated. First of all we define

Definition 2.3.1 ($\| \, \|_\beta$, M_β)**.**
Let $\beta \subset \mathbb{B}(\mathbb{E})$. Let $\mu : \beta \to \mathbb{R}$ be such that $\mu(\emptyset) = 0$ if $\emptyset \in \beta$ and for any collection of disjoint sets $A_i \in \beta$ with $\bigcup_{i\in\mathbb{N}} A_i \in \beta$: $\mu(\bigcup_{i\in\mathbb{N}} A_i) = \sum_{i\in\mathbb{N}} \mu(A_i)$. We define

$$\begin{aligned} \|\mu\|_\beta &:= \sup \left\{ \sum_{i\in\mathbb{N}} \mu(A_i) \middle| A_i \in \beta \text{ disjoint} \right\} \text{ and} \\ M_\beta &:= \left\{ \mu : \beta \to \mathbb{R} \middle| \ \|\mu\|_\beta < \infty \right\}. \end{aligned}$$

Remark 2.3.2.
$\| \, \|_\beta$ is a norm on M_β and the inclusion mapping $i \colon M(\mathbb{E}, \mathbb{B}(\mathbb{E})) \to M_\beta$, $\nu \mapsto \nu_{|\beta}$ is continuous.

Following the idea of [ASF71, p.143], we define

Definition 2.3.3 (β-differentiable w.r.t. subspace)**.**
Let $\beta \subset \mathbb{B}(\mathbb{E})$ be a class of measurable sets. ν is β-differentiable w.r.t. $\mathbb{H}$ iff

1. *$\forall h \in \mathbb{H}$:*

$$\exists \lim_{t\to 0} \frac{\nu(A + th) - \nu(A)}{t} \text{ uniformely for all } A \in \beta \tag{2.9}$$

If the limit exists, it is denoted by $\nu_h^{'\beta}$ and called the β-derivative of ν (w.r.t. $\mathbb{H}$).

2. *and taking β-derivative is linear and continuous, i.e.*

$$\begin{aligned} \nu'^{\beta} \colon (\mathbb{H}, \| \, \|_{\mathbb{H}}) &\to (M_\beta, \ \| \, \|_\beta) \\ h &\mapsto \nu_h^{'\beta} \end{aligned}$$

is linear and continuous.

Definition 2.3.4 (semi-β-differentiable)**.**
If the first condition in Definition 2.3.3 is fulfilled, we call ν semi-β-differentiable.

Remark 2.3.5.

1. *In [ASF71] the definition was restricted to a class of subsets of $\mathbb{H}$, we changed this definition to measurable subsets of $\mathbb{E}$. This will enable us to draw a few connection to the concept of Fomin-differentiability.*

2. *Note that again $'$ indicates that we take a derivative and β that it is the β-derivative. Of course, we can consider different classes of subsets. In order to recognize this concept more easily, we will denote all of these classes by β with an index, i.e. $\beta_1, \beta_2, \beta_b, \beta_A, \beta_{\mathbb{E}}, \beta_f, \ldots$.*

3. *We note that if ν is Fomin-differentiable along $h \in \mathbb{H}$, then the above limit coincides pointwisely with the Fomin-derivative along $h \in \mathbb{H}$. That is*

$$\nu_h'^{\beta}(A) = \nu_h'^{F}(A) \ \forall A \in \beta \subset \mathbb{B}(\mathbb{E}).$$

4. *Comparing the definitions of Fomin- and β-differentiability we notice that: ν is Fomin-differentiable along all $h \in \mathbb{H}$, iff $\forall A \in \mathbb{B}(\mathbb{E})$ ν is β_A(-semi)-differentiable w.r.t. $\mathbb{H}$, where $\beta_A := \{A\}$.*
   If $\beta_f \subset \mathbb{B}(\mathbb{E})$ finite, i.e. $|\beta_f| < \infty$, then Fomin-differentiability along all $h \in \mathbb{H}$ implies β_f-differentiability.

5. *If we chose $\beta_{\mathbb{E}} = \mathbb{B}(\mathbb{E})$, then a signed measure $\mu \in M(\mathbb{E}, \mathbb{B}(\mathbb{E}))$ is semi-$\beta_{\mathbb{E}}$-differentiable w.r.t. $\mathbb{H}$ iff it is (uniformly) Fomin-differentiable along each $h \in \mathbb{H}$.*

Definition 2.3.6 (bounded differentiable)**.**
Let $\beta_b \subset \mathbb{B}(\mathbb{E})$ be the class of all bounded subsets of $\mathbb{H}$. A signed measure ν is bounded differentiable w.r.t. $\mathbb{H}$, if it is β_b-differentiable w.r.t. $\mathbb{H}$.

For the transformation rule we need to introduce the following definition (cf. e.g. [Wer05, Definition VIII.1.3, p.392]):

Definition 2.3.7 (linear topological subspace)**.**
A linear topological subspace is a linear subspace of a topological space, where additivity and multiplication with scalars are continuous .

Similar to [ASF71, Proposition 2.4.1] we have

Proposition 2.3.8 (transformation rule for β-differentiable)**.**
For i=1,2 let $\mathbb{E}_i$ be a linear space, $\mathbb{H}_i$ a linear topological subspace of $\mathbb{E}_i$, and $\sigma(\mathbb{E}_i)$ a σ-algebra of subsets of $\mathbb{E}_i$ that is invariant under translations by elements of $\mathbb{H}_i$. In addition, let $\psi : (\mathbb{E}_1, \sigma(\mathbb{E}_1)) \to (\mathbb{E}_2, \sigma(E_2))$ be a measurable linear mapping such that

1. *$\mathbb{H}_2 = \psi(\mathbb{H}_1)$*

2. *$\psi\ |_{\mathbb{H}_1}$ is an open map, i.e. the image of an open set under ψ is an open set.*

Moreover we assume that β_1 is a class of subsets of $\mathbb{H}_1$, which is invariant w.r.t. translations by elements of $\mathbb{H}_1$ and $\beta_2 = \psi(\beta_1)$.

1. *If a measure $\mu \in M(\mathbb{E}_1, \sigma(\mathbb{E}_1))$ is semi-β_1-differentiable w.r.t. a subspace $\mathbb{H}_1$, then the measure $\psi(\mu)$ is semi-β_2-differentiable w.r.t. $\mathbb{H}_2$, and*

$$\psi\left(\mu_h'^{\beta_1}\right) = \psi(\mu)_{\psi(h)}'^{\beta_2}$$

for all $h \in \mathbb{H}_1$

2. *If μ is even β_1-differentiable, then $\psi(\mu)$ is β_2-differentiable.*

Proof.
For a better readability we use the already introduced notation:

$$\mu_{th}(\cdot) := \mu(\cdot + th) \tag{2.10}$$

1. In order to prove the first assertion, we show that the following limit exists for all $h \in \mathbb{H}$ uniformly in β_2. That is choosing $A \in \beta_2$ arbitrary

$$\lim_{t_n \to 0} \frac{\psi(\mu)_{t_n\psi(h)}(A) - \psi(\mu)(A)}{t_n} - \psi(\mu)_{\psi(h)}'^{\beta_2}(A) \stackrel{!}{=} 0 \tag{2.11}$$

We calculate the parts of the fraction separately and then plug in the semi-β_1-differentiability of μ. By $\beta_2 = \psi(\beta_1)$ we know that $\forall A \in \beta_2\ \exists A_1 \in \beta_1 : A = \psi(A_1)$. Thus $\forall t \in \mathbb{R},\ h \in \mathbb{H}_1$

$$A + t\psi(h) = \psi(A_1) + t\psi(h) \stackrel{\text{linearity}}{=} \psi(A_1 + th) \in \beta_2,$$

because β_1 is invariant w.r.t. translations of $\mathbb{H}_1$. This implies that

$$\begin{aligned} &\emptyset \neq \psi^{-1}(A + t\psi(h)) = \{y \mid \psi(y) \in A + t\psi(h)\} \\ \underset{\psi^{-1}A\neq\emptyset}{\stackrel{\text{linearity}}{=}} &\{y \mid \psi(y - th) \in A\} = \{x + th \mid \psi(x) \in A\} = \psi^{-1}(A) + th \neq \emptyset \end{aligned}$$

For all measures μ and $t \in \mathbb{R}$ we have

$$\begin{aligned} &(\psi(\mu_{th}))(A) \stackrel{\text{def.}}{=} \mu_{th}(\psi^{-1}(A)) \stackrel{\text{def. } \mu_{th}}{=} \mu(\psi^{-1}(A) + th) \\ \stackrel{\text{linearity}}{=} &\mu(\psi^{-1}(A + t\psi(h))) \stackrel{\text{def}}{=} \psi(\mu)(A + t\psi(h)) = \psi(\mu)_{t\psi(h)}(A) \end{aligned} \tag{2.12}$$

Thus

$$\frac{\mu_{th}(\psi^{-1}(A)) - \mu(\psi^{-1}(A))}{t} \stackrel{\text{by (2.12)}}{=} \frac{\psi(\mu)_{t\psi(h)}(A) - \psi(\mu)(A)}{t}$$

Using these identities we obtain:

$$\begin{aligned} &\lim_{t_n \to 0} \frac{\psi(\mu)_{t_n\psi(h)}(A) - \psi(\mu)(A)}{t_n} \stackrel{(2.12)}{=} \lim_{t_n \to 0} \frac{\mu_{t_n h}(\psi^{-1}(A)) - \mu(\psi^{-1}(A))}{t} \\ \stackrel{\text{semi-}\beta_1\text{-diff}}{=} &\mu_h'^{\beta_1}(\psi^{-1}(A)) = \psi(\mu_h'^{\beta_1})(A) =: \psi(\mu)_{\psi(h)}'^{\beta_2}(A), \end{aligned} \tag{2.13}$$

where the semi-β_1-differentiability implies the uniform convergence in β_2 and is well-defined by (2.12).

2. Having β_1-differentiability, we even obtain β_2-differentiability. It remains to show that $\psi(\nu)'^{\beta_2}_{\cdot} : (\mathbb{H}_2, \|\ \|_{\mathbb{E}_2}) \to (M_{\beta_2}, \|\ \|_{\beta_2})$is linear and continuous. The linearity follows by the linearity of ψ and $\mu'^{\beta_1}_{\cdot}(A)$ (which is linear by β_1-differentiability), because μ is β_1-differentiable, i.e. (well-defined by (2.12))

$$\begin{aligned} \psi(\mu)'^{\beta_2}_{\lambda h+h'}(A) &\overset{\mathbb{H}_2=\psi(\mathbb{H}_1)}{=} \psi(\mu)'^{\beta_2}_{\lambda\psi(h_1)+\psi(h_1')}(A) \overset{\psi \text{ linear}}{=} \psi(\mu)'^{\beta_2}_{\psi(\lambda h_1+h_1')}(A) \\ &\overset{(2.13)}{=} \psi(\mu'^{\beta_1}_{\lambda h_1+h_1'})(A) \overset{\mu'^{\beta_1}_{\cdot}(A) \text{ linear}}{=} \psi(\lambda\mu'^{\beta_1}_{h_1} + \mu'^{\beta_1}_{h_1'})(A) \\ &= \lambda\mu'^{\beta_1}_{h_1}(\psi^{-1}(A)) + \mu'^{\beta_1}_{h_1'}(\psi^{-1}(A)) \overset{(2.13)}{=} (\lambda\psi(\mu)'^{\beta_2}_{h} + \psi(\mu)'^{\beta_2}_{h'})(A) \end{aligned}$$

The continuity follows by $\psi(\nu)'^{\beta_2}_{\cdot} : (\mathbb{H}_1, \|\ \|_{\mathbb{E}_1}) \to (M_{\beta_1}, \|\ \|_{\beta_1})$ being continuous and $\psi\ |_{\mathbb{H}_1}$ being open. In detail we have to show that $\forall\psi(h) \in \mathbb{H}_2, \forall\varepsilon > 0,\ \exists\delta_2 > 0 :\ \sup_{A\in\beta_2}\left|\psi(\mu)'^{\beta_2}_{\psi(h)}(A) - \psi(\mu)'^{\beta_2}_{\tilde{h}}(A)\right| \overset{!}{<} \varepsilon\ \forall\tilde{h} \in \mathbb{H}_2, \left\|\psi(h) - \tilde{h}\right\|_{\mathbb{H}_2} < \delta_2$. By the continuity of $\mu'^{\beta_1}_{\cdot}$ we gain $\exists\delta_1 > 0 :\ \forall\tilde{h}_1 \in \mathbb{H}_1, \left\|h - \tilde{h}_1\right\|_{\mathbb{H}_1} < \delta_1 : \sup_{B\in\beta_1}\left|\mu'^{\beta_1}_{h}(B) - \mu'^{\beta_1}_{\tilde{h}_1}(B)\right| < \varepsilon$.

Since $\psi\big|_{\mathbb{H}_1}$ is open, we know that there exists an open set $\tilde{A}$ such that $\psi(\{\tilde{h}_1 \in \mathbb{H}_1| \left\|h - \tilde{h}_1\right\|_{\mathbb{H}_1} < \delta_1\}) = \tilde{A} \ni \psi(h)$. Thus ($\tilde{A}$ being open) $\exists\delta_2 :$ $\{\tilde{h} \in \mathbb{H}_2| \left\|\psi(h) - \tilde{h}\right\|_{\mathbb{H}_2} < \delta_2\} \subset \tilde{A}$. Finally $\forall\tilde{h} \in \mathbb{H}_2, \left\|\psi(h) - \tilde{h}\right\|_{\mathbb{H}_2} < \delta_2 :$

$$\begin{aligned} &\sup_{A\in\beta_2}\left|\psi(\mu)'^{\beta_2}_{\psi(h)} - \psi(\mu)'^{\beta_2}_{\tilde{h}}\right| = \sup_{A\in\beta_2}|\mu'^{\beta_2}_{h}(\underbrace{\psi^{-1}(A)}_{\in\beta_1}) - \mu'^{\beta_2}_{\psi^{-1}(\tilde{h})}(\psi^{-1}(A))| \\ \leq\ &\sup_{B\in\beta_2}|\mu'^{\beta_2}_{h}(\underbrace{\psi^{-1}(A)}_{\in\beta_1}) - \mu'^{\beta_2}_{\psi^{-1}(\tilde{h})}(\psi^{-1}(A))| < \varepsilon. \end{aligned}$$

□

2.4 Properties on a finite dimensional space

For this section we fix $\mathbb{E} = \mathbb{R}^n$ and we derive a few properties of the Fomin-differentiation on finite dimensional spaces and state a connection to the β-differentiability.

In [ASF71, Propositon 3.1.1] we find the following:

Proposition 2.4.1.
Let $h_1, \ldots, h_n$ be a basis for $\mathbb{R}^n$ and $\nu \in M(\mathbb{R}^n, \mathbb{B}(\mathbb{R}^n))$, i.e. a signed measure

with finite total variation. If ν is Fomin-differentiable in each of the directions $h_1, \dots, h_n$, then it is absolutely continuous w.r.t. Lebesgue measure λ on $\mathbb{R}^n$, i.e.

$$\nu \ll \lambda.$$

Proof.
For every Borel set $A \subset \mathbb{R}^n$ we define a function $\Psi_A \colon \mathbb{R}^n \to \mathbb{R}$, $x \mapsto \nu(A+x)$. By the Fomin-differentiability the function Ψ_A is differentiable in the directions $h_1, \dots, h_n$ at every point $x \in \mathbb{R}^n$, and

$$\Psi_\cdot(x)'^F_{h_i}(A) = \frac{d}{dt}\nu(A + x + th_i)\Big|_{t=0} = \nu'^F_{h_i}(A + x) = \nu'^F_{h_i\,x}(A).$$

Since $\nu'^F_{h_i}$ is of bounded variation on $\mathbb{R}^n$ (cf. Theorem 2.2.6), $\sup_{B \subset \mathbb{R}^n} |\nu'^F_{h_i}(B)| < \infty$. Therefore $\Psi_{A}{}'_{h_i} := \nu'^F_{h_i\,\cdot}(A)$ is bounded:

$$\sup_{x \subset \mathbb{R}^n} |\nu'^F_{h_i\,x}(A)| = \sup_{x \subset \mathbb{R}^n} |(\nu'^F_{h_i}(A + x))| \leq \sup_{B \subset \mathbb{R}^n} |\nu'^F_{h_i}(B)| < \infty \quad \forall A \in \mathbb{B}(\mathbb{R}^n),$$

Ψ_A has bounded derivatives in the basis directions and hence is continuous on $\mathbb{R}^n$, i.e. choosing $h := \sum_{i=1}^n \lambda_i h_i \in \mathbb{H}$ we obtain that $\exists\ (\xi_i)_{1 \leq i \leq n} \subset \mathbb{R}^n$:

$$\begin{aligned}
&\Psi_A(x + h) = \Psi_A(x + \sum_{i=1}^{n-1} \lambda_i h_i + \lambda_n h_n) \\
\overset{\text{part. diff}}{=}\ &\Psi_A\left(x + \sum_{i=1}^{n-1} \lambda_i h_i\right) + \frac{d\Psi_A}{dh_n}(\xi_n)\lambda_n h_n \overset{\text{induction}}{=} \Psi_A(x) + \sum_{i=1}^{n} \frac{d\Psi_A}{dh_n}(\xi_i)\lambda_i h_i
\end{aligned}$$

Thus, since we have bounded derivatives, we see

$$|\Psi_A(x + h) - \Psi_A(x)| \ \ \leq\ \ \sum_{i=1}^{n} \|\lambda_i h_i\|_{\mathbb{R}^n} \underbrace{|\max_{1 \leq i \leq n} \frac{d\Psi_A}{dh_n}(\xi_i)|}_{=:M} \leq Mh \underset{h \to 0}{\longrightarrow} 0$$

In other words, $\nu_x(A) := \nu(A+x) \to \nu(A)$ uniformly for all $A \in \mathbb{B}(\mathbb{R}^n)$, as $x \to 0$. By a theorem of Saks(cf. [Sak64, Theorem 11.2, p.91]), this implies that ν is absolutely continuous w.r.t. Lebesgue measure. □

We attain (the third statement is the assertion of [ASF71, Theorem 3.2.1]):

Theorem 2.4.2.
Let $h_1, \dots, h_n$ be a basis for $\mathbb{R}^n$. Let $\nu \in M(\mathbb{R}^n, \mathbb{B}(\mathbb{R}^n))$ be Fomin-differentiable in each of the directions h_i.

1. *If ν is Fomin-differentiable in the direction $h \in \mathbb{H}$, then*

$$\lim_{t \to 0} \|\frac{\nu_{th} - \nu}{t} - \nu'^F_h\|_{tv} = \lim_{t \to 0} \sup_{0 \leq \tau \leq t} \|\nu'^F_{h\ \tau h} - \nu'^F_h\|_{tv} = 0 \quad (2.14)$$

2. *Each linear combination of directions, in which ν is Fomin-differentiable, gives a new direction, in which ν is Fomin-differentiable and the linearity holds.*

3. *The measure ν is boundedly differentiable w.r.t. $\mathbb{R}^n$. In other words, the following translation mapping is boundedly differentiable*

$$\begin{aligned} T\colon \mathbb{R}^n &\to (M(\mathbb{R}^n, \mathbb{B}(\mathbb{R}^n)), \|\ \|_{tv}) \\ x &\mapsto \nu_x. \end{aligned}$$

Proof.

1. Let $h \in \mathbb{H}$. The assumptions and Proposition 2.4.1 imply that ν is absolutely continuous (w.r.t. Lebesgue measure). Thus the measure $\nu_h'^F$ is also absolutely continuous, because $\nu_h'^F \ll \nu$ (cf. Theorem 2.2.6). Therefore, by the theorem of Saks ([Sak64, Theorem 11.2, p.91] (and Remark 2.2.3)),

$$\begin{aligned} D_h\colon (\mathbb{R}, |\ |) &\to (M(\mathbb{R}^n, \mathbb{B}(\mathbb{R}^n)), \|\ \|_{tv}) \\ \tau &\mapsto {\nu_h'^F}_{\tau h} \end{aligned} \tag{2.15}$$

is continuous and

$$\begin{aligned} T_\nu\colon (\mathbb{R}, |\ |) &\to (M(\mathbb{R}^n, \mathbb{B}(\mathbb{R}^n)), \|\ \|_{tv}) \\ \tau &\mapsto (\nu_h)_{\tau h} \end{aligned} \tag{2.16}$$

has continuous partial derivatives w.r.t. every argument (by assumption they exist): Since ν is Fomin-differentiable along h, we have for all sets $B \in \mathbb{B}(\mathbb{R}^n)$ and $x \in \mathbb{R}^n$:

$$\begin{aligned} &\left| \frac{\nu(B+x+th) - \nu(B+x)}{t} - \nu_h'^F(B+x) \right| \\ \leq\ & \frac{1}{|t|}\ \|\nu_{th} - \nu - t\nu_h'^F\|_{tv} \overset{\text{Thm 2.2.12}}{\leq} \sup_{0<\tau<t} \|{\nu_h'^F}_{\tau h} - \nu_h'^F\|_{tv} \end{aligned} \tag{2.17}$$

Since D_h is continuous, $\forall \varepsilon > 0\ \exists \delta > 0 : \forall \tau < \delta :\ \|{\nu_h'^F}_{\tau h} - \nu_h'^F\|_{tv} < \varepsilon$. Thus choosing $t = \delta$, we gain that the supremum in (2.17) is less than ε.

2. Let $h, h_1, h_2 \in \mathbb{H}, \lambda \in \mathbb{R}$. We prove the multiplication with scalars

$$\nu_{\lambda h}'^F \overset{t'=\lambda t}{=} \lim_{\frac{t'}{\lambda}\to 0} \frac{\nu_{t'\lambda^{-1}\lambda h} - \nu}{t'\lambda^{-1}} = \lambda \lim_{\frac{t'}{\lambda}\to 0} \frac{\nu_{t'h} - \nu}{t'} \overset{\substack{\text{limit}\\ \text{unique}}}{=} \lambda \nu_h'^F \tag{2.18}$$

and the additivity

$$\begin{aligned} &\lim_{t\to 0}\ \left\|\frac{\nu_{th_1+th_2} - \nu}{t} - \nu_{h_1}'^F - \nu_{h_2}'^F\right\|_{tv} \\ \leq\ & \lim_{r\to 0}\lim_{s\to 0}\lim_{t\to 0}\ \left\|\frac{\nu_{th_1+th_2} - \nu}{t} - \frac{\nu_{sh_1} - \nu}{s} - \frac{\nu_{rh_2} - \nu}{r}\right\|_{tv} \\ &+ \lim_{s\to 0}\ \left\|\frac{\nu_{sh_1} - \nu}{s} - \nu_{h_1}'^F\right\|_{tv} + \lim_{r\to 0}\ \left\|\frac{\nu_{rh_2} - \nu}{r} - \nu_{h_2}'^F\right\|_{tv} \end{aligned}$$

Using (2.14) and that $\|\nu_{-th_2}\|_{tv} = \|\nu\|_{tv}$ we have

$$
\begin{aligned}
&= \lim_{r\to 0}\lim_{s\to 0}\lim_{t\to 0} \|\frac{\nu_{th_1} - \nu_{-th_2}}{t} - \frac{\nu_{sh_1-th_2} - \nu_{-th_2}}{s} - \frac{\nu_{rh_2-th_2} - \nu_{-th_2}}{r}\|_{tv} \\
&= \lim_{r\to 0}\lim_{s\to 0}\lim_{t\to 0} \|\frac{\nu_{th_1} - \nu}{t} + \frac{\nu - \nu_{-th_2}}{t} - \frac{\nu_{sh_1-th_2} - \nu_{-th_2}}{s} \\
&\quad -\frac{\nu_{rh_2-th_2} - \nu_{-th_2}}{r}\|_{tv}, \\
&\overset{\triangle}{\leq} \lim_{s\to 0}\lim_{t\to 0} \|\frac{\nu_{th_1} - \nu}{t} - \frac{\nu_{sh_1-th_2} - \nu_{-th_2}}{s}\|_{tv} \\
&\qquad + \lim_{r\to 0}\lim_{t\to 0} \|\frac{\nu - \nu_{-th_2}}{t} - \frac{\nu_{rh_2-th_2} - \nu_{-th_2}}{r}\|_{tv} \\
&\overset{\triangle}{\leq} \lim_{t\to 0} \|\frac{\nu_{th_1} - \nu}{t} - \nu'^F_{h_1}\|_{tv} + \lim_{t\to 0} \|\nu'^F_{h_1} - \nu'^F_{h_1\, -th_2}\|_{tv} \\
&\quad + \lim_{s\to 0} \|\nu'^F_{h_1} - \frac{\nu_{sh_1} - \nu}{s}\|_{tv} + \lim_{t\to 0} \|-\frac{\nu_{-th_2} - \nu}{t} - \nu'^F_{h_2}\|_{tv} \\
&\quad + \lim_{t\to 0} \|\nu'^F_{h_2} - \nu'^F_{h_2\, -th_2}\|_{tv} + \lim_{r\to 0} \|\nu'^F_{h_2} - \frac{\nu_{rh_2} - \nu}{r}\|_{tv} \overset{(2.14)}{=} 0
\end{aligned}
\tag{2.19}
$$

3. We know by the second part that ν is Fomin-differentiable along $\mathbb{R}^n$ and linear on $\mathbb{R}^n$, therefore the continuity remains to show in order to gain that it is boundedly differentiable w.r.t. $\mathbb{R}^n$(cf. Definition 2.3.6).

   W.l.o.g. $h_1, \dots, h_n$ is an orthonormal base. If this wasn't the case, we would just choose one and gain by the second statement of this theorem that ν is as well Fomin-differentiable along these new directions.
   Since T_ν is linear (cf. (2.16)), it is enough to show continuity at 0: For $\varepsilon > 0$ choose $h, \tilde{h} \in \mathbb{H} : \left\|h - \tilde{h}\right\|_{\mathbb{H}} < \frac{\varepsilon}{Mn}$, where $M := \max_{1\leq i\leq n} \|\nu'^F_{h_i}\|_{tv}$.
   Fixing $h - \tilde{h} =: \sum_{i=1}^n \lambda_i h_i$ and analyzing

$$
\left\|h - \tilde{h}\right\|_{\mathbb{H}} \underset{h_i \text{ ONB}}{\overset{\text{Pythagoras}}{=}} \left(\sum_{i=1}^n |\lambda_i|^2 \underbrace{\|h_i\|^2_{\mathbb{H}}}_{=1}\right)^{\frac{1}{2}} = \left(\sum_{i=1}^n |\lambda_i|^2\right)^{\frac{1}{2}} \geq \left(|\lambda_i|^2\right)^{\frac{1}{2}} = |\lambda_i| \quad \forall i
$$

   we gain the continuity

$$
\|\nu'^F_{\tilde{h}-h}\|_{tv} \overset{\triangle}{\leq} \sum_{i=1}^n |\lambda_i| \underbrace{\|\nu'^F_{h_i}\|_{tv}}_{\leq M} \leq nM \left\|h - \tilde{h}\right\|_{\mathbb{H}} < \varepsilon.
$$

□

Remark 2.4.3.

1. *We have shown that ν is even $\beta_{\mathbb{R}^n}$-differentiable, where $\beta_{\mathbb{R}^n} = \mathbb{B}(\mathbb{R}^n)$.*
2. *We note that the assertion of [Sak64, Theorem 11.2, p.91] is only stated for finite dimensional spaces. Thus we cannot use this method for the infinite dimensional case without further considerations.*

2.5 Taking Fomin-derivative is linear and continuous

We return to $\mathbb{E}$ being an arbitrary separable Banach space (cf. Section 2.1) and prove (cmp. [ASF71, Proposition 4.1.1, p.167]):

Proposition 2.5.1 ($\nu_{\cdot}^{\prime F}$ linear).
Suppose that ν is uniformly Fomin-differentiable in every direction $h \in \mathbb{H}$. Then

$$\begin{array}{rcl} \nu_{\cdot}^{\prime F}: \mathbb{H} & \to & M(\mathbb{E}, \mathbb{B}(\mathbb{E})) \\ h & \mapsto & \nu_h^{\prime F} \end{array}$$

is linear.

Proof.
We know that $\sigma(\mathbb{E}') = \mathbb{B}(\mathbb{E})$. Let $h \in \mathbb{H}$. The cylindrical sets are generated by the elements of $\mathbb{E}'$. Since $\nu_h^{\prime F}$ is a signed measure we know ($A_i \in \mathbb{B}(\mathbb{E})\ \forall i \in \mathbb{N}$):

$$\nu_h^{\prime F}(\dot{\bigcup_{i \in \mathbb{N}}} A_i) = \sum_{i=1}^{\infty} \nu_h^{\prime F}(A_i).$$

Thus it is sufficient to verify the following equation for cylindrical sets A only.

$$\nu_{\lambda_1 h_1 + \lambda_2 h_2}^{\prime F}(A) \stackrel{!}{=} \lambda_1 \nu_{h_1}^{\prime F}(A) + \lambda_2 \nu_{h_2}^{\prime F}(A) \tag{2.20}$$

We choose $f_1, \ldots, f_n \in \mathbb{E}'$. Let $\Psi = \bigcap_{i=1}^n \ker f_i$, $\mathbb{E}_{f_1 \cdots f_n}$ be the factor space $\mathbb{E}/\Psi$,

$$\begin{array}{rcl} \psi := \psi_{f_1 \cdots f_n}: \mathbb{E} & \to & \mathbb{E}_{f_1 \cdots f_n} \\ x & \mapsto & x + \Psi, \end{array}$$

$B \in \mathbb{B}(\mathbb{E}_{f_1, \ldots, f_n})$ be a Borel subset of the finite dimensional space $\mathbb{E}_{f_1 \cdots f_n}$ and $A = \{x : x \in \mathbb{E}, \psi(x) \in B\}$. Thus we will simplify our task: Using Remark 2.3.5 we have that ν is uniformly Fomin-differentiable along each $h \in \mathbb{H}$ implies semi-$\beta_{\mathbb{E}}$-differentiability, where $\beta_{\mathbb{E}} = \mathbb{B}(\mathbb{E})$. Furthermore, choosing $\beta_2 := \psi(\beta_{\mathbb{E}}) = \mathbb{B}(\mathbb{E}_{f_1 \cdots f_n})$, we obtain using Proposition 2.3.8 for $h_1, h_2 \in \mathbb{H}, \lambda_1, \lambda_2 \in \mathbb{R}$:

$$\begin{array}{rcl} & & (\psi(\nu))_{\lambda_1 \psi(h_1) + \lambda_2 \psi(h_2)}^{\prime F}(B) \overset{\substack{\mathbb{H} \text{ linear subset} \\ \text{and } \psi(x) = x + \Psi}}{=} (\psi(\nu))_{\psi(\lambda_1 h_1 + \lambda_2 h_2)}^{\prime F}(B) \\ & \overset{\text{Prop 2.3.8}}{=} & \nu_{\lambda_1 h_1 + \lambda_2 h_2}^{\prime F}(\psi^{-1}(B)) \overset{\text{def. A}}{=} \nu_{\lambda_1 h_1 + \lambda_2 h_2}^{\prime F}(A) \end{array}$$

and for $i = 1, 2$

$$\lambda_i \psi(\nu)_{\psi(h_i)}^{\prime F}(B) \overset{\text{Prop 2.3.8}}{=} \lambda_i \nu_{h_i}^{\prime F}(\psi^{-1}(B)) \overset{\text{def. of } A}{=} \lambda_i \nu_{h_i}^{\prime F}(A)$$

Thus equation (2.20) is implied by

$$\begin{aligned} &\psi(\nu)'^{F}_{\lambda_1\psi(h_1)+\lambda_2\psi(h_2)}(B) \\ &\stackrel{!}{=} \lambda_1\psi(\nu)'^{F}_{\psi(h_1)}(B) + \lambda_2\psi(\nu)'^{F}_{\psi(h_2)}(B)\ \forall B \in \mathbb{B}(\mathbb{E}_{f_1,\cdots f_n},) \end{aligned} \tag{2.21}$$

Since we work on the finite dimensional space $\mathbb{E}_{f_1,\dots,f_n} \cong \mathbb{R}^m$, each Borel set B is of the form $B_1 \times B_2$, where B_1 is a Borel subset of the space M_1 generated by the vector $\psi(h_1)$ and $\psi(h_2)$ and B_2 is a Borel subset of the subspace M_2 such that $M_1 \oplus M_2 = \mathbb{E}_{f_1\dots f_n}$. With B_2 we associate the measure ν_{B_2}:

$$\begin{aligned} \nu_{B_2} \colon \mathbb{B}(M_1) &\to \mathbb{R} \\ B_1 &\mapsto (\psi(\nu))(B_1 \times B_2) \end{aligned}$$

ν_{B_2} is differentiable in every direction of M_1(because ν is differentiable in every direction): $\tilde{h} \in B_1 \hookrightarrow B_1 \times B_2$

$$\begin{aligned} \nu_{B_2}(B_1 + t\tilde{h}) &\stackrel{\text{def.}}{=} \nu(\psi^{-1}((B_1 + t\tilde{h}) \times B_2)) \\ &\stackrel{M_1 \oplus M_2}{=} \nu(\underbrace{\psi^{-1}(B + t\tilde{h})}_{\substack{= \{x \in A \mid \psi(x) \in B + t\tilde{h}\} \\ = \{x \in A \mid \psi(x - t\psi^{-1}(\tilde{h})) \in B\} \\ = \{x + t\psi^{-1}(\tilde{h}) \mid \psi(x) \in B\}}}) = \nu(A + t\psi^{-1}(\tilde{h})) \end{aligned}$$

Thus we gain

$$\begin{aligned} \nu_{B_2}{}'^{F}_{h_1}(B_1) &= \lim_{t\to 0} \frac{\nu_{B_2}(B_1 + th_1) - \nu_{B_2}(B_1)}{t} \\ &= \lim_{t\to 0} \frac{\nu(A + t\psi^{-1}(h_1)) - \nu(A)}{t} \\ &\stackrel{\substack{\nu \text{ is Fomin-} \\ \text{differentiable}}}{=} \nu'^{F}_{\psi^{-1}(h_1)}(A) \end{aligned} \tag{2.22}$$

Hence we are in the case of Theorem 2.4.2 and statement 2 shows the linearity. □

The next Proposition is only cited from [ASF71, Propositon 4.13]:

Proposition 2.5.2 (Continuity).
If ν is Fomin differentiable w.r.t. to all $h \in \mathbb{H}'$, then ν is continuous w.r.t. $\mathbb{H}' \subset \mathbb{H}$, i.e. we have that

$$\begin{aligned} \nu'^{F}_{\cdot} \colon (\mathbb{H}', \|\ \|_{\mathbb{H}}) &\to (M(\mathbb{E}, \mathbb{B}(\mathbb{E})),\ \|\ \|_{tv}) \\ h &\mapsto \nu'^{F}_{h} \end{aligned}$$

is continuous.

2.6 Logarithmic gradient

Definition 2.6.1 (${}_F\beta^\nu_{\mathbb{H}}$).
We suppose that ν *is Fomin-differentiable along every* $h \in \mathbb{H}$ *and define*

$$\begin{array}{rcl} {}_F\beta^\nu_{\mathbb{H}} \colon \mathbb{H} & \to & L^1(\nu) \\ h & \mapsto & {}_F\beta^\nu(h,\cdot) \end{array}$$

${}_F\beta^\nu_{\mathbb{H}}$ *is called the logarithmic gradient of* ν *w.r.t.* $\mathbb{H}$ *and is unique except for a* ν*-null set* $N(h) \in \mathbb{B}(\mathbb{E})$.

Corollary 2.6.2.
Let $(\tilde{\mathbb{H}}, \| \|_{\tilde{\mathbb{H}}})$ *be a Banach space, where* $\tilde{\mathbb{H}} \subset \mathbb{E}$. *If* ν *is Fomin-differentiable along* h *for all* $h \in \tilde{\mathbb{H}}$ *and*

$$\begin{array}{rcl} \nu'^F_{\cdot} \colon (\tilde{\mathbb{H}}, \| \|_{\tilde{\mathbb{H}}}) & \to & (M(\mathbb{E}, \mathbb{B}(\mathbb{E})),\ \| \|_{tv}) \\ h & \mapsto & \nu'^F_h \end{array}$$

is continuous, then the logarithmic gradient

$$\begin{array}{rcl} {}_F\beta^\nu_{\tilde{\mathbb{H}}} \colon (\tilde{\mathbb{H}}, \| \|_{\tilde{\mathbb{H}}}) & \to & (L^1(\nu), \| \|_{L^1(\nu)}) \\ h & \mapsto & {}_F\beta^\nu(h,\cdot) \end{array}$$

is continuous, where $\frac{d\nu'^F_h}{d\nu} = {}_F\beta^\nu(h,\cdot)$.

Proof.
By the continuity of $\nu'^F_{\cdot}$ (cf. Proposition 2.5.2)we know that $\forall \varepsilon > 0, \forall h \in \tilde{\mathbb{H}}$ exists $\delta > 0$ such that for all $h' \in \tilde{\mathbb{H}}$ with $\|h - h'\|_{\tilde{\mathbb{H}}} < \delta$ we have

$$\begin{array}{rl} \varepsilon > & \|\nu'^F_h - \nu'^F_{h'}\|_{tv} \overset{\nu'^F_{\cdot} \ll \nu}{=} \|{}_F\beta^\nu(h,\cdot)\,\nu - {}_F\beta^\nu(h',\cdot)\,\nu\|_{tv} \\ \overset{\text{Lemma 1.0.18}}{=} & \|{}_F\beta^\nu(h,\cdot) - {}_F\beta^\nu(h',\cdot)\|_{L^1(\nu)} \end{array}$$

Thus the claim is proved. □

Definition 2.6.3 (ν-quasi-linear, $N(\lambda, h_1, h_2)$).
A function $f : \mathbb{H} \to L^1(\nu)$ *is* ν*-quasi-linear, iff* $\forall\ h_1, h_2 \in \mathbb{H}$ *and* $\forall \lambda \in \mathbb{R}\ \exists N = N(\lambda, h_1, h_2) : \nu(N) = 0$ *and*

$$f(\lambda h_1 + h_2)(x) = \lambda f(h_1)(x) + f(h_2)(x),\ \forall x \in N^C.$$

A function $\tilde{f} : \mathbb{H} \times \mathbb{E} \to \mathbb{R}$ *is* ν*-quasi-linear in the first component, if the function*

$$\begin{array}{rcl} f \colon \mathbb{H} & \to & L^1(\nu) \\ h & \mapsto & \tilde{f}(h,\cdot) \end{array}$$

is ν*-quasi-linear* $\forall h \in \mathbb{H}$.

Lemma 2.6.4.
The logarithmic gradient ${}_F\beta^\nu_{\mathbb{H}}$ *is* ν*-quasi-linear.*

Proof.
This follows by the linearity of taking Fomin-derivative (cf. Proposition 2.5.1):

$$\begin{aligned} {}_F\beta^\nu_{\mathbb{H}}(\lambda h + h')(\cdot)\nu &\overset{\text{by def. of } {}_F\beta^\nu_{\mathbb{H}}}{=} \nu'^F_{\lambda h+h'} \overset{\text{Prop 2.5.1}}{=} \lambda\nu'^F_h + \nu'^F_{h'} \\ &\overset{\text{def. } {}_F\beta^\nu_{\mathbb{H}}}{=} \lambda\, {}_F\beta^\nu_{\mathbb{H}}(h)(\cdot)\nu + {}_F\beta^\nu_{\mathbb{H}}(h')(\cdot)\nu, \end{aligned} \tag{2.23}$$

where the first holds $\forall x \in N^C(\lambda h+h')$ and the last $\forall x \in N^C(h)\cap N^C(h')$. Thus we gain that ${}_F\beta^\nu_{\mathbb{H}}(\cdot)$ is ν-quasi-linear, where $N(h,h',\lambda) := N(\lambda h+h')\cup N(h)\cup N(h')$.
□

Following [SvW95] we define the logarithmic gradient for vector fields. In [SvW95] there were no conditions or further clues given to show that the logarithmic gradient for vector fields exists. We will give a condition, under which the logarithmic gradient exists for a vector field. Furthermore we change the definition, such that it will be independent of the choice of the orthonormal base.

Definition 2.6.5 (respects null sets)**.**
A vector field $g : \mathbb{E} \mapsto \mathbb{E}$ *is said to respect null sets (or* $g(\nu) \ll \nu$*), iff*

$$\forall N \in \mathbb{B}(\mathbb{E}) : \nu(N) = 0\ \exists N_0 \in \mathbb{B}(\mathbb{E}) : g^{-1}(N) \subset N_0 \wedge \nu(N_0) = 0.$$

Definition 2.6.6 (logarithmic gradient of ν w.r.t a vector field h)**.**
Let $g : \mathbb{E} \mapsto \mathbb{E}$ *respect null sets. Assume that the functions* ${}_F\beta^\nu_{\mathbb{H}}(h)(g(\cdot))$ *with* $h \in \mathbb{H}$ *are in* $L^2(\nu)$ *and depend continuously on* $h \in \mathbb{H}$*. Furthermore we assume that it is bounded, i.e.* $\| {}_F\beta^\nu_{\mathbb{H}}(h)(g(\cdot))\|_{L^2} \le M^2(g(\cdot))\, \|h\|_{\mathbb{H}}$ *.*
If for a vector field $h : \mathbb{E} \mapsto \mathbb{H}$ *with* $\int \|h(x)\|^2_{\mathbb{H}}\nu(dx) < \infty$ *the limit of the* $L^2(\nu)-$*convergent series* $\sum (h(\cdot), e_i)_{\mathbb{H}}\, {}_F\beta^\nu_{\mathbb{H}}(e_i)(g(\cdot))$ *exists independently of any orthogonal base* $\{e_i\}_{i\in\mathbb{N}}$ *of* $\mathbb{H}$*, we define the function*

$${}_F\beta^\nu_{\mathbb{H}}(h(\cdot))(g(\cdot)) \tag{2.24}$$

ν*-a.e. as this limit. It is called the (g-)logarithmic gradient of* ν *w.r.t. to a vector field* h*.*

From now on till the end we assume that the logarithmic gradient for vector fields exists for the vector fields, with which we will work. As announced, we give a condition to see that this object exists for finitely based vector fields:

Definition 2.6.7 (finitely based vector field)**.**
Let $h : \mathbb{H} \to \mathbb{E}$ *be a vector field. If there exist* $N \in \mathbb{N}$*: for* $1 \le i \le N$ $\exists h_i \in \mathbb{H}$ *orthogonal,* $f_i \in L^2(\nu)$ *such that*

$$h(x) \;=\; \sum_{i=1}^{N} f_i(x) h_i$$

then h *is called finitely based.*

Lemma 2.6.8.
Let $h : \mathbb{H} \to \mathbb{E}$ be a finitely based vector field. If $g : \mathbb{E} \to \mathbb{E}$ respects null sets, then there exists ${}_F\beta^\nu_{\mathbb{H}}(h(\cdot))(g(\cdot))$ and it is independent of the choice of the orthonormal base.

Proof.
First of all we have

$$\int_{\mathbb{E}} \|h(\cdot)\|^2_{\mathbb{H}} d|\nu| = \int_{\mathbb{E}} \|\sum_{i=1}^N f_i(x)h_i\|^2_{\mathbb{H}} d|\nu| = \int_{\mathbb{E}} \sum_{i,j=1}^N f_i f_j < h_i, h_j > d|\nu|$$

$$= \int_{\mathbb{E}} \sum_{i=1}^N |f_i(x)|^2 \|h_i\|^2_{\mathbb{H}} |\nu|(dx) = \sum_{i=1}^N \|h_i\|_{\mathbb{H}} \|f_i\|^2_{L^2(\nu)} < \infty,$$

Let $\{e_k\}_{k\in\mathbb{N}}$ be any orthogonal base, then by ${}_F\beta^\nu_{\mathbb{H}}$ being ν-quasi linear (Lemma 2.6.4) and continuous (Corollary 2.6.2) and g respecting null sets we gain

$$\begin{aligned}
& \int_{\mathbb{E}} \left| \sum_{e_k} \left(\sum_{i=1}^N f_i(x)h_i, e_k \right) {}_F\beta^\nu_{\mathbb{H}}(e_k)(g(\cdot)) \right| d|\nu| \\
= \quad & \int_{\mathbb{E}} \left| \sum_{i=1}^N f_i(\cdot) \, {}_F\beta^\nu_{\mathbb{H}}(h_i)(g(\cdot)) \right| d|\nu| && (2.25) \\
\leq \quad & \sum_{i=1}^N \int_{\mathbb{E}} \left| f_i(\cdot) \, {}_F\beta^\nu_{\mathbb{H}}(h_i)(g(\cdot)) \right| d|\nu| && (2.26) \\
\overset{\substack{\text{Cauchy} \\ \text{Schwartz}}}{\leq} \quad & \sum_{i=1}^N \|f_i\|_{L^2(\nu)} \| {}_F\beta^\nu_{\mathbb{H}}(h_i)(g(\cdot)) \|_{L^2(\nu)} \\
\overset{\text{Cor 2.6.2}}{\leq} \quad & M(g) \sum_{i=1}^N \|f_i\|_{L^2(\nu)} \|h_i\|_{\mathbb{H}} \overset{\substack{\text{finitely} \\ \text{based}}}{<} \infty
\end{aligned}$$

In the first equation, we have seen that the series is independent of the orthogonal basis. Furthermore it is independent of the representation of $h(\cdot)$, because ${}_F\beta^\nu_{\mathbb{H}}$ is ν-quasi linear in the first component. □

Remark 2.6.9.
If one justifies that the sums in (2.25) can be interchanged, the same calculation can be done for $h(x) = \sum_{i=1}^\infty f_i(x)h_i$. Fubini can be applied in (2.26) if one assumes that $\sum_{i=1}^\infty \|f_i\|_{L^2(\nu)} \|h_i\|_{\mathbb{H}} < \infty$.

Chapter 3

Different derivative notations

The main idea of this chapter is to generalize the notion of Fomin-differentiability (introduced in Section 2.2) to C-differentiability and general differentiability and to state some connection between C-, $\| \|_{tv}-$ and Fomin-differentiability.

In Section 3.1 we state the definition of stably ν-integrable function and a kind of integration by parts formula for the Fomin-derivative.

Motivated by the idea of the integration by parts formula, we introduce in Section 3.2 the C-differentiability and prove that it is well defined. Furthermore we define the logarithmic gradient for C-differentiability and show that it is well defined.

In Section 3.3 we outline the general definition of a derivative (w.r.t. a topology) of a sequence of measures, of a measure w.r.t. a subspace and along a vector field. The τ_S-, τ_C- and τ_{tv}-topology are presented as examples.

In Section 3.4 we illustrate a few connections between the τ_S-, τ_C- and τ_{tv}-derivative and state further properties, including a kind of main theorem of calculus and a formula for the logarithmic gradient of a certain family of measures. The main connections are summarized in a graphic.

Though most of the ideas can be found in [SvW95], we find it helpful to motivate them. The Lemma 3.5.1 has not been stated there and a few details of Proposition 3.5.2 and Proposition 3.5.3 of the proofs were omitted.

3.1 Formula for integration by parts

We need the following definition (cf. [ASF71, p.150]):

Definition 3.1.1 (stably ν-integrable w.r.t. h).
A function $f : \mathbb{E} \to \mathbb{R}$ is stably ν-integrable with respect to translations in the direction $h \in \mathbb{H}$ if there exist $\delta > 0$ and $g \in L^1(\nu)$: $\forall t \in \mathbb{R}, |t| < \delta$ we obtain

$$|f(x + th)| \leq g(x) \quad \forall x \in \mathbb{E}.$$

Example 3.1.2.
For example every bounded function is stably ν-integrable w.r.t. translations in any direction.

We only repeat the assertion of [ASF71, Corollary 2.2.2], its proof can be found in [ASF71, p. 150 ff]:

Theorem 3.1.3 (integration by parts)**.**
Let ν be Fomin-differentiable in the direction $h \in \mathbb{H}$ and f be a $\mathbb{B}(\mathbb{E})$-measurable function, which is Gâuteaux-differentiable in the direction h at every point. If f is stably $\nu_h'^F$-integrable and f_h' is stably ν-integrable w.r.t. translation in the direction h, then

$$\int_E f_h'(x)\nu(dx) = -\int_E f(x)\nu_h'^F(dx).$$

3.2 C-differentiable

Inspired by the formula for integration by parts we define a new concept of differentiability, which is not calculated setwise. A further motivation is to gain a concept for the derivative of a measure along a vector field. The concept of C-differentiability that we introduce following [SvW95] slightly differs from the concept of an integration by parts operator introduced in [Bel90].

Similar to [SvW95, p.105] we define

Definition 3.2.1 (norm-defining)**.**
Let $\tilde{C}$ denote any (arbitrary) set, which consists of bounded functions $\phi : \mathbb{E} \mapsto \mathbb{R}$, i.e. $\tilde{C} \subset B_b(\mathbb{E})$. The set $\tilde{C}$ is called norm-defining, if for all $\nu \in M(\mathbb{E}, \mathbb{B}(\mathbb{E}))$:

$$\|\nu\|_{tv} = \sup\left\{\int_{\mathbb{E}} \phi d\nu, \|\phi\|_\infty \leq 1, \phi \in \tilde{C}\right\} \tag{3.1}$$

Remark 3.2.2.

1. *Of course, in (3.1) the l.h.s is always bigger than or equal to the r.h.s..*
2. *If we add more bounded measurable functions to a norm-defining set, it will remain norm-defining.*
3. *For the β-differentiability we have chosen a set (or class) of subsets of $\mathbb{E}$. Now we choose a set consisting of functions $\phi : \mathbb{E} \to \mathbb{R}$.*
4. *In Theorem 4.1.15 we show that $C_b^1 := C_b^1(\mathbb{E}, \mathbb{R})$ is norm-defining.*

Slightly differing from [SvW95], where the elements of the following norm-defining set are assumed to be smooth, we define

Definition 3.2.3 (C-differentiable along a vector field)**.**
Let C be a norm-defining set, which elements ϕ are at least once Gâuteaux-differentiable in each direction $h' \in \mathbb{H}$.
A measure ν is called C-differentiable along a vector field $h : \mathbb{E} \to \mathbb{H}$ with logarithmic derivative ${}_C\beta_h^\nu \in L^1(\nu)$ and derivative $\nu_h^{\prime C} := {}_C\beta_h^\nu \nu$, if for every $\phi \in C$ one has

$$-\int \phi'_{h(x)}(x)\nu(dx) = \int \phi(x)\, {}_C\beta_h^\nu(x)\nu(dx) \tag{3.2}$$

Remark 3.2.4.
Let $h : \mathbb{E} \to \mathbb{R}$ be a vector field and C be a norm-defining set.

1. *As for the notation of the β-differentiability, we remark that if we use the symbol C with indices, we mean the concept of C-differentiability, i.e. ν'^C.*

   The same is true for the logarithmic derivative, i.e. ${}_C\beta_h^\nu$.

   Note that the right lower index is meant as a vector field $h : \mathbb{E} \to \mathbb{H}$. If h is constant, we mean the vector field $h : \mathbb{E} \to \mathbb{H}$, such that $\forall x \in \mathbb{E}\ h(x) = h$.

2. *If h is constant, each Fomin-differentiable measure ν is C-differentiable along h by the integration by parts formula (Theorem 3.1.3) and the derivatives (and ${}_C\beta_h^\nu = {}_F\beta^\nu(h, \cdot)$) coincide.*

3. *Being familiar with [Bel90] one notices that this definition slightly differs form the definition of an integration by parts operator (IPO) (cf. [Bel90, p.17]), i.e. the IPO is the negative of the logarithmic derivative ${}_C\beta_h^\nu$.*

4. *We notice that*

$$\|\nu_h^{\prime C}\|_{tv} \overset{Def.\ 3.2.1}{=} |\sup_{\substack{\|\phi\|_\infty \leq 1 \\ \phi \in C}} \int_{\mathbb{E}} \phi d\nu_h^{\prime C}| \leq \int_{\mathbb{E}} |\, {}_C\beta_h^\nu\, |d|\nu| \overset{{}_C\beta_h^\nu \in L^1(\nu)}{<} \infty$$

Lemma 3.2.5 (${}_C\beta_h^\nu$ is well defined)**.**
The logarithmic derivative ${}_C\beta_h^\nu$ is ν-a.e. unique, and hence well defined.

Proof.
Assume there existed ${}_C\beta_h^\nu$ and ${}_C\tilde{\beta}_h^\nu$ such that definition (3.2) were fulfilled. By Lemma 1.0.17 $({}_C\beta_h^\nu - {}_C\tilde{\beta}_h^\nu)\nu$ is a signed measure, because ${}_C\beta_h^\nu - {}_C\tilde{\beta}_h^\nu \in L^1(\nu)$.

Thus using (3.2) and Definition 3.2.1 we get:

$$\int |\,{}_C\beta_h^\nu - {}_C\tilde{\beta}_h^\nu| d|\nu| \overset{\text{Lemma 1.0.18}}{=} \|(\,{}_C\beta_h^\nu - {}_C\tilde{\beta}_h^\nu)\nu\|_{tv}$$

$$= \sup_{\substack{\phi \in C \\ \|\phi\|_\infty \le 1}} \int \phi(\,{}_C\beta_h^\nu - {}_C\tilde{\beta}_h^\nu) d\nu$$

$$\overset{(3.2)}{=} \sup_{\substack{\phi \in C \\ \|\phi\|_\infty \le 1}} \int_{\mathbb{E}} \phi'_{h(x)}(x)\nu(dx) - \underbrace{\int_{\mathbb{E}} \phi'_{h(x)}(x)\nu(dx)}_{\substack{\text{finite, by (3.2)} \\ \text{and } {}_C\beta_h^\nu \in L^1(\nu)}} = 0$$

Thus ${}_C\beta_h^\nu = {}_C\tilde{\beta}_h^\nu$ ν-a.e.. □

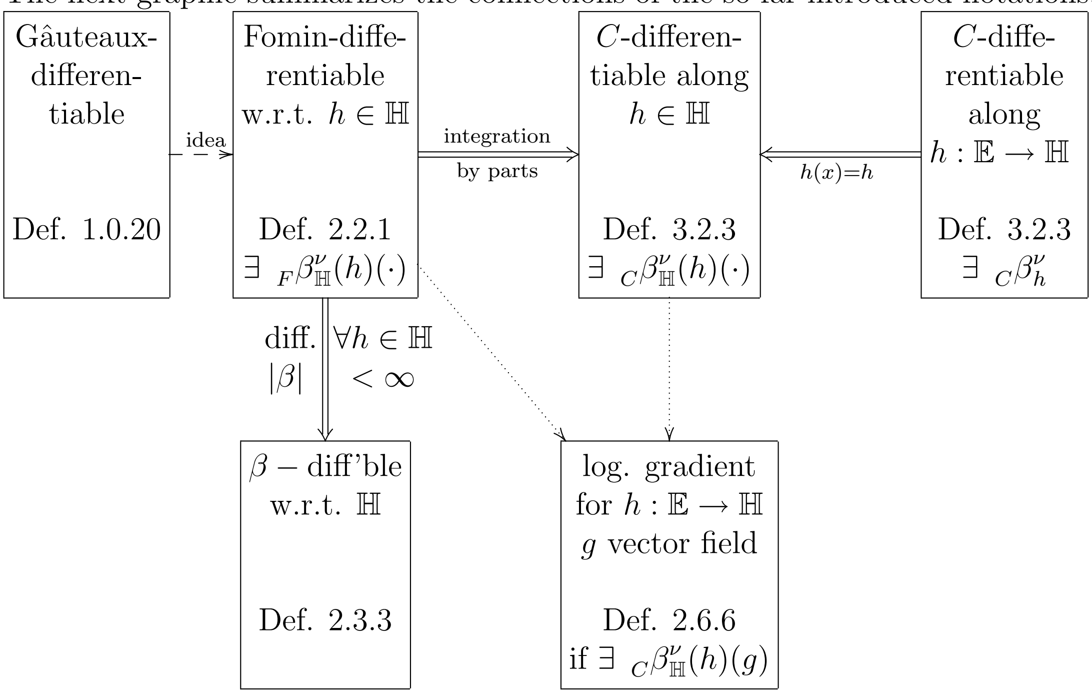
Remark 3.2.6.
Analogously to ${}_F\beta_{\mathbb{H}}^\nu$ *we define for* $h \in \mathbb{H}$ ${}_C\beta_{\mathbb{H}}^\nu$ *by* ${}_C\beta_{\mathbb{H}}^\nu(h)(\cdot) = {}_C\beta^\nu(h,\cdot) := {}_C\beta_h^\nu$ *and obtain that it is also* ν*-quasi-linear:*

$$\int_{\mathbb{E}} \phi\, {}_C\beta^\nu(h + \lambda h', \cdot) d\nu = \int_{\mathbb{E}} \phi'_{h+\lambda h'} d\nu$$

$$\overset{\text{Def. 1.0.20}}{=} \int_E \phi'_h d\nu + \lambda \int_E \phi'_{h'} d\nu = \int_E \phi\left(\,{}_C\beta^\nu(h,\cdot) + \lambda\, {}_C\beta^\nu(h',\cdot)\right) d\nu$$

Noticing that it is continuous in the first component we conclude as in Corollary **??** *that it is even* ν*-a.e. linear. Moreover, one can easily define the logarithmic gradient* ${}_C\beta_{\mathbb{H}}^\nu(h)(g)$ *for* $h : \mathbb{E} \to \mathbb{H}$ *and* g *a vector field with the properties mentioned in Definition 2.6.6.*

The next graphic summarizes the connections of the so far introduced notations.



3.3 General definition of the derivative

After having introduced several different notions for the derivative of a measure, we present the general concept to differentiate a family of measures.

Throughout this section τ denotes a (Hausdorff) topology on $M(\mathbb{E}, \mathbb{B}(\mathbb{E}))$, which is compatible with the vector space structure.

Before we state the general defintion of a measures, we introduce (cmp. [Sie00, P.72]):

Definition 3.3.1 (τ-limit)**.**
An infinite sequence $\mu_1, \mu_2, \ldots$ of elements of any topological space (T, τ) is said to have the τ-limit $\nu \in T$, if for every neighborhood V of μ there exists a natural number $N \in \mathbb{N}$:

$$\mu_n \in V \quad \forall n > N$$

In symbols,

$$\tau - \lim_{n \to \infty} \mu_n = \mu$$

If the sequence is indexed by $h \in \mathbb{R}$, we define that ν is the τ-limit, iff it is the τ-limit for every countable subsequence.

Definition 3.3.2 (τ-differentiable at t)**.**
Let $I \subset \mathbb{R}$ be an interval. A family $(\nu^t)_{t \in I}$ of signed measures $\nu_t \in M(\mathbb{E}, \mathbb{B}(\mathbb{E}))$ is τ-differentiable at t iff there exists the τ-limit

$$\tau - \lim_{\varepsilon \to 0} \frac{\nu^{t+\varepsilon h} - \nu^t}{\varepsilon} \in M(\mathbb{E}, \mathbb{B}(\mathbb{E})).$$

If it exists, it is called τ-derivative of ν and is denoted by $\nu^{t'^{\tau}}$.
If in addition $\nu^{t'^{\tau}} \ll \nu^t$, then there exists the Radon-Nikodỳn derivative denoted by ${}^{t}_{\tau}\beta^{\nu^{\clubsuit}}$. In this case we call it the logarithmic derivative of $(\nu^t)_{t \in I}$ w.r.t. to the topology τ at t, denoted by ${}^{t}_{\tau}\beta^{\nu^{\clubsuit}}$, and have $\nu^{t'^{\tau}} = {}^{t}_{\tau}\beta^{\nu^{\clubsuit}}\, \nu^t$.

Remark 3.3.3.

1. *Note that, whenever the symbol $\clubsuit$ occurs in the notation of the logarithmic derivative, we treat a sequence of signed measures. Thus this is an extension of the notation for the logarithmic derivative, which we used so far.*

2. *In full detail the logarithmic derivative ${}^{t}_{\tau}\beta^{\nu^{\clubsuit}}$ would have to be denoted by*

$$ {}^{t}_{\tau}\beta^{(\nu^t)_{t \in I}}.$$

But for a better readability we omit the range of definition of the sequence.

Definition 3.3.4 (τ-differentiable along h).
A measure $\nu \in M(\mathbb{E}, \mathbb{B}(\mathbb{E}))$ is τ-differentiable along $h \in \mathbb{H}$, if

$$\nu^t := \nu_{th} := \nu(\cdot + th)$$

is τ-differentiable at 0. The τ-derivative is denoted by ${\nu'_h}^{\tau}$.

In [SvW93, p.473] we spot

Definition 3.3.5 (τ-differentiable along $\mathbb{H}$).
A measure $\nu \in M(\mathbb{E})$ is τ-differentiable along $\mathbb{H}$, if

1. *$\mathbb{H} \subset \mathbb{E}$ is a Hilbert subspace*
2. *For each $h \in \mathbb{H}$ the τ-derivative of ν along h exists.*
3. *The following function is continuous: ${\nu'_\cdot}^{\tau} : \mathbb{H} \to M(\mathbb{E}, \mathbb{B}(\mathbb{E}))$, $h \mapsto {\nu'_h}^{\tau}$.*

If ${\nu'_h}^{\tau} \ll \nu$ for all h we denote, as before, the logarithmic derivative of ν along $\mathbb{H}$ by ${}_{\tau}\beta^{\nu}(h)(\cdot)$.

Remark 3.3.6.
As before (cf. Definition 2.6.1)we define the logarithmic derivative ${}_{\tau}\beta^{\nu}_{\mathbb{H}}(h)(\cdot) := {}_{\tau}\beta^{\nu}(h)(\cdot)$ and for functions $h : \mathbb{E} \to \mathbb{H}$, whenever it exists in the sense of Definition 2.6.6. Though we do not assume any linearity, it will be given for the standard examples.

Differing from [SvW93, p.471f] we assume in the following definition that there exists a unique solution on $[-\varepsilon, \varepsilon]$, such that the derivative is well defined. Later (Theorem 4.2.11) we state a condition under which such a unique solution exists.

Definition 3.3.7 (τ-differentiable along a vector field h).
A measure $\nu \in M(\mathbb{E}, \mathbb{B}(\mathbb{E}))$ is said to be differentiable along the vector field $h : \mathbb{E} \to \mathbb{H}$ with derivative ${\nu'}^{\tau}_h$, if and only if

1. *There exists $\varepsilon > 0$ and an unique, in both components differentiable mapping $a : [0-\varepsilon, 0+\varepsilon] \times \mathbb{E} \to \mathbb{E}$ such that*
   (a) $a(t_1 + t_2, x) = a(t_1, a(t_2, x))$
   (b) $a(0, x) = x$
   (c) $D_1 a(t, x) = h(a(t, x))$
2. *$\left\{ {}^h\nu^t \right\}_{t \in [-e,e]}$ is τ-differentiable at t=0, where ${}^h\nu^t(B) := \nu(a(t, B))$ $\forall B \in \mathbb{B}(\mathbb{E})$. If the τ-derivative exists, it is denoted by ${\nu'}^{\tau}_h$.*

If ${\nu'}^{\tau}_h \ll \nu$, then we denote the logarithmic derivative of ν along the vector field h by ${}_{\tau}\beta^{\nu}_h$.

Remark 3.3.8.
This is a generalization of Definition 3.3.4, because for constant h the unique solution on $\mathbb{R}$ with properties (1a) to (1c) is $a(t,x) = x + th$. Thus the notation is justified.

Example 3.3.9 ($\tau_S, \tau_C, \tau_{tv}$)**.** *Following [SvW93, p.456] we have*

1. *Let us turn to the topology of setwise convergence on $M(\mathbb{E}, \mathbb{B}(\mathbb{E}))$, denoted by τ_S. Then we have by definition that a measure $\nu \in M(\mathbb{E}, \mathbb{B}(\mathbb{E}))$ is Fomin-differentiable along $h \in \mathbb{H}$, iff it is τ_S-differentiable along h.*
2. *Let C be a norm-defining set. We consider the weak topology on $M(\mathbb{E}, \mathbb{B}(\mathbb{E}))$ defined by the duality between $M(\mathbb{E}, \mathbb{B}(\mathbb{E}))$ and C, which is denoted by τ_C. Then a measure $\nu \in M(\mathbb{E}, \mathbb{B}(\mathbb{E}))$ is C-differentiable along $h \in \mathbb{H}$, iff it is τ_C-differentiable along h. A connection for vector fields is established in Proposition 4.4.1. For details of this example we refer to the proof of Proposition 4.4.1.*
3. *The topology τ_{tv} denotes the topology generated by the $\| \|_{tv}$-norm of the Banach space $(M(\mathbb{E}, \mathbb{B}(\mathbb{E})), \| \|_{tv})$. Thus ν is τ_{tv}-differentiable iff ν is uniformly Fomin differentiable (cf. Definition 2.2.2):*

$$\tau_{tv} - \lim_{t \to 0} \frac{\nu_{th} - \nu}{t} = \nu_h^{\prime tv}, \text{ iff } \|\frac{\nu_{th} - \nu}{t} - \nu_h^{\prime tv}\|_{tv} \underset{t\to 0}{\to} 0$$

From now on τ_S, τ_C and τ_{tv} denote the above defined topologies, where C can be replaced by $C_b^1, \tilde{C}_b^1$, etc. and a derivative w.r.t. to them can be denoted by $\nu'^{\tau_S}, \nu'^{\tau_C}, \nu'^{\tau_{C_b^1}}, \nu^{t\prime\tau_C}$ etc., where $\nu, \nu^t \in M(\mathbb{E}, \mathbb{B}(\mathbb{E}))$.

Remark 3.3.10.
Let τ_s, τ_w be topologies and τ_s be stronger or finer than τ_w (i.e. τ_s-convergence implies τ_w-convergence). If the τ_s-derivative exists, then the τ_w-derivative exists as well.

Therefore we may use the same notation for all limits and logarithmic derivatives, if the topologies can be compared in the above way. Mainly we will consider the topologies mentioned in the last example. These topologies are comparable in the above sense (cf. [SvW93, Remark 2.2]), i.e. τ_{tv} finer than τ_S, τ_S finer than τ_C. We note that for $h \in \mathbb{H}$, where the equations hold ν-a.e,

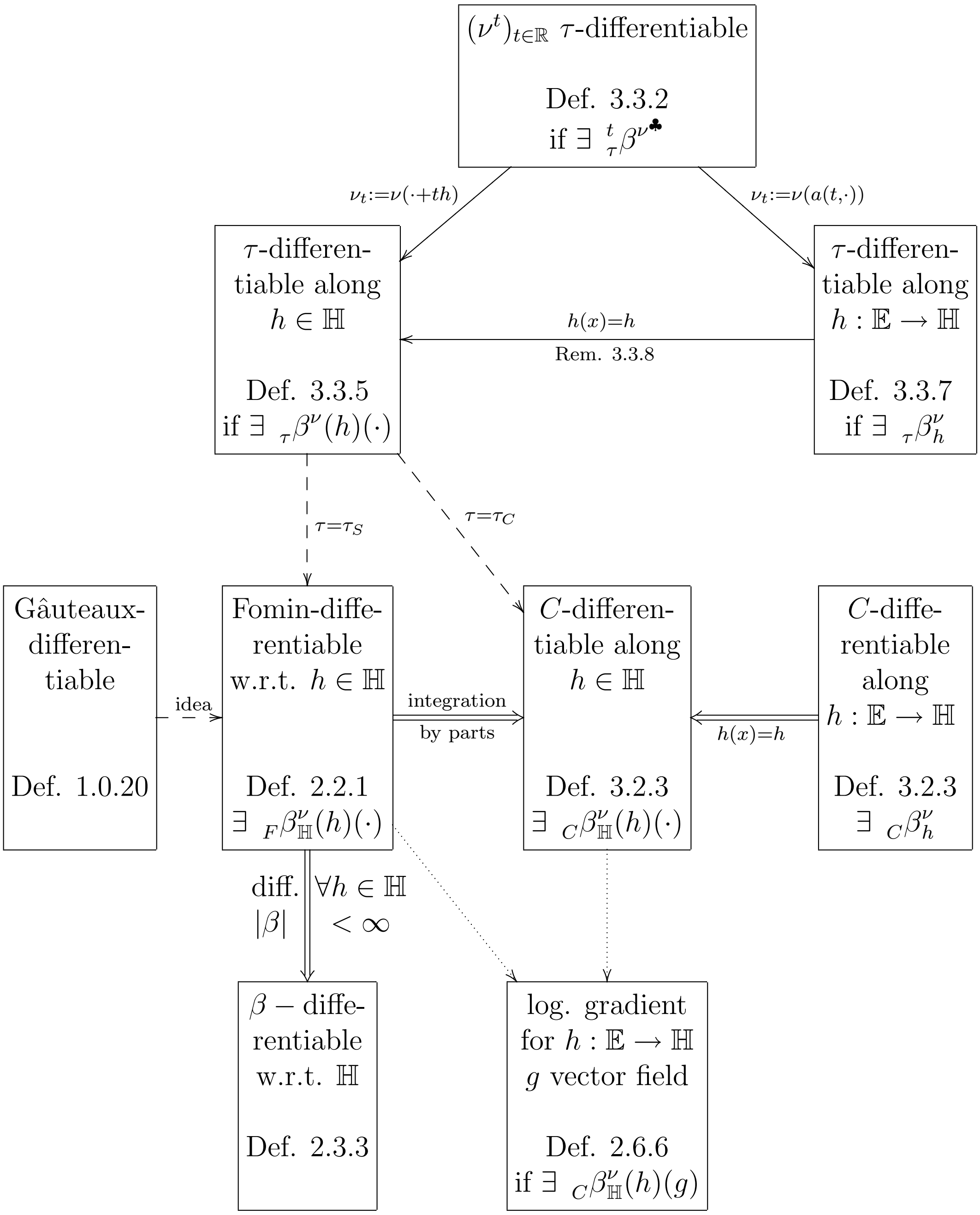
$${}_{tv}\beta^{\nu}_{\mathbb{H}}(h)(\cdot) = {}_{F}\beta^{\nu}_{\mathbb{H}}(h)(\cdot) = {}_{C}\beta^{\nu}_{\mathbb{H}}(h)(\cdot), \text{ if all exist.}$$
$${}_{\tau_{tv}}\beta^{\nu}_{\mathbb{H}}(h)(\cdot) = {}_{\tau_S}\beta^{\nu}_{\mathbb{H}}(h)(\cdot) = {}_{\tau_C}\beta^{\nu}_{\mathbb{H}}(h)(\cdot), \text{ if all exist.}$$

Thus we do not need to distinguish within the notation, if all of them exist. However we will use different notations to express the topology we are considering.

3.4 Overview of the introduced notations

Now we observe, how the definition mentioned so far fit in the general picture.

In Proposition 4.4.1 we show a connection between the τ_C-differentiability and the C-differentiability for a special topology and a special norm-defining set. But to prove and to understand this relation, there is still some work to do. Since this connection is not as obvious as the others, we omit it in this graphic (cf. Remark 4.4.2 for details).



3.5 Connections

Within this section, we outline some connections between the different notions of differentiability (i.e. τ_S, τ_C, τ_{tv}, cf. Example 3.3.9). At the end we summarize them in a grafic. For this section let $I \subset \mathbb{R}$ be an open interval and C be a norm-defining set.

Lemma 3.5.1 (τ_S-τ_C-Lemma)**.**
Let $t_0 \in I$, then $(\nu^t)_{t\in I}$ being τ_S-differentiable at t_0 implies that $(\nu^t)_{t\in I}$ is τ_C-differentiable at t_0.

Proof.
Let $\gamma_s^{t_0} : \mathbb{B}(\mathbb{E}) \to \mathbb{R}$, $A \mapsto \frac{\nu^{t_0+s}(A)-\nu^{t_0}(A)}{s}$ be a sequence. By the τ_S-differentiability of ν^t it is bounded for every set A. This is the assumption of [DS57, Theorem IV.9.8, p.309f] and hence we gain the boundedness of $\gamma_s^{t_0}$. In addition we use the τ_S-differentiability and obtain by [DS57, Theorem IV.9.5, p.308] that it converges weakly. But in our case this is exactly the τ_C-differentiability.

□

In [SvW93, Proposition 2.5] we have

Proposition 3.5.2.
Let $(\nu^t)_{t\in I}$ be τ_C-differentiable and suppose that either C is complete under the sup-norm or that the map $t \mapsto \|\nu^{t'^{\tau_C}}\|_{tv}$ is bounded from above by a locally integrable function on I. Then

1. *$(\nu^t)_{t\in I}$ is τ_{tv}-continuous.*
2. *There exists a probability measure $\nu^\star \in M(\mathbb{E}, \mathbb{B}(\mathbb{E}))$ such that $\nu^t \ll \nu^\star$ for all $t \in I$. For every such $\nu^\star$ one can choose the derivatives $f_t = d\nu^t/d\nu^\star$ such that $f_t(x)$ is a $\mathbb{B}(I) \otimes \mathbb{B}$-measurable function of (t, x).*
3. *If $(\nu^t)_{t\in I}$ is τ_S-differentiable, then the measure $\nu^\star$ in part 2 dominates even $\nu_t'^F$ for all t.*

Proof.
Succeeding the proof of [SvW93, Proposition 2.5] we have

1. Suppose first that C is complete under the sup-norm. Fix t and $\varepsilon > 0$ such that $(t - \varepsilon, t + \varepsilon) \subset I$. We show that the τ_C-differentiability at t implies that the set $A = \{(\nu^{t+h} - \nu^t)/h : 0 < h < \varepsilon\}$ is bounded for τ_C and then use Banach Steinhaus([Wer05, p.141]) to obtain the continuity:
   We know that by the τ_C-differentiability at t there exists $h_0 : \forall h < h_0 :$

$$\left| \int \phi(x)\nu^{t'^{\tau_C}}(dx) - \left(\frac{1}{h} \int \phi(x)(\nu^{t+h}(dx) - \nu^t(dx)) \right) \right| < 1,$$

which implies

$$\left|\left(\frac{1}{h}(\nu^{t+h}-\nu^{t})\right)(\phi)\right| < \left|\int \phi(x)\nu^{t'\tau_C}(dx)\right| + 1$$
$$\underset{\text{Def. 3.2.1}}{\overset{C \text{ norm-def.}}{\leq}} \|\phi\|_\infty \ \|\nu^{t'\tau_C}\|_{tv} + 1 < \infty.$$

And $\forall h \geq h_0$:

$$\left|\frac{1}{h}(\nu^{t+h}-\nu^{t})(\phi)\right| \leq \frac{1}{h_0}\left(\|\nu^{t+h}-\nu^{t}\|_{tv}\right) \underbrace{\|\phi\|_\infty}_{\substack{\text{finite,}\in C,\\ \text{by Def. 3.2.1}}} < \infty$$

Therefore we have proved that for all $\phi \in C$

$$\sup_{h\in]0,\varepsilon[}\left|\left(\frac{1}{h}(\nu^{t+h}-\nu^{t})\right)(\phi)\right| < \infty \tag{3.3}$$

Since $(C, \|\cdot\|_\infty)$ is a Banach space and $(\mathbb{R}, |\cdot|)$ is a normed space, the equation (3.3) implies (by Banach-Steinhaus) that

$$\infty > \sup_{h\in]0,\varepsilon[}\left\|\frac{1}{h}(\nu^{t+h}-\nu^{t})\right\| := \sup_{h\in]0,\varepsilon[}\ \sup_{\phi\in C,\|\phi\|_\infty\leq 1}\left|(\frac{1}{h}(\nu^{t+h}-\nu^{t}))(\phi)\right|$$
$$\overset{C \text{ norm-def.}}{=} \sup_{h\in]0,\varepsilon[} \|\frac{1}{h}(\nu^{t+h}-\nu^{t})\|_{tv} \tag{3.4}$$

Reformulating (3.4) we get

$$\|\nu^{t+h}-\nu^{t}\|_{tv} \leq Mh \tag{3.5}$$

Thus $\|\nu^{t+h}-\nu^{t}\|_{tv} \to 0$ as $h \to 0$.

Suppose now that $\|\nu^{t'\tau_C}\|_{tv} \leq g(t)$ for some locally integrable function g. Then

$$\begin{aligned}
\|\nu^{t+h}-\nu^{t}\|_{tv} &\overset{C \text{ norm-def.}}{=} \sup\left\{\int \phi d(\nu^{t+h}-\nu^{t}) : \phi \in C, \|\phi\|_\infty \leq 1\right\} \\
&= \sup\left\{\int_t^{t+h} \underbrace{\int \phi d\nu^{s'\tau_C}}_{\leq \|\nu^{s'\tau_C}\|_{tv}} ds : \phi \in C, \|\phi\|_\infty \leq 1\right\} \\
&\leq \int_t^{t+h} g(s)ds,
\end{aligned}$$

which converges to 0 as $h \to 0$. This proves part 1.

2. Choose any positive probability measure $\nu^\star$ which dominates ν^t for all rational t, e.g. $\nu^\star = \sum_{i=1}^\infty c_i|\nu^{t_i}|$, where $(t_i)_{i\in\mathbb{N}}$ is an enumeration of the rational elements of I and c_i is chosen so that $\sum_{i=1}^\infty c_i \ \|\nu^{t_i}\|_{tv} = 1$, that is $c_i := \frac{1}{2^i \ \|\nu^{t_i}\|_{tv}}$.

   Since ν^t is τ_{tv}-continuous (cf. part 1), $\nu^\star$ then dominates even every ν^t, because by the τ_{tv}-continuity of ν^t in t, there exists $t_n \in \mathbb{Q}$ such that $\|\nu^t\|_{tv} \leq \ \|\nu^{t_n}\|_{tv} + \frac{1}{n}$. Since $\nu^{t_n} \ll \nu^\star$ for every rational t_n, this becomes arbitrary small if $\|\nu^\star\|_{tv} \to 0$.

   Then every choice of the densities f_t (they exist, because $\nu^t \ll \nu^\star$) defines a stochastic process on the probability space $(\mathbb{E}, \mathbb{B}(\mathbb{E}), \nu^\star)$ (e.g. cf. [PZ92, 3.1, p.70], f_t is $\mathbb{B}$-measurable).

   Claim: This process is stochastically continuous in measure.

   Proof.
   We have to show (stochastically continuous): $\forall\varepsilon > 0 \ \forall\delta > 0 \ \exists n \in \mathbb{N}$: $\forall h \in [-\frac{1}{n}, \frac{1}{n}]$

$$\begin{aligned} \delta & \overset{!}{\geq} & \nu^\star(\{|f_{t+h} - f_t| \geq \varepsilon\}) \\ & = & \int_{\mathbb{E}} \mathbb{1}_{\{|f_{t+h}-f_t|\geq\epsilon\}}(x)\nu^\star(dx) \\ & \overset{\text{Tschebychef}}{\leq} & \frac{1}{\varepsilon}\int_{\mathbb{E}} |f_{t+h} - f_t| d|\nu^\star| \\ & \overset{\text{Lemma 1.0.18}}{=} & \frac{1}{\varepsilon} \ \|f_{t+h}\nu^\star - f_t\nu^\star\|_{tv} \\ & = & \frac{1}{\varepsilon} \ \|\nu^{t+h} - \nu^t\|_{tv} \underset{h\to 0}{\longrightarrow} 0 \end{aligned} \tag{3.6}$$

   Thus the last term tends to zero, because ν^t is τ_{tv}-continuous in t.

   From the claim we deduce that the process has a jointly measurable modification (cf. [Nev65, Theorem III.4.4., p.91] or [PZ92, Proposition 3.2, p.72]), which proves the second assertion.

3. Let $\nu^\star$ dominate ν^t for all t and let N be a $\nu^\star$-nullset. Then

$$\nu^{t'F}(N) = \lim_{h\to 0} \frac{1}{h}(\nu^{t+h}(N) - \nu^t(N)) \overset{\nu^\cdot \ll \nu^\star}{=} 0.$$

□

3.5.1 Kind of main theorem of calculus

In [SvW93, Theorem 2.7] a kind of main theorem of calculus is postulated, i.e.

Proposition 3.5.3.
Suppose that one of the following assumptions is fulfilled

1. *The family* $(\nu^t)_{t\in I}$ *is* τ_S*-differentiable.*
2. *The family* $(\nu^t)_{t\in I}$ *is* τ_C*-differentiable and* $\nu^{t'\tau_C} \ll \nu^t$ *for* λ*-a.a.* $t \in I$.

Assume further that $\|\nu^{t'\tau_C}\|_{tv} \leq g(t)$ *on* $[a,b]$ *for some a,b in I and some* $g \in L^1([a,b],\lambda)$. *Then*

1. *For Lebesgue a.e.* $t \in [a,b]$ *the family* $(\nu^t)_{t\in I}$ *is* τ_{tv}*-differentiable at t and* $\nu^{t'\tau_{tv}} \ll \nu^t$. *The logarithmic derivative is* ${}^{t}_{\tau_{tv}}\beta^{\nu^\clubsuit} := \frac{d\nu^{t'\tau_{tv}}}{d\nu^t}$.
2. *There exist a probability measure* $\nu^\star$ *and two* $\mathbb{B}([a,b]) \otimes \mathbb{B}(\mathbb{E})$*-measurable functions* f, f' *such that* $f_t = \frac{d\nu^t}{d\nu^\star}$, $f'_t = \frac{d\nu^{t'\tau_C}}{d\nu^\star}$ *for a.e. t, and*
$$f_t(x) - f_a(x) = \int_a^t f'_s(x)ds. \tag{3.7}$$
holds for all $x \in \mathbb{E}$ *and all* $t \in [a,b]$.
3. *For all* $t \in [a,b]$
$$\nu^t - \nu^a = \int_a^t \nu^{s'\tau_{tv}} ds \tag{3.8}$$
as a $(M(\mathbb{E},\mathbb{B}(\mathbb{E})), \|\cdot\|_{tv})$*-valued Bochner integral. (For the definition see e.g. [PR07, Appendix A] or [DJU77, II.2, p. 44 ff])*

In the first assertion the absolute continuity is already assumed in the case of the τ_C-differentiability.

Proof of Proposition 3.5.3.
We carry out the proof as in [SvW93, Theorem 2.7]. W.l.o.g. ν is τ_C-differentiable (eventually using in addition Lemma 3.5.1). Choose a measure $\nu^\star$ according to Proposition 3.5.2 and denote by λ the Lebesgue measure on $[a,b]$. Define a measure m' on $\mathbb{B}(I) \otimes \mathbb{B}(\mathbb{E})$ by $m'(dt,dx) = \nu^{t'\tau_C}(dx)dt$.
Then $\|m'\| := \int_a^b \|\nu^{t'\tau_C}\|_{tv}dt \leq \int_a^b g(t)dt < \infty$.
Claim: $m' \ll \lambda \otimes \nu^\star$
Proof. Let $A = I' \times A' \in \mathbb{B}(I) \times \mathbb{B}(\mathbb{E})$

$$\begin{aligned} m'(A) &= \int_{[a,b]\times\mathbb{E}} \mathbb{1}_A(t,x)m'(dt,dx) \\ &\overset{\text{Fubini}}{=} \int_a^b \mathbb{1}_{I'}(t) \int_{A'} \underbrace{\nu^{t'\tau_C}\ (dx)}_{\leq\|\nu^{t'\tau_C}\ \|} dt = \int_a^b \nu^{t'\tau_C}\ (A')dt \end{aligned} \tag{3.9}$$

where Fubini is justified, because $\|\nu^{\cdot'\tau_C}\|_{tv}$ is locally integrable over $[a,b]$.

Let $t \in [a,b]$. In the case of τ_S-differentiability by Proposition 3.5.2 part 3 we have $\nu^{t'F} \ll \nu^\star$ and in the other one by assumption $\nu^{t'\tau_C} \ll \nu^t (\ll \nu^\star)$. Thus $m' \ll \lambda \otimes \nu^\star$. Let $f' \in L^1(\lambda \otimes \nu^\star)$ be a version of $\frac{dm'}{d\lambda\otimes\nu^\star}$, which is of course jointly measurable. Furthermore as in the proof of Proposition 3.5.2 part 2 there exists f_t, a jointly measurable version of $\frac{d\nu^t}{d\nu^\star}$.

Now we prove the first assertion:
Choose $\phi \in C$ in the case of τ_C-differentiability or $\phi = \mathbb{1}_B, B \in \mathbb{B}(\mathbb{E})$, in the case of τ_S-differentiability. Then by the definition of differentiability $t \mapsto \int \phi d\nu^t$ is a differentiable function with a (by assumption) integrable derivative over $[a,b]$. Since $f' \in L^1(\lambda \otimes \nu^\star)$ and ϕ is bounded, we may apply Fubini

$$\begin{aligned}\int_{\mathbb{E}} \phi(x) \int_a^t f'_s(x) ds \; \nu^\star(dx) &\overset{\text{Fubini}}{=} \int_{[a,t]\times\mathbb{E}} \phi(x) f'_s(x) \lambda \otimes \nu^\star(ds,dx) \\ &\overset{\text{def. } f'}{=} \int_{[a,t]\times\mathbb{E}} \phi(x) m'(ds,dx) \\ &\overset{\text{def. } m'}{=} \int_a^t \int_{\mathbb{E}} \phi(x) \nu^{s'\tau_C} \; (dx) ds \qquad (3.10)\end{aligned}$$

Using in addition the fundamental theorem of calculus for Bochner integrals (cf. [PR07, A.2.3]) we gain that (3.10) equals

$$\begin{aligned}&\int_{\mathbb{E}} \phi(x)\nu^t(dx) - \int_{\mathbb{E}} \phi(x)\nu^a(dx) = \int_{\mathbb{E}} \phi(x) d(\underbrace{\nu^t - \nu^a}_{\int_a^t \nu^{s'\tau_C} \; ds}) \\ = &\int_{\mathbb{E}} \phi(x)(f_t(x) - f_a(x))\nu^\star(dx), \qquad (3.11)\end{aligned}$$

where we used in the last equation the assumptions and the 2. statement of Propositioin 3.5.2. Hence (3.7) holds $\nu^\star$-a.e., because C is norm-defining.

We redefine f by $f_t(x) = \int_a^t f'_s(x) ds$. This changes each f_t only on a $\nu^\star$-nullset and hence still $f_t = \frac{d\nu^t}{d\nu^\star}$. But with this new f, (3.7) holds everywhere.
The Banach space $L^1(\lambda \otimes \nu^\star)$ can be identified with $L^1_{L^1(\nu^\star)}([a,b])$, the space of equivalence classes of $L^1(\nu^\star)$-valued Bochner integrable function on $[a,b]$, because

$$\infty > \int_{[a,b]\times\mathbb{E}} |f'_t(x)| (\lambda \otimes \nu^\star) d(t,x) \overset{\substack{\text{Fubini} \\ f'_t \in L^1(\lambda\otimes\nu^\star)}}{=} \int_{[a,b]} \int_{\mathbb{E}} |f'_t(x)| \nu^\star(dx) \lambda(dt)$$

and thus (3.11) implies

$$f_t - f_a = \int_a^t f'_s ds \qquad (3.12)$$

where the r.h.s. is a Bochner integral, which exists by Fubini and the construction.

The Lebesgue differentiation theorem for Bochner integrals (cf. e.g. [DJU77, Theorem 9, p.49]) yields: Since f' is Bochner integrable over $[a, b]$, for Lebesgue almost all $t \in [a, b]$,

$$\lim_{h\to 0}(f_{t+h} - f_t)/h = \lim_{h\to 0} \frac{1}{h} \int_t^{t+h} f_s' ds \overset{\substack{\text{Lebesgue}\\ \text{for Bochner}}}{=} f_t'$$

in the $L^1(\nu^\star)$-norm.
This implies that $\lim_{h\to 0} \frac{\nu^{t+h}-\nu^t}{h}$ exists in τ_{tv} and is given by the measure with $\nu^\star$-density f_t':

$$\begin{aligned} 0 &\leq \lim_{h\to 0} \|\frac{f_{t+h}\nu^\star - f_t\nu^\star}{h} - f_t'\nu^\star\|_{tv} \\ &\overset{\text{Lemma 1.0.18}}{=} \lim_{h\to 0} \left\| \frac{f_{t+h} - f_t}{h} - f_t' \right\|_{L^1(\nu^\star)} \overset{\text{Lebesgue}}{\underset{Bochner}{=}} 0 \end{aligned}$$

Thus

$$\tau_{tv} - \lim_{h\to 0}(\nu^{t+h} - \nu^t)/h = f_t'\nu^\star.$$

On the other hand by the weaker differentiability (τ_S or τ_C) of our assumption this measure must be $\nu^{t'^{\tau_C}}$ (cf. Remark 3.3.10). Thus $f_t' = \frac{d\nu^{t'^{\tau_{tv}}}}{d\nu^\star}$ for almost all t and (3.12) implies (3.8).
It remains to prove $\nu^{t'^F} \ll \nu^t$ in the case of τ_S-differentiability. For this we compare the zero sets of f and f'. For each x the set $J := \{t : f_t(x) = 0\}$ is contained in the set $\{t : f_t'(x) = 0\}$ up to a Lebesgue nullset:
Chose t' such that $f_{t'}(x) = 0$

$$\begin{aligned} 0 &= \int_{\mathbb{E}} \int_I \underbrace{\mathbb{1}_{\{t:f_t(x)=0\}}(t) f_t(x)}_{=\mathbb{1}_{\{x:f_t(x)=0\}}(x) f_t(x)} - f_{t'}(x) dt \nu^\star(dx) \\ &\overset{\substack{f.(x)\text{ diff}\\ \text{Fubini}}}{=} \int_I \int_{\mathbb{E}} \underbrace{\mathbb{1}_{\{x:f_t(x)=0\}}(x)}_{=:A} \underbrace{\int_{t'}^t f_s'(x) ds}_{=:B} \nu^\star(dx) dt \end{aligned}$$

If $A = 1$, this implies for Lebesgue-a.e. $t \in \mathbb{R}$ that B is 0 $\nu^\star$-a.e.. Thus for every density point t_0 (where $f_{t_0}(x)$ is differentiable) and every sequence $(t_n)_{n\in\mathbb{N}}$ in J=$\{t : f_t(x) = 0\}$ towards it, we know that

$$\begin{aligned} f_{t_0}(x) - f_{t_n}(x) &= \int_{t_n}^{t_0} f_s'(x) ds \\ \overset{\substack{\text{Bochner}\\ \text{Thm. Integral}}}{\Rightarrow} \quad f_{t_0}' &= \lim_{t_n\to t_0} \frac{f_{t_0}(x) - f_{t_n}(x)}{t_0 - t_n} = 0 \end{aligned} \tag{3.13}$$

Let $\frac{1}{0}$ denote the length of I, i.e. $\frac{1}{0} := sup\{a - b | a, b \in I\}$. We have

$$J = \dot{\bigcup_{n\in\mathbb{N}}} \left\{ t \,\middle|\, \frac{1}{n-1} \geq \|t-y\| > \frac{1}{n} \ \forall y \in J \right\}$$
$$\dot{\cup}\{t | t \text{ density point}\}$$

The sets in the first union are finite, because $I \subset J$ is finite, i.e. $\frac{1}{0} < \infty$. Thus there exist only countably many non density points in the set $I \subset \mathbb{R}$. Hence we have the property (3.13) for Lebesgue-a.e. t. Therefore

$$\{t_0 : f_{t_0}(x) = 0\} \subset \{t_0 : f'_{t_0}(x) = 0\} \ \lambda\text{-a.e. } \nu^\star\text{-a.e}$$

This just means that $\nu^{t'^F} \ll \nu^t$ for a.e. t.

□

The following theorem can be found in [SvW93, Theorem 3.3]:

Theorem 3.5.4.
Suppose that

1. $(\nu^t)_{t\in I}$ *is differentiable for* τ_S *or* τ_C.
2. $\int_a^b \|\nu^{t'^{\tau_C}}\|_{tv} dt < \infty$ *for some a, b in I*
3. *the map* $(t, x) \mapsto {}^{\,t}_{\tau_C}\beta^{\nu^\clubsuit}(x)$ *is* $\mathbb{B}(I) \otimes \mathbb{B}(\mathbb{E})$*-measurable and for Lebesgue-a.e. t:* ${}^{\,t}_{\tau_C}\beta^{\nu^\clubsuit}(\cdot) = \frac{d\nu^{t'^{\tau_C}}}{d\nu^t}$
4. $\int_a^b |{}^{\,t}_{\tau_C}\beta^{\nu^\clubsuit}(x)| dt < \infty$ *holds* $|\nu^a| + |\nu^b|$*-a.e.*

then all measures ν^t, $a \leq t \leq b$, *are equivalent and*

$$\frac{d\nu^t}{d\nu^a}(x) = \exp\left(\int_a^t {}^{\,s}_{\tau_{tv}}\beta^{\nu^\clubsuit}(x) ds\right). \tag{3.14}$$

Proof.
Instead of a differential equation we consider the corresponding integral equation and apply Gronwall:

Because of assumptions 1 to 3 we may apply Proposition 3.5.3 and gain a dominating probability measure $\nu^\star$ and two $\mathbb{B}([a, b]) \otimes \mathbb{B}(\mathbb{E})$-measurable functions f and f' such that for $\nu^\star$-a.e $x \in \mathbb{E}$ the following holds λ-a.e.:

$$f_t(\cdot)\, {}^{\,t}_{\tau_{tv}}\beta^{\nu^\clubsuit}(\cdot) = \frac{d\nu^t}{d\nu^\star} \frac{d\nu^{t'^{\tau_{tv}}}}{d\nu^t} = \frac{d\nu^{t'^{\tau_{tv}}}}{d\nu^\star} = f'_t(\cdot) \ \lambda\text{-a.e,} \tag{3.15}$$

where $f_t = \frac{d\nu^t}{d\nu^\star}$ and $f_t' = \frac{d\nu^{t'\tau_{tv}}}{d\nu^\star}$.

Using this in the integral of (3.7) in Proposition 3.5.3 we get for $\nu^\star$-a.e. $x \in \mathbb{E}$ and all $t \in I$ the relation

$$f_t(x) - f_a(x) = \int_a^t f_s(x)\, {}^{s}_{\tau_{tv}}\beta^{\nu^\clubsuit}(x)ds \tag{3.16}$$

Moreover, assumption 4 and Remark 3.3.10 imply that for $\nu^\star$-a.e. $x \in \mathbb{E}$

$$f_a(x) = f_b(x) = 0 \text{ or } \int_a^b |\, {}^{s}_{\tau_C}\beta^{\nu^\clubsuit}(x)|ds < \infty.$$

Together with Gronwall (cf. [Ama90, p.89]) and the last formula this shows that

$$f_t(x) = f_a(x) \exp\left(\int_a^t {}^{s}_{\tau_{tv}}\beta^{\nu^\clubsuit}(x)ds\right) \quad \forall t \in [a, b]. \tag{3.17}$$

for $\nu^\star$-a.e. x for which either $f_a(x) \neq 0$ or $f_b(x) \neq 0$. Since $\int_a^b |\, {}^{t}_{\tau_C}\beta^{\nu^\clubsuit}(x)|dt < \infty$ equation (3.17) implies that f_a and f_t vanish on $\nu^\star$-almost the same points. Thus the measure ν^a and ν^t are equivalent and

$$\frac{d\nu^t}{d\nu^a}(x) = \frac{f_t}{f_a}(x) = \exp\left(\int_a^t {}^{s}_{\tau_{tv}}\beta^{\nu^\clubsuit}(x)ds\right) \tag{3.18}$$

□

This theorem can be found in [SvW93, Theorem 6.2].

Theorem 3.5.5.
Let $(\{\gamma_t : t \in \mathbb{R}\}, \circ)$ be a group of bimeasurable bijections of $(\mathbb{E}, \mathbb{B}(\mathbb{E}))$. Define $\forall t \in \mathbb{R} : \nu^t(A) = \nu({\gamma_t}^{-1}(A))\ \forall A \in \mathbb{B}(\mathbb{E})$. Suppose that for some point s, the family $\{\nu^t\}_{t\in\mathbb{R}}$ is

1. *τ_C-differentiable at s with $\nu^{s'\tau_C} \ll \nu^s$ and*
2. *$\phi \circ \gamma_t \in C$ for all $t \in \mathbb{R}$ and all $\phi \in C$, C being norm-defining*

Then

1. *the family itself is τ_{tv}-differentiable on $I = \mathbb{R}$,*
2. *$\nu^{t'\tau_{tv}} \ll \nu^t$ and*
3. *${}^{t}_{\tau_{tv}}\beta^{\nu^\clubsuit}(x) = {}^{0}_{\tau_{tv}}\beta^{\nu^\clubsuit}(\gamma_{-t}(x))$ ν-a.e. for all t.*

Remark 3.5.6.
Of course, we can formulate the above theorem for $I \subset \mathbb{R}$ open, $0 \in I$ and the proof still holds. One just has to take care that all expressions are well-defined. E.g. the group condition is replaced by $\forall s, t \in I : s+t \in I$ we have $\gamma_{t+s} = \gamma_t \circ \gamma_s$. For the sake of clarity we prove it for $I = \mathbb{R}$.

Proof of Theorem 3.5.5.
First of all we show the following
Claim: for all $t \in \mathbb{R}$ the derivative $\nu^{t'\tau_C}$ exists in any of the topologies $\tau_C, \tau_S, \tau_{tv}$ and equals $\nu^{s'\tau_C} \circ {\gamma_{t-s}}^{-1}$, whenever $\nu^{s'\tau_C}$ exists for some s in the topologies.
Proof.
First we prove the claim for τ_C and τ_S: Choose $\phi \in C$ or $\phi = \mathbb{1}_B$, where $B \in \mathbb{B}(\mathbb{E})$. Using the group property of the flow it follows that

$$\begin{aligned}\int \phi d\nu^{t'\tau_C} &= \lim_{h\to 0} \frac{1}{h} \int \phi d(\nu^{t+h} - \nu^t) \\ &\overset{\substack{\text{group}\\ \text{property}}}{=} \lim_{h\to 0} \frac{1}{h} \int \phi \circ \gamma_{t-s} d(\nu^{s+h} - \nu^s) \\ &= \int \phi \circ \gamma_{t-s} d(\nu^{s'\tau_C}) = \int \phi d(\nu^{s'\tau_C} \circ {\gamma_{t-s}}^{-1}), \end{aligned} \tag{3.19}$$

where we used the τ_S respectivly τ_C-differentiability and that $\phi \circ \gamma_{t-s} \in C$. Thus (3.19) implies the claim for τ_S and τ_C.

Thus it remains to prove the claim for τ_{tv}. By (3.19) the norm $\|\nu^{t'\tau_C}\|_{tv}$ is constant for all t. As in Remark 3.2.4 we see that $\|\nu^{s'\tau_C}\|_{tv}$ is finite:

$$\|\nu^{s'\tau_C}\|_{tv} = |\sup_{\|\phi\|_\infty \le 1} \int_{\mathbb{E}} \phi d\nu^{s'\tau_C}| = |\sup_{\|\phi\|_\infty \le 1} \int_{\mathbb{E}} \phi(x) \, {}^{s}_{\tau_C}\beta^{\nu^\clubsuit}(x) \nu^s(dx)|$$

$$\le \int_{\mathbb{E}} |{}^{s}_{\tau_C}\beta^{\nu^\clubsuit}(x)| |\nu^s|(dx) \overset{{}^{s}_{\tau_C}\beta^{\nu^\clubsuit} \in L^1(\nu^s)}{\underset{\text{cf. Def. 3.3.2}}{<}} \infty$$

Thus we even gain that $\|\nu^{t'\tau_C}\|_{tv}$ is locally integrable.
Furthermore $\nu^{s'\tau_C} \ll \nu^s$ for some s implies $\nu^{s+t'\tau_C} \ll \nu^{s+t}$ for all t: We have

$$\sup\{A | A \in \mathbb{B}(\mathbb{E})\} = \sup\{\gamma_{t-s}^{-1}(A) | A \in \mathbb{B}(\mathbb{E})\}, \tag{3.20}$$

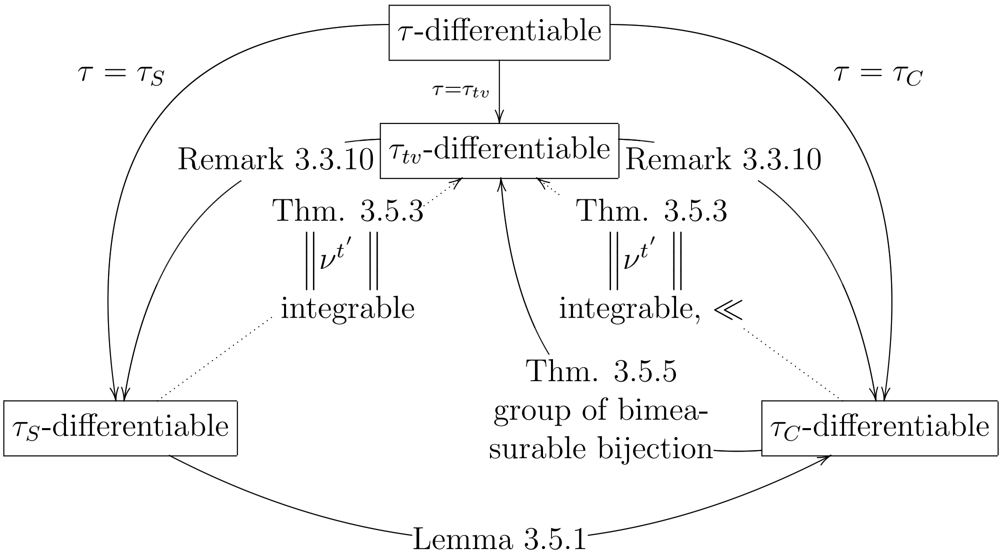
because γ_t is a bijection, which is bimeasurable. Thus we have

$$\begin{aligned}\|\nu^{t'\tau_C}\|_{tv} &\overset{(3.19)}{=} \|\nu^{s'\tau_C} \circ {\gamma_{t-s}}^{-1}\|_{tv} \overset{(3.20)}{=} \|\nu^{s'\tau_C}\|_{tv} \\ &\ll \|\nu^s\|_{tv} \overset{(3.20)}{=} \|\nu^s \circ \gamma_{t-s}\|_{tv} = \|\nu^t\|_{tv} \end{aligned} \tag{3.21}$$

Hence the assumptions of Proposition 3.5.3 are fulfilled and the family is τ_{tv}-differentiable at Lebesgue-a.e. t. Using (3.20) for $\nu^{t'\tau_{tv}}$ this is even true for all t. □

3.5.2 Connections of τ-, τ_{tv}-, τ_C- and τ_S-differentiable

We summarize some of the proven relations in the following graphic.



Chapter 4

Applicability of concepts

This chapter is a key step to obtain properties, with which we will prove the key properties. In Chapter 3 we postulated that the set C_b^1 is norm-defining. In this chapter this is proved. Moreover we give a condition, such that the existence of the derivative along a vector field $h : \mathbb{E} \to \mathbb{H}$ can be checked, i.e. we give a condition for the existence of the local flow.

In Section 4.1 we present the proof for the existence of a norm-defining set with the postulated properties.

In Section 4.2 we derive a condition for mappings such that the existence of the derivative along this mapping of a measure can be checked. If a mapping has this property and if the derivative along this mapping exists, it is well defined. For this end we apply methods of the theory of evolutionary equations.

In Section 4.3 we prove an adaption of the Lebesgue Dominated Convergence Theorem. We deduct a corollary, which we will use to prove the Key Proposition. Furthermore we derive a nice integration property of a local flow and an element of the norm-defining set $\tilde{C}_b^1$.

In Section 4.4 the correspondence between C_b^1-differentiability and the $\tau_{C_b^1}$-differentiability is established.

In [SvW95] only one direction of Proposition 4.4.1 was stated, though both are needed for the proof of the results. Neither the ideas of the proofs nor the conditions mentioned in this chapter have been indicated in [SvW95].

4.1 C_b^1 is norm-defining

Now we will show that C_b^1 is indeed a norm-defining set. First of all, we will summarize a few properties of C_b^1. The following assertions are used to prove that C_b^1 is norm-defining (cf. Theorem 4.1.15). The set $\tilde{C}_b^1$ is heavily used in Chapter 5.

Definition 4.1.1 ($\tilde{C}^1$,$\tilde{C}^1(F, G; F_h, G_H)$)**.**
Let $(F, \| \ \|_F)$ *and* $(G, \| \ \|_G)$ *be Banach spaces and* $F_H \subset F$ *and* $G_H \subset G$ *be linear subspaces, such that* $(F_H, \| \ \|_{F_H})$ *respectively* $(G_H, \| \ \|_{G_H})$ *are Banach spaces, where* $\| \ \|_F \leq \| \ \|_{F_H}$ *and* $\| \ \|_G \leq \| \ \|_{G_H}$. *We define*

$$\tilde{C}^1(F, G; F_H, G_H) := \Big\{ \phi : F \to G \Big| \ \phi \text{ is continuously Gâuteaux-differentiable} \\ \text{w.r.t. } (F_H, \| \ \|_F)\big(\to (G_H, \| \ \|_G)\big), \text{ i.e. } D\phi(x) : (F_H, \| \ \|_F) \to (G_H, \| \ \|_G) \Big\}$$

If $G = G_H = \mathbb{R}$, *we abbreviate* $\tilde{C}^1(F) := \tilde{C}^1(F; F_H) := \tilde{C}^1(F, \mathbb{R}; F_H, \mathbb{R})$. *Moreover we set* $\tilde{C}^1 := \tilde{C}^1(\mathbb{E}, \mathbb{H})$.

Definition 4.1.2 ($\tilde{C}^1_b$)**.**
We define

$$\tilde{C}^1_b(F, G; F_H, G_H) := \{ \phi \in \tilde{C}^1(F, G; F_H, G_H) \big| \ \phi \text{ is bounded and} \\ \exists \tilde{M}_\phi < \infty : \ \forall x \in F, \ \forall h \in F_H : \| \phi'_h(x) \|_{G_H} \leq \tilde{M}_\phi \, \| h \|_{F_H} \}$$

If $G = G_H = \mathbb{R}$, *we abbreviate* $\tilde{C}^1_b(F) := \tilde{C}^1_b(F; F_H) := \tilde{C}^1_b(F, \mathbb{R}; F_H, \mathbb{R})$. *Moreover we set* $\tilde{C}^1_b := \tilde{C}^1_b(\mathbb{E}, \mathbb{H})$.

Remark 4.1.3.
We note that that $C^1_b(F) \subset \tilde{C}^1_b(F; F_H)$ *and if* $F_H = F$, *then* $\tilde{C}^1_b(F) = C^1_b(F)$. *In chapter 5* F_H *will be the Hilbert subspace* $\mathbb{H}$ *(cf. Definition 4.1.1).*

Definition 4.1.4.
Abbreviating $\| \phi \|_\infty := \sup_{x \in F} \| \phi(x) \|$ *for* $\phi \in \tilde{C}^1_b(F, G; F_H, G_H)$, *we know that there exist* $M_\phi < \infty$ *and* $\tilde{M}_\phi < \infty : \ \forall h \in F_H$

$$\begin{aligned} \| \phi \|_\infty = \| \phi \|_{\infty, \ G} &\leq M_\phi \\ \| \phi'_h \|_\infty = \| \phi \|_{\infty, \ G_H} &\leq \tilde{M}_\phi \, \| h \|_F \end{aligned}$$

For this subsection, we fix this notation for any function $\phi \in \tilde{C}^1_b(F, G; F_H, G_H)$.

Lemma 4.1.5.
The set $\tilde{C}^1_b(F, F_H)$ *is closed under multiplication.*

Proof.
Let $f, g \in \tilde{C}^1_b(F)$. We obtain $\| fg \|_\infty \leq M_f M_g$
and $\big\| (fg)'_h \big\|_\infty = \| f'_h g + f g'_h \|_\infty \leq \tilde{M}_f \, \| h \|_{F_H} M_g + M_f \tilde{M}_g \, \| h \|_{F_H} \quad \forall h \in F_H.$ □

Proposition 4.1.6.
Let $\phi \in \tilde{C}^1_b(G; G_H)$ *and* $f \in \tilde{C}^1_b(F, G; F_H, G_H)$, *then* $\phi \circ f \in \tilde{C}^1_b(F; F_H)$.

Remark 4.1.7.
Note, that it is not necessary that f is bounded.

Proof of Proposition 4.1.6.
$\|\phi \circ f\|_\infty < M_\phi$ and $\forall h \in F_H$

$$\left\|(\phi \circ f)_h'\right\|_\infty = \left\|\phi_{f_h'}'(f)\right\|_\infty \leq \tilde{M}_\phi \left\|f_h'\right\|_\infty \leq \tilde{M}_\phi \tilde{M}_f \left\|h\right\|_{F_H}. \qquad \square$$

Remark 4.1.8.
We notice that the last two assertions hold for $\mathfrak{F}\tilde{C}_b^1(F)$ as well: For $f, g \in \mathfrak{F}\tilde{C}_b^1(F)$ we have $f = \tilde{f}(l_1^f, \ldots, l_{n_f}^f)$, where $\tilde{f} \in \tilde{C}_b^1(\mathbb{R}^{n_f})$ and $l_i^f \in F'$ $\forall 1 \leq i \leq n_f$, and $g = \tilde{g}(l_1^g, \ldots, l_{n_g}^g)$, where $\tilde{g} \in \tilde{C}_b^1(\mathbb{R}^{n_g})$ and $l_i^g \in F'$ $\forall 1 \leq i \leq n_g$. We define

$$\begin{aligned} \tilde{C}_b^1(\mathbb{R}^{n_f+n_g}) \ni f' &:= f(l_1^f, \ldots, l_{n_f}^f, 0, \ldots, 0) \quad \text{and} \\ \tilde{C}_b^1(\mathbb{R}^{n_f+n_g}) \ni g' &:= g(0, \ldots, 0, l_1^g, \ldots, l_{n_g}^g). \end{aligned}$$

Thus we may apply the results for $f', g' \in \tilde{C}_b^1(\mathbb{R}^{n_f+n_g})$.

Lemma 4.1.9.
There exists $\chi_n \in C_b^1(\mathbb{R}), \chi_n : \mathbb{R} \to \mathbb{R}$, with
$\chi_n|_{[-1,1]} = id$, $\left|\chi_n|_{[-1-\frac{1}{n}, 1+\frac{1}{n}]}\right| \leq 1 + \frac{1}{n}$ *and* $\left|\chi_n|_{[-1-\frac{1}{n}, 1+\frac{1}{n}]^c}\right| = 1 + \frac{1}{n}$.

Proof.
We know that for

$$\begin{aligned} g \colon \mathbb{R} &\to \mathbb{R} \\ x &\mapsto \mathbb{1}_{\mathbb{R}^-}(x)x + \mathbb{1}_{[0,\frac{1}{n}]}(x)\left(x + nx^2 - n^2x^3\right) + \mathbb{1}_{]\frac{1}{n},\infty[}(x)\frac{1}{n} \end{aligned}$$

we have $g(0) = 0$, $g'(0) = 1$, $g(\frac{1}{n}) = \frac{1}{n}$, and $g'(\frac{1}{n}) = 0$. We define

$$\chi_n(x) := \mathbb{1}_{\mathbb{R}^+}(x)\frac{1}{3}(1 + g(x-1)) - \mathbb{1}_{\mathbb{R}^-}(x)\frac{1}{3}(1 + g(-x-1)) \qquad (4.1)$$

and show that $\chi_n \in C_b^1(F)$. We have $g \leq \frac{3}{n} \Rightarrow \|\chi_n\|_\infty \leq 1 + \frac{3}{3n}$ and $\|g_h'\|_\infty \leq (5n^2+1)|h| \Rightarrow \|\chi_n{}_h'\|_\infty \leq (5n^2+1)|h|$. The other properties of χ_n are clear. $\square$

4.1.1 C_b^1 is norm defining

The main idea of the proof that C_b^1 is norm-defining is to consider a bigger set of functions and then show that the supremum remains the same. For this end we show that $\mathfrak{F}C_b^1$ is dense w.r.t. $L^1(\nu)$ in $L^\infty(\nu)$ by a monotone class argument.

We use Parthasarathy (Theorem 4.1.13) for this proof. In order to understand it, we will have to introduce some definitions, which can be found in [Pat67, Definiton I.1.2, p.6; Definition V.2.1, p.132].

Definition 4.1.10 (Borel space)**.**
A Borel space $(X, \mathbb{B})$ *is a pair, where* X *is an abstract set and* $\mathbb{B}$ *is a* σ*-algebra of subsets of* X*.*

We remind ourselves of

Definition 4.1.11 (denumerable)**.**
A set or a class of sets $\mathbb{D}$ *is called denumerable, if there exists a surjective map* $j : \mathbb{N} \to \mathbb{D}$*.*

Definition 4.1.12 (countably generated, separable)**.**
A Borel space $(X, \mathbb{B})$ *is said to be countably generated if there exists a denumerable class* $\mathbb{D} \subset \mathbb{B}$ *such that* $\mathbb{D}$ *generates* $\mathbb{B}$*.* $(X, \mathbb{B})$ *is called separable if it is countably generated and for each* $x \in X$*, the single point set* $\{x\} \in \mathbb{B}$*.*

By [Pat67, Theorem V.2.4, p.135] we have:

Theorem 4.1.13 (Parthasarathy)**.**
Let X *and* Y *be separable metric spaces. Let* X *be complete and* $(X, \mathbb{B}_X)$*,* $(Y, \mathfrak{A})$ *be separable Borel spaces. Moreover suppose that* ϕ *is a one-to-one map of* X *into* Y*, which is measurable.*
Then $Y' := \phi(X)$ *is a Borel subset of* Y *and* ϕ *is an isomorphism between the spaces* $(X, \mathbb{B}_X)$ *and* $(Y', \mathfrak{A}_{Y'})$*, where* $\mathfrak{A}_{Y'} := \{A \cap Y' \mid A \in \mathfrak{A}\}$*.*

By [Sie00, p.115] we have

Proposition 4.1.14 (Lindelöf property)**.**
Let (E, d) *be a metric space, then* E *possesses the Lindelöf property iff* E *is separable. This means that every set* $A \subset E$ *has the Lindelöf property, i.e. for every aggregate of open sets whose sum contains* A *there exists a countable (or even finite) aggregate of these sets whose union contains* E*.*

Theorem 4.1.15.
The set $\mathfrak{F}C_b^1 \subset C_b^1$ *is norm-defining.*

Remark 4.1.16.
We note that $C_b^1 \subset \tilde{C}_b^1 \subset C_b$ *and thus Theorem 4.1.15 implies that* $\tilde{C}_b^1$ *is norm-defining (cf. Remark 3.2.2).*

Proof of Theorem 4.1.15.
To prove

$$\|\nu\|_{tv} = \int_{\mathbb{E}} \mathbb{1}_{\mathbb{E}^+} d\nu + \int_{\mathbb{E}} \mathbb{1}_{\mathbb{E}^-} d\nu = \sup\left\{ \int_{\mathbb{E}} \phi d\nu, \|\phi\|_\infty \leq 1, \phi \in \mathfrak{F}C_b^1 \right\}$$

we use

1. **Claim**: Let ν be a positive, finite measure. Then any function in $L^\infty(\nu)$ can be approximated by functions from $\mathfrak{F}C_b^1$ in $\| \ \|_{L^1(\nu)}$. I.e. $\mathfrak{F}C_b^1$ is dense (w.r.t. $\| \ \|_{L^1(\nu)}$) in $L^\infty(\nu)$.
   Proof:
   We use a monotone class argument(cf. [Röc05b, Definition 1.11.7, Satz 1.11.11, p.54f] or [Pro04, I Theorem 8]):
   On the one hand $\mathfrak{H} := \overline{\mathfrak{F}C_b^1}^{L^1(\nu)} \subset L^1(\nu)$ is a monotone vector space, i.e.
   (a) $1 \in \mathfrak{H}$
   (b) Let $f_n, n \in \mathbb{N}$ be a sequence in $\mathfrak{H}$ such that $0 \le f_1 \le f_2 \le \cdots \le f_n \nearrow f$ and f bounded. We have to prove that $f \in \mathfrak{H}$. We show that by the Lebesgue dominated convergence theorem and a diagonal argument there exists a sequence $g_n \in \mathfrak{F}C_b^1$ such that $f = L^1(\nu) - \lim_{n\to\infty} g_n \in \mathfrak{H}$: Since $f_n (\in L^1(\nu))$ converges pointwisely monotone increasing to f and are bounded by $f \in L^\infty(\nu) \subset L^1(\nu)$, the Lebesgue dominated convergence theorem gives us that

   $$L^1(\nu) - \lim_{n\to\infty} f_n = f$$

   and thus w.l.o.g (eventually we have to consider a subsequence) we know that $\|f_n - f\|_{L^1(\nu)} \le \frac{1}{n}$. For each $f_n \in \mathfrak{H}$ there exists a sequence $g_{n,m} \in \mathfrak{F}C_b^1$ such that $f_n = L^1(\nu) - \lim_{m\to\infty} g_{n,m}$. Furthermore w.l.o.g for all $m \in \mathbb{N} : m \ge n$ we have $\|f_n - g_{n,m}\|_{L^1(\nu)} \le \frac{1}{n}$.
   Defining $g_n := g_{n,n}$ we obtain

   $$\|g_n - f\|_{L^1(\nu)} \le \|g_{n,n} - f_n\|_{L^1(\nu)} + \|f_n - f\|_{L^1(\nu)} \le \frac{2}{n}$$

   and hence $f \in \mathfrak{H}$.

   On the other hand we know by Lemma 4.1.5 and Remark 4.1.8 that $\mathfrak{M} := \mathfrak{F}C_b^1$ is a set of bounded functions, which is closed under multiplication. Then (by monotone classes, e.g. [Röc05b, Satz 11.1.11] or [Pro04, I Theorem 8]) $\sigma(\mathfrak{F}C_b^1)_b \subset \mathfrak{H}$. $\sigma(\mathfrak{F}C_b^1)_b$ denotes the set of all bounded, $\sigma(\mathfrak{F}C_b^1)$-measurable functions.

   We will prove that

   $$\sigma(\mathfrak{F}C_b^1) = \mathbb{B}(\mathbb{E}) \tag{4.2}$$

   Thus

   $$\mathfrak{F}C_b^1 \subset L^\infty(\nu) \overset{(4.2)}{=} \sigma(\mathfrak{F}C_b^1)_b \overset{\substack{\text{monoton}\\ \text{classes}}}{\subset} \mathfrak{H} = \overline{\mathfrak{F}C_b^1}^{L^1(\nu)} \tag{4.3}$$

and hence we are done, because the indicator functions are measurable.

Proof of (4.2).

(a) **Claim**: There exists $\{f_m\}_{m\in\mathbb{N}} \subset \mathfrak{F}C_b^\infty \subset \mathfrak{F}C_b^1 \subset C_b^1$, which separates the points of $\mathbb{E}$.
Proof.
Choose $l \in \mathbb{E}'$ and define: $(\sin l, \sin l), (\cos \pi l, \cos \pi l) : \mathbb{E} \to \mathbb{R}^2$. Then the union of open sets

$$\begin{aligned} &\bigcup_{l\in\mathbb{E}'} (\sin l, \sin l)^{-1}(\mathbb{R}^2\backslash\triangle_{\mathbb{R}}) \cup (\cos \pi l, \cos \pi l)^{-1}(\mathbb{R}^2\backslash\triangle_{\mathbb{R}}) \\ \stackrel{!}{=} \ &\mathbb{E}\times\mathbb{E}\backslash\triangle_{\mathbb{E}} \end{aligned} \tag{4.4}$$

is an open set, where $\triangle_{\mathbb{R}} := \{(x,x)|x\in\mathbb{R}\}$ and $\triangle_{\mathbb{E}} := \{(u,u)|u\in\mathbb{E}\}$.
$\subset$: We have for $u = v \in \mathbb{E}$ that $(\sin lu, \sin lu) \in \triangle_{\mathbb{R}}$ and
$(\cos lu, \cos lu) \in \triangle_{\mathbb{R}}$. Thus (u,u) is not an element of the l.h.s.. Therefore (by counter proposition) the l.h.s. is a subset of the r.h.s..
$\supset$: If $u \neq v \in \mathbb{E}$, then we know by Hahn Banach (cf. [Wer05, Theorem III.2.4, P.103]), that there exists an l, which separates u and v, i.e. $l(u) \neq l(v)$. Thus

$$\begin{aligned} sin(l(u)) &= \sin(l(v)) \text{ iff } l(u) - l(v) \in 2\pi\mathbb{Z} \\ \cos(\pi l(u)) &= \cos(\pi l(v)) \text{ iff } l(u) - l(v) \in 2\mathbb{Z} \end{aligned}$$

Hence $\sin l(u) \neq \sin l(v)$ or $\cos l(u) \neq \cos l(v)$ and thus the superset property is proved.

We use that $\mathbb{E}\times\mathbb{E}$ is separable and thus by the Lindelöf property (cf. Proposition 4.1.14) the l.h.s. of (4.4) can be represented by a countable union. Furthermore sin, $\cos(\pi\cdot) \in \mathfrak{F}C_b^\infty(\mathbb{E})$ and the (1a) Claim is proved.

(b) **Claim**: $\sigma(\mathfrak{F}C_b^\infty) = \mathbb{B}(\mathbb{E})$
Proof.
$\mathfrak{A} := \sigma(\{f_n \mid n \in \mathbb{N}\})$ is countable generated, where the f_n are as in (1a). Consider

$$\begin{aligned} \mathrm{id}\colon (\mathbb{E}, \mathbb{B}(\mathbb{E})) &\to (\mathbb{E}, \mathfrak{A}) \\ A &\mapsto A \end{aligned}$$

On the one side, the function id is one-to-one and measurable, because the f_n are continuous. On the other side, we know that $(\mathbb{E}, \mathfrak{A})$ is even a separable Borel space (cf. Definition 4.1.12), because

$$\forall x \in \mathbb{E}\colon \{x\} \stackrel{!}{=} \bigcap_n \{f_n = f_n(x)\} \in \mathfrak{A}$$

The subset inclusion is obvious. x is the only element in the intersection, because the set $\{f_n \mid n \in \mathbb{N}\}$ is point separating (cf. Claim 1a), and thus for every $y \neq x$ there exists a $n' \in \mathbb{N}$, such that $f_{n'}(y) \neq f_{n'}(x)$.
Thus by Theorem 4.1.13 id^{-1} is an isomorphism and hence $\mathbb{B}(\mathbb{E}) = \mathfrak{A}$.
Therefore we obtain $\mathbb{B}(\mathbb{E}) = \mathfrak{A} \subset \sigma\{\mathfrak{F}C_b^\infty\} \subset \mathbb{B}(\mathbb{E})$.

2. **Claim**:

$$\sup_{\tilde{\phi}_n \in \mathfrak{F}C_b^1, \left\|\tilde{\phi}_n\right\|_\infty \leq 1} \int \tilde{\phi}_n d\nu = \sup_{f \in L^\infty(|\nu|), \|f\|_\infty \leq 1} \int f d\nu$$

Proof:
The l.h.s. is less or equal to the r.h.s., because $\mathfrak{F}C_b^1 \subset L^\infty(|\nu|)$ (cf. (4.3)) and thus the supremum is taken over more functions. Let $f \in L^\infty(|\nu|)$, $\|f\|_\infty \leq 1$ and $\varepsilon > 0$, then by the 1. Claim there exists $\phi_1 \in \mathfrak{F}C_b^1$:

$$\int |f - \phi_1| \, d|\nu| < \frac{\varepsilon}{2}$$

Thus

$$\begin{aligned} |\int f - \phi_1 d\nu| &\overset{\triangle}{\leq} \int |f - \phi_1| \mathbb{1}_{\mathbb{E}^+} d\nu - \int |f - \phi_1| \mathbb{1}_{\mathbb{E}^-} d\nu \\ \leq & \int |f - \phi_1| \, d|\nu| < \frac{\varepsilon}{2} \end{aligned}$$

In addition by Lemma 4.1.9 we have a function $\chi_n \in \mathfrak{F}C_b^1(\mathbb{R})$, with $\chi_n|_{[-1,1]} = id$, $|\chi_n\Big|_{[-1-\frac{1}{n}, 1+\frac{1}{n}]}| \leq 1 + \frac{1}{n}$ and $\chi_n|_{[-1-\frac{1}{n}, 1+\frac{1}{n}]^C} = 1 + \frac{1}{n}$. Hence, using Proposition 4.1.6, we obtain $\phi_n := \chi_n(\phi_1) \in \mathfrak{F}C_b^1$. We consider $\tilde{\phi}_n := \frac{\phi_n}{1+\frac{1}{n}}$ and have

$$\int |f - \tilde{\phi}_n| d|\nu| \quad \overset{\triangle}{\leq} \quad \underbrace{\int |f - \phi_n| d|\nu|}_{\underset{n\to\infty}{\to} \frac{\varepsilon}{2}} + \underbrace{\int (1 - \frac{1}{1+\frac{1}{n}}) \overbrace{|\phi_n|}^{\leq 1+\frac{1}{n}} d|\nu|}_{\leq \frac{1}{n} \, \|\nu\|_{tv}} \underset{n\to\infty}{\to} \frac{\varepsilon}{2}$$

Thus w.l.o.g there exists $\tilde{\phi}_n \in \mathfrak{F}C_b^1 : \left\|\tilde{\phi}_n\right\|_\infty \leq 1$ and $\int |f - \tilde{\phi}_n| d|\nu| \leq \frac{\varepsilon}{2}$.
Therefore the claim is proved (Using that there exist sup approximating sequences in $L^\infty(|\nu|)$).

With the last claim we see:

$$\begin{aligned}\|\nu\|_{tv} &= \int \mathbb{1}_{\mathbb{E}^+} - \mathbb{1}_{\mathbb{E}^-} d\nu \leq \sup_{f\in L^\infty(|\nu|),\|f\|_\infty\leq 1} \int f d\nu \\ &\overset{2.\text{ claim}}{=} \sup_{\phi\in\mathfrak{F}C_b^1,\|\phi\|_\infty\leq 1} \int \phi d\nu \leq \int 1 d|\nu| = \|\nu\|_{tv}\end{aligned}$$

□

Remark 4.1.17.
We have not included that $\mathfrak{F}C_b^\infty$ is norm-defining, because it is not needed to prove the key results, in fact, if we restrict the norm defining set C, we would have to restrict the functions, for which the key results are applicable (cf. Chapter 5 and Definition 5.1.1) and we did not find a good reference of a nice representation for $\chi_n \in \mathfrak{F}C_b^\infty$.

4.2 A condition to check τ-differentiability

In this section we give a condition, under which exactly one local flow a (as used in Definition 3.3.7) exists (compare the properties 1 to 3 and 5 of Definition 4.2.1).

First of all we define the set *Sol*, which is motivated by a condition for functions, for which we like to apply our Main Theorem. Then we state the known assertions in Proposition 4.2.4. After doing a few preparations like a generalization of the Inverse Mapping and Implicit Mapping Theorem we demand a stronger condition, which is sufficent for a vector field h being an element of *Sol* (cf. Theorem 4.2.11).

Compare Proposition 5.2.1, where we use the following:

Definition 4.2.1 (Sol , local flow).
Let Sol denote the set of all $f : \mathbb{E} \to \mathbb{H}$ continuous such that there exists exactly one local flow a, i.e. $\exists\varepsilon > 0,\ a :]0-\varepsilon, 0+\varepsilon[\times\mathbb{E} \to \mathbb{E} : \forall t \in]-\varepsilon, \varepsilon[, \forall\tilde{x} \in \mathbb{E}$:

1. *$a(t_1 + t_2, \tilde{x}) = a(t_1, a(t_2, \tilde{x}))$*
2. *$a(0, \tilde{x}) = \tilde{x}$*
3. *$a(t, \tilde{x})$ is differentiable in t and $D_1a(t, \tilde{x}) = f(a(t, \tilde{x}))$*
4. *$\exists\tilde{M} < \infty : f(x) \leq \tilde{M}\ \forall x \in \mathbb{E}$*
5. *The derivative of $a(t, \tilde{x})$ in $\tilde{x}$ exists and is bounded, i.e. defining $\gamma_t(\tilde{x}) := a(t, \tilde{x})$ we have $\gamma_t \in \tilde{C}^1$ and $\exists M_t < \infty$:*

$$\|{\gamma_t}'_h(\tilde{x})\|_{\mathbb{H}} \leq M_t \|h\|_{\mathbb{H}} \quad \forall h \in \mathbb{H}.$$

Example 4.2.2.
If $h \in \mathbb{H}$ is constant, all the conditions are fulfilled by choosing the unique solution

$$\begin{array}{rcl} a\colon \mathbb{R}\times\mathbb{E} & \to & \mathbb{E} \\ (t,\tilde{x}) & \mapsto & \tilde{x}+th \end{array}$$

Definition 4.2.3 (Lipschitz)**.**
A continuous function $f:\mathbb{E}\to\mathbb{E}$ is called (globally) Lipschitz continuous with Lipschitz constant $L<\infty$, if

$$\|f(x+h)-f(x)\|_{\mathbb{E}} \le L\,\|h\|_{\mathbb{E}} \;\; \forall x,h\in\mathbb{E}$$

By [Lan93, chapter XIV] we conclude

Proposition 4.2.4.
If $f:\mathbb{E}\to\mathbb{E}$ is Lipschitz with Lipschitz constant $L<\infty$, bounded and continuously differentiable, then there exists exactly one solution

$$a:]-\frac{1}{2L},\frac{1}{2L}[\times\mathbb{E}\to\mathbb{E},$$

which fulfills the conditions 1 to 4 of Definition 4.2.1 and for which $a(t,\cdot)$ is continuously differentiable for all $t\in]-\frac{1}{2L},\frac{1}{2L}[$.

Proof.
We abbriviate $J:=]-\frac{1}{2L},\frac{1}{2L}[$. By [Lan93, Theorem XIV.3.1, p.367] we obtain $a:J\times\mathbb{E}\to\mathbb{E}$ such that

$$a(0,\tilde{x})=\tilde{x} \text{ and } D_1a(t,\tilde{x})=f(a(t,\tilde{x})) \;\; \forall t\in J,\tilde{x}\in\mathbb{E}.$$

Taking a closer look at the proof (which is the standard one using the integral representation and finding a fixed point by the shrinking lemma or Banach fixed point theorem) we find using $\exists\tilde{M}\in\mathbb{R}:\sup_{x\in\mathbb{E}}|f(x)|<\tilde{M}$

$$\begin{aligned} \frac{1}{s}\left(a(s,x)-a(0,x)\right) &= \frac{1}{s}\left(x+\int_0^s f(a(\tau))d\tau - x - \int_0^0 f(a(\tau))d\tau\right) \\ = \;\; & \frac{1}{s}\int_0^s \underbrace{f(a(\tau))}_{\le\tilde{M}}d\tau \le \tilde{M} \;\; \forall s\in J,\forall x\in\mathbb{E} \end{aligned}$$

Furthermore, by [Lan93, Theorem XIV.5.1, p.377] we conclude

$$a(t_1+t_2,\tilde{x})=a(t_1,a(t_2,\tilde{x})) \qquad \forall x\in\mathbb{E},t_1,t_2,t_1+t_2\in J$$

In addition [Lan93, Theorem XIV.5.2, p.377] tells us that $a(t,\tilde{x})$ is continuously differentiable in $\tilde{x}$ for $t\in J$. □

By Proposition 4.2.4 we gain almost all desired conditions. We demand more to gain a condition, under which a function $f : \mathbb{E} \to \mathbb{H} \subset \mathbb{E}$ belongs to the set *Sol*. Following the proof of [Lan93, Theorem XIV.4.3, p.373] and transforming it appropriately we will obtain this result. As a preparation for finding the desired condition, we adapt the Inverse Mapping Theorem to our needs. We introduce

Definition 4.2.5 (toplinear)**.**
A map is called toplinear, if it is invertible as a continuous linear map.

Proposition 4.2.6 (Inverse Mapping Theorem)**.**
Let $\phi \in \tilde{C}^1(E, F; E_H, F_H)$, $\phi(E_H) \subset F_H$, ϕ be continuously differentiable w.r.t. $(E_H, \| \ \|_{E_H}) \to (F_H, \| \ \|_{F_H})$,

$$\phi'(x) : (E_H, \| \ \|_{E_H}) \to (F_H, \| \ \|_{F_H}) \text{ be toplinear} \tag{4.5}$$
$$\phi'(x) : (E_H, \| \ \|_{E}) \to (F_H, \| \ \|_{F}) \text{ be toplinear,} \tag{4.6}$$

and $\phi'(x)(E_H) = F_H$ for all $x \in E$. If for any $x_0 \in E$

$$\left\| id_{E_H} - \phi'(x_0)^{-1} \circ \phi'(x) \right\|_{L\left((E_H, \| \ \|_{E_H})\right)} \leq \alpha < 1 \ \forall x \in E_H \text{ and} \tag{4.7}$$
$$\left\| id_{E_H} - \phi'(x_0)^{-1} \circ \phi'(x) \right\|_{L\left((E_H, \| \ \|_{E})\right)} \leq \alpha < 1 \ \forall x \in E_H \tag{4.8}$$

then ϕ is a global $C^1(E, F; E_H, F_H)$-isomorphism and $(\phi^{-1})'(\cdot) = \phi'(\phi^{-1}(\cdot))^{-1}$.

Proof.
Let $\lambda := \phi'(x_0)$ and $g \colon E_H \to E_H$, $x \mapsto x - \lambda^{-1}\phi(x)$, which is well-defined by $\phi(E_H) \subset F_H$ and $\phi'(x_0)(E_H) = F_H$. By assumption $\forall x \in E_H$

$$\|g'(x)\|_{L\left((E_H, \| \ \|_{E_H})\right)} \overset{(4.5)}{=} \left\|id - \lambda^{-1} \circ \phi'(x)\right\|_{L\left((E_H, \| \ \|_{E_H})\right)} \overset{(4.7)}{\leq} \alpha < 1$$
$$\|g'(x)\|_{L\left((E_H, \| \ \|_{E})\right)} \overset{(4.6)}{=} \left\|id - \lambda^{-1} \circ \phi'(x)\right\|_{L\left((E_H, \| \ \|_{E})\right)} \overset{(4.8)}{\leq} \alpha < 1$$

By the Mean Value Theorem (cf. Theorem 1.0.21) we see that $\forall x_1, x_2 \in E_H$

$$\|g(x_1) - g(x_2)\|_{E_H} \overset{\text{Thm 1.0.21}}{\leq} \alpha \|x_1 - x_2\|_{E_H}, \tag{4.9}$$
$$\|g(x_1) - g(x_2)\|_{E} \overset{\text{Thm 1.0.21}}{\leq} \alpha \|x_1 - x_2\|_{E}. \tag{4.10}$$

Claim: For each $y \in F$ there exists an unique element $x \in E$ such that $\phi(x) = y$.
Proof.
We prove this by considering for all $\tilde{y} \in F_H$ the map
$g_{\tilde{y}} \colon E_H \to E_H$, $x \mapsto \lambda^{-1}\tilde{y} + g(x)$ and have for all $\tilde{x}_1, \tilde{x}_2 \in E_H$

$$\|g_{\tilde{y}}(\tilde{x}_1) - g_{\tilde{y}}(\tilde{x}_2)\|_{E_H} = \|g(\tilde{x}_1) - g(\tilde{x}_2)\|_{E_H} \overset{(4.9)}{\leq} \alpha \|\tilde{x}_1 - \tilde{x}_2\|_{E_H} \tag{4.11}$$

Thus by the shrinking lemma [Lan93, p.361, Lemma 1.1] or the Banach Fixed Point Theorem ($(E_H, \| \ \|_{E_H})$ is a Banach space), it follows that $g_{\tilde{y}}$ has a unique fix point $\tilde{x}$, i.e. $\tilde{x} = \lambda^{-1}\tilde{y} + \tilde{x} - \lambda^{-1}\phi(\tilde{x})$. It is precisely the solution of $\lambda^{-1}\phi(\tilde{x}) = \lambda^{-1}\tilde{y}$, respectively (by the linearity of λ^{-1}) of $\phi(\tilde{x}) = \tilde{y}$. Hence we have received a (global) inverse for $\phi|_{E_H}$, which we denote by $\phi^{-1}|_{F_H}$.
We write $\tilde{x} = \tilde{x} - \lambda^{-1} \circ \phi(\tilde{x}) + \lambda^{-1} \circ \phi(\tilde{x}) = \lambda^{-1} \circ \phi(\tilde{x}) + g(\tilde{x})\ \forall \tilde{x} \in E_H$. Using $E_H \subset E$ dense (w.r.t. $\| \ \|_E$), ϕ continuous and $F_H \subset F$ dense w.r.t. $\| \ \|_F$ we conclude that for $i \in \{1,2\}$ $\forall \varepsilon > 0$ $\forall x_i \in E$

$$\exists \tilde{x}_i \in E_H : \ \|\tilde{x}_i - x_i\|_E < \varepsilon, \ \|\phi(x_i) - \phi(\tilde{x}_i)\|_E < \varepsilon, \ \phi(\tilde{x_1}) - \phi(\tilde{x_2}) \in F_H \quad (4.12)$$

Thus

$$\begin{aligned} \|x_1 - x_2\|_E &\underset{(4.12)}{\overset{\triangle}{\leq}} 2\varepsilon + \|\tilde{x}_1 - \tilde{x}_2\|_E \\ &\underset{\triangle}{\overset{(4.12)}{\leq}} 2\varepsilon + \left\|\lambda^{-1} \circ (\phi(\tilde{x}_1) - \phi(\tilde{x}_2))\right\|_E + \underbrace{\|g(\tilde{x}_1) - g(\tilde{x}_2)\|_E}_{\overset{(4.10)}{\leq} \alpha\|\tilde{x}_1 - \tilde{x}_2\|_E \leq 2\alpha\varepsilon + \alpha\|x_1 - x_2\|_E} \end{aligned}$$

Defining $y := \phi(x_i)$ and $\tilde{y}_i := \phi(\tilde{x}_i)$ for $i \in \{1,2\}$, we see

$$\begin{aligned} \|x_1 - x_2\|_E &\leq \frac{2\varepsilon(1+\alpha)}{1-\alpha} + \frac{1}{1-\alpha}\left\|\lambda^{-1} \circ (\phi(\tilde{x}_1) - \phi(\tilde{x}_2))\right\|_E \\ &\overset{(4.6)}{\leq} \frac{2\varepsilon(1+\alpha)}{1-\alpha} + \frac{1}{1-\alpha}\underbrace{\left\|\phi'(x_0)^{-1}\right\|_{L\left((F_H, \| \ \|_F),(E_H, \| \ \|_E)\right)}}_{<\infty} \|\tilde{y}_1 - \tilde{y}_2\|_F \\ &\overset{(4.12)}{\leq} \frac{2\varepsilon(1+\alpha)}{1-\alpha} + \frac{1}{1-\alpha}\left\|\phi'(x_0)^{-1}\right\|_{L\left((F_H, \| \ \|_F),(E_H, \| \ \|_E)\right)} \left(2\varepsilon + \|y_1 - y_2\|_F\right), \end{aligned}$$

Letting $\varepsilon \to 0$ and $y_1 \to y_2$ the r.h.s. tends to 0 for and thus for $y \in F$ exists $x \in E : \phi^{-1}(y) = \{x\}$. $\phi^{-1}(y_1) = \{x_1\}$ is well-defined (on $\phi(E)$). Hence the continuity and the claim are proved.
Furthermore it is differentiable at $y_1 \in F$: We choose $y \in F$ such that $y - y_1 \in F_H$ and set $x := \phi^{-1}(y)$ and $x_1 := \phi^{-1}(y_1)$. Then we choose for $\varepsilon > 0$ $\tilde{x} \in E$: $\|\phi(\tilde{x}) - \phi(x)\|_{F_H} \leq \varepsilon$, $\tilde{x} - x_1 \in E_H$ and $\|\tilde{x} - x\|_E \leq \varepsilon_2 \leq \varepsilon$. We have

$$\begin{aligned} &\left\|\phi^{-1}(y) - \phi^{-1}(y_1) - \phi'(x_1)^{-1}(y - y_1)\right\|_E \\ = &\left\|x - x_1 - \phi'(x_1)^{-1}(\phi(x) - \phi(x_1))\right\|_E \end{aligned} \quad (4.13)$$

From the differentiability of ϕ, we can write ($\tilde{x} - x_1 \in E_H$)

$$\phi(\tilde{x}) = \phi(x_1) + \phi'(x_1)(\tilde{x} - x_1) + o(\tilde{x} - x_1)$$

If we substitute this in (4.13), (4.13) equals

$$\begin{aligned}
&\left\| x - x_1 - \phi'(x_1)^{-1}\big(\phi(x) - \phi(\tilde{x}) + \phi'(x_1)(\tilde{x} - x_1) + o(\tilde{x} - x_1)\big) \right\|_E \\
\overset{(4.6)}{\leq}\ & \|x - \tilde{x}\|_E + \left\| \phi'(x_1)^{-1}(\phi(x) - \phi(\tilde{x})) \right\|_E + \left\| \phi'(x_1)^{-1} o(\tilde{x} - x_1) \right\|_E \\
\underset{(4.12)}{\overset{(4.6)}{\leq}}\ & \varepsilon + M\varepsilon + \left\| \phi'(x_1)^{-1} o((\tilde{x} - x) + x - x_1) \right\|_E,
\end{aligned}$$

Let $\varepsilon \to 0$ and $y \to y_1$. Using the continuity of ϕ^{-1}, that for $\|x\|_E \to 0$ $\left\| \frac{o(x)}{\|x\|_E} \right\|_F \to 0$ and (4.6) we obtain that ϕ^{-1} is differentiable at y_1 (w.r.t. $(F_H, \| \, \|_F)$ by the definition of differentiability (of ϕ w.r.t. $(E, \| \, \|_E)$. Its derivative is given by

$$(\phi^{-1})'(y_1) = \phi'(\phi^{-1}(y_1))^{-1}.$$

Since ϕ^{-1} is continuous, ϕ continuously differentiable w.r.t. $(E_H, \| \, \|_E)$ and ϕ' is toplinear, $(\phi^{-1})'$ is continuous and $\phi^{-1} : (F, \| \, \|_F) \to (E, \| \, \|_E)$ is continuously differentiable w.r.t. $(F_H, \| \, \|_F)$. □

We define

Definition 4.2.7 ($D(E, F, \| \, \|_1, \| \, \|_2; E_H, F_H, \| \, \|_3, \| \, \|_4)$)**.**
Let $(F, \| \, \|_1)$, $(G, \| \, \|_2)$, $(F_H, \| \, \|_3)$, $(G, \| \, \|_4)$ be normed spaces and $F_H \subset F$ and $G_H \subset G$ be linear subspaces (Then $(F_H, \| \, \|_1)$ and $(G_H, \| \, \|_2)$ are normed spaces).

$$\phi : (F, \| \, \|_1) \to (G, \| \, \|_2) \in D(E, F, \| \, \|_1, \| \, \|_2; E_H, F_H, \| \, \|_3, \| \, \|_4),$$

iff for all $x \in E$ holds

ϕ is continuously differentiable w.r.t. $(E_H, \| \, \|_1) \to (F_H, \| \, \|_2)$,
ϕ is continuously differentiable w.r.t. $(E_H, \| \, \|_3) \to (F_H, \| \, \|_4)$,
$\phi'(x)(E_H) = F_H$,
$\phi'(x) : (E_H, \| \, \|_1) \to (F_H, \| \, \|_2)$ is toplinear and
$\phi'(x) : (E_H, \| \, \|_3) \to (F_H, \| \, \|_4)$ is toplinear.

If the norms and subspaces are clear, we abbreviate
$D(E, F) := D(E, F, \| \, \|_1, \| \, \|_2; E_H, F_H, \| \, \|_3, \| \, \|_4)$.

The next preparation is to state an adequate version of the Implicit Mapping Theorem.

Proposition 4.2.8 (Implicit Mapping Theorem)**.**
Let $T(E_H \times F_H) \subset G_H$, where

$$T \in C^1(E \times F, G; E_H \times F_H, G_H).$$

Let $(\tilde{x}, \tilde{\sigma}) \in E \times F$ with $T(\tilde{x}, \tilde{\sigma}) = 0$ and assume that

1. *$\forall \tilde{x} \in E_H$ $T(\tilde{x}, F_H) \subset G_H$, $\forall x \in E$ $T(x,\cdot) \in D(F, G, \| \|_F, \| \|_G; F_H, G_H, \| \|_{F_H}, \| \|_{G_H})$ and $\forall \sigma \in F$ $T(\cdot,\sigma)$ is differentiable w.r.t. $(E_H, \| \|_E) \to (F_H, \| \|_F)$ and w.r.t. $(E_H, \| \|_{E_H}) \to (F_H, \| \|_{F_H})$.*

2. *$\exists M < \infty : \forall w \in E_H$ $\sup_{(x,\sigma)\in E\times F} \|D_1 T(x,\sigma) w\|_{G_H} \leq M \|w\|_{F_H}$*

3. *$\exists \tilde{M} < \infty : \forall w \in G_H$ $\sup_{(x,\sigma)\in E\times F} \left\|D_2 T(x,\sigma)^{-1} w\right\|_F \leq \tilde{M} \|w\|_{G_H}$*

4. *$\exists \alpha > 0 : \sup_{(x,\sigma)\in E\times F} \left\|D_2 T(\tilde{x},\tilde{\sigma})^{-1}(-D_1 T(\tilde{x},\tilde{\sigma}) + D_1 T(x,\sigma))\right\| + \left\|D_2 T(\tilde{x},\tilde{\sigma})^{-1} D_2 T(x,\sigma) - I_{F_H}\right\| \leq \alpha < 1$, where the norm is the $L\big((E_H, \| \|_{E_H}), (F_H, \| \|_{F_H})\big)$, respectively $L\big((E_H, \| \|_{E_H})\big)$ and $L\big((E_H, \| \|_E), (F_H, \| \|_F)\big)$, respectively $L\big((E_H, \| \|_E)\big)$.*

Then there exists exactly one continuous map $g : E \to F$ such that

1. *$g(\tilde{x}) = \tilde{\sigma}$ and $T(x, g(x)) = 0$ for all $x \in E$,*

2. *$g \in C^1(E, F; E_H, F)$ is uniquely determined and*

3. *$\forall h \in E_H : \sup_{x\in E} \|Dg(x)h\|_{F_H} \leq M\tilde{M} \|h\|_{E_H}$*

Proof.
Consider the map

$$\begin{array}{rcl} \phi\colon E\times F & \to & E\times G \\ (x,\sigma) & \mapsto & (x, T(x,\sigma)) \end{array}$$

Then $D\phi(\tilde{x},\tilde{\sigma}) : E_H \times F_H \to E_H \times G_H$ and

$$D\phi(\tilde{x},\tilde{\sigma}) = \begin{pmatrix} I_{E_H} & 0 \\ D_1 T(\tilde{x},\tilde{\sigma}) & D_2 T(\tilde{x},\tilde{\sigma}) \end{pmatrix},$$

the inverse of $D\phi(x,\sigma)$ exists and is

$$(D\phi(\tilde{x},\tilde{\sigma}))^{-1} = \begin{pmatrix} I_{E_H} & 0 \\ -D_2 T(\tilde{x},\tilde{\sigma})^{-1} \circ D_1 T(\tilde{x},\tilde{\sigma}) & D_2 T(\tilde{x},\tilde{\sigma})^{-1} \end{pmatrix} \quad (4.14)$$

Furthermore $\phi(E_H \times F_H) = E_H \times T(E_H \times F_H) \subset E_H \times G_H$, for any $(x,\sigma) \in E\times F$

$$\begin{aligned} & D\phi(x,\sigma)(E_H \times F_H) = \{(\tilde{x}, D_1 T(x,\sigma)(\tilde{x}) + D_2 T(x,\sigma)(F_H)) : \tilde{x} \in E_H\} \\ = \ & E_H \times G_H \end{aligned}$$

and
$\phi \in D\big(E\times F, E\times G, \| \|_{E\times F}, \| \|_{E\times G}; E_H \times F_H, E_H \times G_H, \| \|_{E_H\times F_H}, \| \|_{E_H\times G_H}\big)$.

Abbreviating $B := (D_2T(\tilde{x},\tilde{\sigma}))^{-1}$ we find

$$\begin{array}{cl} & \left\| \mathrm{id}_{E_H\times F_H} - D\phi(\tilde{x},\tilde{\sigma})^{-1}\circ D\phi(x,\sigma)\right\| \\ \overset{B \text{ toplinear isomorphism}}{=} & \left\| \mathrm{id}_{E\times F} - \begin{pmatrix} I_{E_H} & 0 \\ B(D_1T(x,\sigma) - D_1T(\tilde{x},\tilde{\sigma})) & B\circ D_2T(x,\sigma)\end{pmatrix}\right\| \\ = & \|B(D_1T(x,\sigma) - D_1T(\tilde{x},\tilde{\sigma}))\| + \|B\circ D_2T(x,\sigma) - I_{F_H}\| \\ \overset{\text{assump}}{\leq} & \alpha < 1. \end{array} \tag{4.15}$$

Thus the global inverse exists (cf. Proposition 4.2.6) and is denoted by ψ. We write

$$\psi(x,z) = (x, h(x,z)) \;\forall z \in T(E\times F),$$

where $h \in \tilde{C}^1(E\times T(E,F), F; E_H\times G_H, F_H)$, and define

$$\begin{array}{rcl} g\colon E & \to & F \\ x & \mapsto & h(x,0) \end{array}$$

Thus $\forall x \in E$

$$(x, T(x,g(x))) \overset{\text{def.}}{=} \phi(x,g(x)) = \phi(x,h(x,0)) = \phi(\psi(x,0)) = (x,0)$$

Furthermore, denoting by A_{21} the value a_{21} if $A = (a_{ij})_{1\leq i\leq n, 1\leq j\leq n}$, we know

$$\begin{array}{rcl} Dg(x) & = & D_1h(x,0) = (D\psi(x,0))_{21} = ((D\phi(\psi(x,0)))^{-1})_{21} \\ & = & ((D\phi(x,h(x,0)))^{-1})_{21} = -D_2T(x,g(x))^{-1}\circ D_1T(x,g(x)) \end{array}$$

and hence $\forall h \in E_H$

$$\|Dg(x)h\|_{F_H} = \left\|-D_2T(x,g(x))^{-1}\circ D_1T(x,g(x))h\right\|_{F_H} \overset{\text{assump.}}{\leq} \tilde{M}M\,\|h\|_{E_H}$$

□

4.2.1 Deriving the desired condition

After having adapted the well know theorems to our needs, we state the framework to obtain the desired condition:

Let $\mathbb{H}$ be dense in a Banach space $(\mathbb{E}, \|\;\|_{\mathbb{E}})$,$(\mathbb{H}, \|\;\|_{\mathbb{H}})$ be a Banach space with $\|\;\|_E \leq \|\;\|_{\mathbb{H}}$ and $f \in \tilde{C}^1(\mathbb{E},\mathbb{H};\mathbb{H},\mathbb{H})$. We abbreviate $I_b := [-b,b]\;\forall b\in\mathbb{R}$ and define

$$V_b := V_b(\mathbb{E}) := \{\sigma : I_b \to \mathbb{E} \,|\, \sigma \text{ is continuous}\},\;\; \|\;\|_{V_b(\mathbb{E})} := \sup_{t\in I_b}\|\sigma(t)\|_{\mathbb{E}}$$
$$V_b(\mathbb{H}) := \{\sigma : I_b \to \mathbb{H} \mid \sigma \text{ is a continuous curve}\}\;\; \|\;\|_{V_b(\mathbb{H})} := \sup_{t\in I_b}\|\sigma(t)\|_{\mathbb{H}}\,.$$

We note that $\| \|_{V_b(\mathbb{E})} \leq \| \|_{V_b(\mathbb{H})}$ and that $(V_b(\mathbb{H}), \| \|_{V_b(\mathbb{H})})$ is complete (analogue to [Wer05, Satz II.1.4]). Furthermore

$$\begin{aligned} T\colon \mathbb{E} \times V_b &\to V_b \\ (x, \sigma) &\mapsto x + \int_0^{\cdot} f(\sigma(u))du + \sigma(\cdot) \end{aligned}$$

$$DT(x, \sigma) : \mathbb{H} \times V_b(\mathbb{H}) \to V_b(\mathbb{H}) \quad \forall x \in \mathbb{E}, \sigma \in V_b.$$

We note that $T(\tilde{x}, \tilde{\sigma}) = 0$ iff $-\tilde{\sigma}$ is a solution for the following differential equation

$$\begin{cases} a'(t) &= f(a(t)) \\ a(0) &= \tilde{x} \end{cases}$$

Using the proof of [Lan93, p.371, Lemma 4.1] for elements of $V_b(\mathbb{H})$ we receive

Lemma 4.2.9.
$T \in \tilde{C}^1(\mathbb{E} \times V_b, V_b; \mathbb{H} \times V_b(\mathbb{H}), V_b(\mathbb{H}))$ and its second partial derivative w.r.t. $V_b(\mathbb{H})$ is given by the formula

$$D_2 T(x, \sigma) = \int_0^{\cdot} Df \circ \sigma - id_{V_b(\mathbb{H})}$$

In terms of $t \in I_b$, this reads:

$$D_2 T(x, \sigma)h(t) = \int_0^t Df(\sigma(u))h(u)du - h(t) \quad \forall h \in V_b(\mathbb{H}). \tag{4.16}$$

Taking a closer look at the proof of [Lan93, Lemma 4.2, p.373] we attain

Lemma 4.2.10.
Suppose that $Df(x)(\mathbb{H}) \subset \mathbb{H}$ $\forall x \in \mathbb{E}$ and $\exists C_1 : 0 < C_1 < \infty$:

$$\begin{aligned} &\sup_{x \in \mathbb{E}} \|Df(x)h\|_{\mathbb{E}} \leq C_1 \|h\|_{\mathbb{E}} \quad \forall h \in \mathbb{H} \\ \text{and} \quad &\sup_{x \in \mathbb{E}} \|Df(x)h\|_{\mathbb{H}} \leq C_1 \|h\|_{\mathbb{H}} \quad \forall h \in \mathbb{H}, \end{aligned}$$

*then $\forall x \in \mathbb{E}$
$T(x, \cdot) \in D(V_b, V_b, \| \|_{V_b}, \| \|_{V_b}; V_b(\mathbb{H}), V_b(\mathbb{H}), \| \|_{V_b(\mathbb{H})}, \| \|_{V_b(\mathbb{H})})$, where $b < \frac{1}{C_1}$ and $\left\| D_2 T(x, \sigma)^{-1} \right\| \leq \frac{1}{1 - bC_1}$.*

Proof.
Using these conditions we observe applying the estimate for $h \in V_b(\mathbb{H}),\ t \in I_b$

$$\left\| \int_0^{\cdot} Df(\sigma(u))h(u)du \right\| \leq \int_0^b C_1 \|h(u)\| \ du \leq bC_1 \|h\| \tag{4.17}$$

in Lemma 4.2.9 that $\|D_2T(x,\sigma)+\mathrm{id}\|_{L(V_b)} < 1$, and hence that $D_2T(x,\sigma)$ is invertible as a continuous linear map (cf. Neumann in [Wer05, Satz II.1.11, p.56]). We have $\forall w \in V_b(\mathbb{H})$:

$$(D_2T(x,\sigma))^{-1}w \overset{\text{Neumann}}{=} \sum_{n=1}^{\infty}(id + D_2T(x,\tilde{\sigma}))^n w \quad \text{and}$$
$$\left\|(D_2T(x,\sigma))^{-1}w\right\| \overset{(4.17)}{\leq} \sum_{n=1}^{\infty}(bC_1)^n \|w\| \leq \frac{1}{1-bC_1}\|w\| \tag{4.18}$$

□

After having done the preparations we state the desired condition:

Theorem 4.2.11.
Let $f \in C_b^1(\mathbb{E},\mathbb{H};\mathbb{H},\mathbb{H})$, $Df(x)(\mathbb{H}) \subset \mathbb{H}$ $\forall x \in \mathbb{E}$ and exists $C_1 < \infty$:

$$\sup_{x\in\mathbb{E}} \|Df(x)h\|_{\mathbb{E}} \leq C_1 \|h\|_{\mathbb{E}}, \quad \forall h \in \mathbb{H}$$
$$\text{and} \quad \sup_{x\in\mathbb{E}} \|Df(x)h\|_{\mathbb{H}} \leq C_1 \|h\|_{\mathbb{H}}, \quad \forall h \in \mathbb{H} \tag{4.19}$$

If $C_1 = 0$, then $f(x) = x_0$ and $a\colon \mathbb{R}\times\mathbb{E} \to \mathbb{E}$, $(t,x) \mapsto x_0 + tx$ is the unique solution. Otherwise let $b < \frac{1}{3C_1}$. Then there exists exactly one solution a : $I_b \times \mathbb{E} \to \mathbb{E}$, which is continuously differentiable and we have $\forall t \in I_b, x \in \mathbb{E}$:

$$\|D_2a(t,x)h\|_{\mathbb{H}} \leq \frac{1}{1-bC_1}\|h\|_{\mathbb{H}} < \frac{3}{2}\|h\|_{\mathbb{H}} \quad \forall h \in \mathbb{H}.$$

Proof.
We have $\|f(x+h)-f(x)\|_{\mathbb{E}} \overset{\text{Thm. 1.0.21}}{\leq} \sup_{x\in\mathbb{E}}\|Df(x)\|\ \|h\|_{\mathbb{E}}$. The first part we have shown in Proposition 4.2.4. In order to prove the rest, we will check the assumptions of Proposition 4.2.8:
By Proposition 4.2.4 there exists a solution $a\colon I_b\times\mathbb{E}\to\mathbb{E}$, $a(\cdot,\tilde{x}) = -\tilde{\sigma}$, such that $T(\tilde{x},\tilde{\sigma}) = 0$. We know $T(\mathbb{H}\times V_b(\mathbb{H})) \subset V_b(\mathbb{H})$.

1. We obtain by Lemma 4.2.10 that $\forall x \in \mathbb{E}$
   $T(x,\cdot) \in D(V_b, V_b, \|\ \|_{V_b}, \|\ \|_{V_b}; V_b(\mathbb{H}), V_b(\mathbb{H}), \|\ \|_{V_b(\mathbb{H})}, \|\ \|_{V_b(\mathbb{H})})$. We have that $D_1T(\bullet,\bullet) = \mathrm{id}$ and $\forall h \in \mathbb{H}$ $T(h, V_b(\mathbb{H})) \subset V_b(\mathbb{H})$.

2. $\|D_1T(\cdot,\bullet)w\| \overset{D_1T=\mathrm{id}}{=} \|w\|_{\mathbb{H}} \quad \forall w \in \mathbb{H}$

3. This is given by Lemma 4.2.10.

4. The following $\| \ \|$ is again meant for the norm of the space and the contained Banach subspace.

$$\begin{aligned} \sup_{(x,\sigma)\in\mathbb{H}\times V} \quad & \left\| D_2T(\tilde{x},\tilde{\sigma})^{-1}(-D_1T(\tilde{x},\tilde{\sigma}) + D_1T(x,\sigma))\right\| \\ & + \left\| D_2T(\tilde{x},\tilde{\sigma})^{-1} D_2T(x,\sigma) - I_{V_b}\right\| \\ \overset{\text{assump. 1}}{\underset{D_1T=id}{=}} \quad & 0 + \left\| D_2T(\tilde{x},\tilde{\sigma})^{-1}(D_2T(x,\sigma) - D_2T(\tilde{x},\tilde{\sigma}))\right\| \overset{!}{\leq} \alpha < 1 \end{aligned}$$

By considering (4.18) and (4.16) it is sufficient to show that

$$\frac{2bC_1}{1-bC_1} \leq \alpha < 1 \overset{bC_1<1}{\Leftrightarrow} b < \frac{1}{3C_1}$$

Thus by Proposition 4.2.8 we obtain that there exists a uniquely determined g such that
$\sup_{x\in\mathbb{E}} \|Dg(x)h\|_{\mathbb{H}} \leq \frac{1}{1-bC_1} \|h\|_{\mathbb{H}} \quad \forall h \in \mathbb{H}$ and $T(x, g(x)) = 0 \ \forall x \in \mathbb{E}$.
Thus $g(x) \in V_b$ is a solution and having $a(t,x) = g(x)(t)$ the assertion is shown. □

4.3 Lebesgue Dominated Convergence

We state a Lebesgue Dominated Convergence Theorem for signed measures and then deduct a corollary, which will be applied to prove the main result.

Proposition 4.3.1 (Lebesgue dominated convergence for signed measure)**.** *Fix $t \in I$ and let $f : I \to (L_1(\nu), \| \ \|_{L^1})$ be bounded, i.e. $\forall x \in \mathbb{E}$ $|f(s)(x)| \leq M(x) \in L^1(\nu)$ $\forall s \in I$, and pointwise differentiable in t with $f'(t)(\cdot) \in L^1(\nu)$ and suppose that there exists a function $\tilde{M}_t \in L^1(\nu)$ and $\delta_t > 0$ such that $[t-\delta_t, t+\delta_t] \subset I$, for all $s \in \mathbb{R}$: $|s| \leq \delta_t$ and for all $x \in \mathbb{E}$:*

$$\left| \frac{f(t+s)(x) - f(t)(x)}{s} \right| \leq \tilde{M}_t(x).$$

Then we obtain

$$\lim_{s\to 0} \int \frac{f(t+s)(x) - f(t)(x)}{s} \nu(dx) = \int \underbrace{\lim_{s\to 0} \frac{f(t+s)(x) - f(t)(x)}{s}}_{=f'(t)(x)} \nu(dx)$$

Proof.
For any $|s| > \delta_t$ we derive

$$\left| \frac{f(t+s)(x) - f(t)(x)}{s} \right| \leq \frac{2M(x)}{\delta_t} \in L^1(\nu)$$

and using in addition the second assumption we gain

$$\left|\frac{f(t+s)(x)-f(t)(x)}{s}\right| \leq \frac{2}{\delta_t}M(x)+\tilde{M}_t(x) \in L^1(\nu) \tag{4.20}$$

We may apply Lebesgue (e.g. [Bau01, Thm 15.6]) to each summand in the following calculation, because ν^+ and ν^- are finite positive measures and because there exists an integrable dominating function for $\left|\frac{f(t+s)(x)-f(t)(x)}{s}\right|$ and obtain

$$\begin{aligned}
& \lim_{s\to 0}\int \frac{f(t+s)-f(t)}{s}d\nu \\
= \quad & \lim_{s\to 0}\int \frac{f(t+s)-f(t)}{s}d\nu^+ - \lim_{s\to 0}\int \frac{f(t+s)-f(t)}{s}d\nu^- \\
\overset{\text{Lebesgue}}{\underset{(4.20)}{=}} \quad & \int \lim_{s\to 0}\frac{f(t+s)-f(t)}{s}d\nu^+ - \int \lim_{s\to 0}\frac{f(t+s)-f(t)}{s}d\nu^- \\
= \quad & \int \lim_{s\to 0}\frac{f(t+s)-f(t)}{s}d\nu
\end{aligned}$$

□

We state a theorem, which we will use for the proof of the Key Proposition and the Main Theorem:

Theorem 4.3.2.
Let $t \in I$ fixed, $\tilde{\phi} \in \tilde{C}_b^1(F,\mathbb{R};F_H,\mathbb{R})$ and $\tilde{\gamma} \in \tilde{C}^1(\mathbb{R}\times\mathbb{E},F;\mathbb{R}\times\mathbb{H},F_H)$, where $\tilde{\gamma}(\tau,x) =: \tilde{\gamma}_\tau(x)$. We assume ($p \geq 1$) that $\tilde{\phi}\circ\tilde{\gamma}_\tau$ is differentiable in $\tau = t$ and $\exists\delta_t > 0\ \exists M_t \in L^p(\nu)$:

$$\sup_{\xi\in[-\delta_t,\delta_t]}\left\|\frac{d}{d\tau}\tilde{\gamma}_\tau(x)\Big|_{\tau=t+\xi}\right\|_{F_H} \leq M_t(x)$$

then $\forall x \in F$:

$$\left|\frac{\tilde{\phi}\circ\tilde{\gamma}_{t+s}(x)-\tilde{\phi}\circ\tilde{\gamma}_t(x)}{s}\right| \leq M_2 M_t(x)$$

and

$$\lim_{s\to 0}\int\frac{\tilde{\phi}\circ\tilde{\gamma}_{t+s}-\tilde{\phi}\circ\tilde{\gamma}_t}{s}d\nu \ = \int \tilde{\phi}'_{\frac{d}{d\tau}\tilde{\gamma}_\tau(x)\big|_{\tau=t}}(\tilde{\gamma}_t(x))\nu(dx) \tag{4.21}$$

Proof.
We check the assumptions of Proposition 4.3.1. First of all there exists $\tilde{M} \in L^1(\nu)$: $\forall x \in \mathbb{E}$ we have $\left|\tilde{\phi}\circ\tilde{\gamma}_{t+s}\right|(x) \leq \tilde{M}$, because $\tilde{\phi}$ is bounded, and, let $x \in \mathbb{E}$,

$$\left|\tilde{\phi}'_{\frac{d}{d\tau}\tilde{\gamma}_\tau(x)\big|_{\tau=t}}(\tilde{\gamma}_t(x))\right| \leq \tilde{M}_\phi\left\|\frac{d}{d\tau}\tilde{\gamma}_\tau(x)\Big|_{\tau=t}\right\|_{F_H} \leq \tilde{M}_\phi M_t(x).$$

Abbreviating $g_x := \tilde{\gamma}_{t+s}(x) - \tilde{\gamma}_t(x) \in F_H$ and $I = \{\tilde{\gamma}_t(x) + \psi g_x : 0 \leq \psi \leq 1\}$, we obtain for $|s| \leq \delta_t$

$$\begin{aligned} &\left\|\tilde{\phi} \circ \tilde{\gamma}_{t+s}(x) - \tilde{\phi} \circ \tilde{\gamma}_t(x)\right\|_G \\ \overset{\text{Thm. 1.0.21}}{\leq} &\sup_{\psi \in I} \left\|\tilde{\phi}'(\psi)\right\| \ \|g_x\|_F \leq \tilde{M}_\phi \|g_x\|_F \\ \overset{\text{Thm. 1.0.21}}{\leq} &\tilde{M}_\phi \sup_{\xi \in [t-\delta_t, t+\delta_t]} \left\|\frac{d}{d\tau}\Big|_{\tau=\xi} \tilde{\gamma}_\tau(x)\right\| \ |s| \leq \tilde{M}_\phi M_t(x)\,|s| \end{aligned}$$

and thus

$$\left|\frac{\tilde{\phi} \circ \tilde{\gamma}_{t+s}(x) - \tilde{\phi} \circ \tilde{\gamma}_t(x)}{s}\right| \leq \tilde{M}_\phi M_t(x) \in L^1(\nu).$$

Thus we may apply Proposition 4.3.1 for $t \mapsto f(t) = \tilde{\phi} \circ \tilde{\gamma}_t$ and obtain the result. □

Corollary 4.3.3.
Let $h \in Sol$, $a : I \times \mathbb{E} \to \mathbb{E}$ its local flow and $\phi \in \tilde{C}^1_b$. Then

$$\lim_{s \to 0} \int_{\mathbb{E}} \frac{\phi \circ a(s,x) - \phi \circ a(0,x)}{s} \nu(dx) = \int_{\mathbb{E}} \lim_{s \to 0} \frac{\phi \circ a(s,x) - \phi \circ a(0,x)}{s} \nu(dx)$$

Proof.
We prove the assumption of Theorem 4.3.2 for ϕ and $a(t,\cdot)$: $\phi \circ a(t,\cdot)$ is differentiable and by Definition 4.2.1 no. 5 $\exists \tilde{M} < \infty$:

$$\sup_{\xi \in I} \left\|\frac{d}{d\tau} a(\tau, x)\big|_{\tau=\xi}\right\|_{\mathbb{E}} = \sup_{\xi \in I} \|h(a(\xi, x))\|_{\mathbb{E}} < \tilde{M}$$

□

4.4 Connection ${}_{\tilde{C}^1_b}\beta^\nu_k \leftrightarrow {}_{\tau_{\tilde{C}^1_b}}\beta^\nu_k$

In Example 3.3.9 we claimed a connection of C- and τ_C-differentiability. Choosing the norm defining set $\tilde{C}^1_b$ we show that they correspond. This result permits us to apply the results gained for general τ_C-differentiability in Chapter 3 for $\tilde{C}^1_b$-differentiability. We will use these results to prove the Main Theorem.

We extend the claim of [SvW95, Propositon 1]:

Proposition 4.4.1 (${}_{\tau_{\tilde{C}^1_b}}\beta^\nu_k = {}_{\tilde{C}^1_b}\beta^\nu_k \leftrightarrow {}^{0}_{\tau_{\tilde{C}^1_b}}\beta^{f^\nu_F}$)**.**
Let $k \in Sol$ and denote its local flow (cf. Definition 4.2.1) by $a :]-b, b[\times \mathbb{E} \to \mathbb{E}$, which satisfies $D_1 a(0,x) = k(x)\ \forall x \in \mathbb{E}$. We define

$$f^\nu_a(t) := \nu(a(t,\cdot)) = a(-t, \nu) \quad \forall t : |t| < b.$$

Then

1. *ν is $\tilde{C}_b^1$-differentiable along the vector field k, iff $(f_a^\nu(t))_{t\in]-b,b[}$ is $\tau_{\tilde{C}_b^1}$-differentiable at $t=0$ (i.e. ν is $\tau_{\tilde{C}_b^1}$-differentiable along k) and $f_a^\nu(0)^{'\tau_{\tilde{C}_b^1}} \ll f_a^\nu(0)$,*

2. *there exists the logarithmic derivative ${}_{\tilde{C}_b^1}\beta_k^\nu$, iff ${}^{0}_{\tau_{\tilde{C}_b^1}}\beta^{f_a^\nu}$ exists. In this case we have*

$$ {}_{\tilde{C}_b^1}\beta_k^\nu = {}^{0}_{\tau_{\tilde{C}_b^1}}\beta^{f_a^\nu} \stackrel{def}{=} \frac{d f_a^\nu(0)^{'\tau_{\tilde{C}_b^1}}}{d f_a^\nu(0)} \left(\stackrel{Def.\ 3.3.7}{=} {}_{\tau_{\tilde{C}_b^1}}\beta_k^\nu \right). $$

Proof.
First of all we recall the setting in which we are working now (cf. Example 3.3.9): The topology $\tau_{\tilde{C}_b^1}$ is generated by the bilinear product

$$ p_\phi(\mu) := < \mu, \phi > := \int_{\mathbb{E}} \phi d\mu, \text{ where } \phi \in \tilde{C}_b^1 \text{ and } \mu \in M(\mathbb{E}, \mathbb{B}(\mathbb{E})) $$

Now we prove that $(M(\mathbb{E},\mathbb{B}(\mathbb{E})), \tilde{C}_b^1, <,>)$ is a dual pair, i.e.

1. $\forall \mu \in M(\mathbb{E},\mathbb{B}(\mathbb{E}))\backslash\{0\}\ \exists \phi \in \tilde{C}_b^1 : \int_{\mathbb{E}} \phi d\mu \neq 0$, because $\tilde{C}_b^1$ is norm defining.

2. $\forall \phi \in \tilde{C}_b^1\backslash\{0\}\ \exists \mu \in M(\mathbb{E},\mathbb{B}(\mathbb{E})) :\ \int_{\mathbb{E}} \phi d\mu \neq 0$, because $\exists y \in \mathbb{E} : \phi(y) \neq 0$ and we choose $\mu(A) = \mathbb{1}_A(y)$ for all $A \in \mathbb{B}(\mathbb{E})$.

We prove: If ν is $\tilde{C}_b^1$-differentiable, then $f_a^\nu(t)$ is $\tau_{\tilde{C}_b^1}$-differentiable at $t=0$. This means in our situation that

$$ \tau_{\tilde{C}_b^1} - \lim_{s\to 0} \frac{f_a^\nu(0+s) - f_a^\nu(0)}{s} = f_a^\nu(0)^{'\tau_{\tilde{C}_b^1}} \in M(\mathbb{E},\mathbb{B}(\mathbb{E})) $$

exists. For this end it is sufficient to show $\forall n \in \mathbb{N}, \forall \phi_1,\dots,\phi_n \in \tilde{C}_b^1, \forall \varepsilon > 0\ \exists s_0 \in \mathbb{R} :\ \forall s < s_0 :$

$$ \varepsilon > p_{\phi_i}\left(\frac{f_a^\nu(s) - f_a^\nu(0)}{s} - f_a^\nu(0)^{'\tau_{\tilde{C}_b^1}} \right) \tag{4.22} $$

We define $f_a^\nu(0)^{'\tau_{\tilde{C}_b^1}} := {}_{\tilde{C}_b^1}\beta_k^\nu\, \nu$, which is by Lemma 1.0.17 a signed measure, and show that $p_\phi(f^\nu_a(0)^{'\tau_{\tilde{C}^1_b}}) = \lim_{s\to 0} p_\phi(\frac{f^\nu_a(s)-f^\nu_a(0)}{s})$:

$$
\begin{aligned}
p_\phi(f^\nu_a(0)^{'\tau_{\tilde{C}^1_b}}) &\overset{\text{Def.}}{=} \int_{\mathbb{E}} \phi(x) f^\nu_a(0)^{'\tau_{\tilde{C}^1_b}}(dx) \\
&= \int_{\mathbb{E}} \phi(x) \left({}_{\tilde{C}^1_b}\beta^\nu_k \, \nu \right)(dx) \overset{\text{Def. 3.2.3}}{=} - \int_{\mathbb{E}} \phi' \underbrace{k(x)}_{=a'_1(0,x)} (x)\nu(dx) \\
&\overset{x=a(0,x)}{=} - \int_{\mathbb{E}} \lim_{-s'\to 0} \frac{\phi\circ a(-s',x) - \phi\circ a(0,x)}{-s'} \nu(dx) \\
&\overset{\substack{\text{[Bau01, Thm 19.1]} \\ \text{Cor. 4.3.3} \\ \text{limit unique}}}{=} \lim_{s\to 0} \int_{\mathbb{E}} \phi(x) \frac{f^\nu_a(s) - f^\nu_a(0)}{s}(dx) \\
&= \lim_{s\to 0} p_\phi \left(\frac{f^\nu_a(s) - f^\nu_a(0)}{s} \right) \qquad (4.23)
\end{aligned}
$$

Furthermore we have $f^\nu_a(0)^{'\tau_{\tilde{C}^1_b}} \ll \nu = f^\nu_a(0)$. Therefore ${}^0_{\tau_{\tilde{C}^1_b}}\beta^{f^\nu_a(0)}$ exists and equals ${}_{\tilde{C}^1_b}\beta^\nu_k$.

For the converse we define

$$
{}_{\tilde{C}^1_b}\beta^\nu_k := {}^0_{\tau_{\tilde{C}^1_b}}\beta^{f^\nu_a},
$$

and obtain by $p_\phi(f^\nu_a(0)^{'\tau_{\tilde{C}^1_b}}) = \lim_{s\to 0} p_\phi(\frac{f^\nu_a(s)-f^\nu_a(0)}{s})$ and equation (4.23) the assertion.

□

Remark 4.4.2.
By Definition 3.2.3 (and Definition 3.2.1) we have that each $\phi \in \tilde{C}^1 \cap C_b$. Therefore we have proved Proposition 4.4.1 for a smaller set ($\tilde{C}^1_b$) than the biggest possible set ($\tilde{C}^1 \cap C_b$). We note that the assertion holds for every norm-defining subset of $\tilde{C}^1_b$ (cf. proof and Corollary 4.3.3). We formulated the assertion only for $\tilde{C}^1_b$, because we will only apply it for $\tilde{C}^1_b$.

With this remark the omitted connection of the graphic in Section 3.4 is obviously true for all norm-defining set $C \subset \tilde{C}^1_b$.

Chapter 5

A transformation rule for measures

The aim of this chapter is to present the key results, which include the Main Theorem and a transformation rule for measures.

In Section 5.1 we present the set of functions, for which the main theorem can be applied. Furthermore we derive a few properties of this set, which we use proving the Main Theorem.

Section 5.2 is reserved for the Key Proposition, which gives a formula for ${}_C\beta_h^\nu$.

In Section 5.3 we prove the Main Theorem. The core arguments of its proof are the Key Proposition, the adapted Lebesgue Dominated Convergence Theorem and Proposition 4.4.1.

Finally in Section 5.4 the transformation rule is presented.

We mainly follow the ideas of [SvW95], whereas we have weakened the conditions given there, changed some of the notation to state clearly the dependence of some parameters, given more details and state further conditions under which the calculations hold. E.G. the proof of the Key Proposition is more detailed than the one given in [SvW93, Proposition 8.2]and for technical reasons we assume further assumptions, which were not postulated in [SvW95].

In this chapter we fix $b_1 \in \mathbb{R}^-$, $b_2 \in \mathbb{R}^+$ and $I :=]b_1, b_2[\ni 0$ an open intervall.

5.1 Preparations

First of all, in order to introduce a few abbreviations and to get familiar to the notation, we state and prove properties, which we will use for the proof of the Main Theorem.

The next definition summarizes conditions needed to apply the main theorem.

Definition 5.1.1 ($\mathbf{F}_{Sol}(\mathbb{E},\mathbb{H})$).
Denote by $\boldsymbol{F}_{Sol}(\mathbb{E},\mathbb{H})$ *the set of mappings* $F : I \times \mathbb{E} \to \mathbb{E}$ *such that*

1. $F(0,x) = x$, $F(t,\cdot) : \mathbb{E} \to \mathbb{E}$ *is bijective and* $F^{\diamondsuit}(t,\cdot)$ *denotes the inverse w.r.t. the second component.*

2. F *and* $F^{\diamondsuit} : I \times \mathbb{E} \to \mathbb{E}$ *and the* D_1, D_2, D_1D_2, D_2D_1 *derivative of* F *and* $F^{\diamondsuit}$ *exist and are continuous, where (in this context)* D_1 *means differentiable w.r.t.* I *and* D_2 *means differentiable w.r.t.* $(\mathbb{H}, \| \, \|_{\mathbb{E}})$ *and* $(\mathbb{H}, \| \, \|_{\mathbb{H}})$. *Furthermore* $\forall t \in I,\ x \in \mathbb{E}\ DF(t,x) : \mathbb{R} \times \mathbb{H} \to \mathbb{H}$, *where* D *denotes the derivative w.r.t.* $I \times \mathbb{H}$. *Moreover* $\forall t \in I\ D_2F(t,x)(\mathbb{H}) = \mathbb{H} \quad \forall x \in \mathbb{E}$ *and*

$$\exists M_2 < \infty : \ \|D_2F(t,x)h\|_{\mathbb{H}} \le M_2 \|h\|_{\mathbb{H}} \quad \forall x \in \mathbb{E}\ \forall h \in \mathbb{H}$$
$$\exists \delta_t > 0, M_t \in L^1(\nu) : \quad \sup_{\xi \in [-\delta_t, \delta_t]} \left\|D_1F^{\diamondsuit}(t+\xi, x)\right\|_{\mathbb{H}} \le M_t(x) \quad \forall x \in \mathbb{E}.$$
$$\exists M_t < \infty : \ \left\|D_2F^{\diamondsuit}(t,x)h\right\|_{\mathbb{H}} \le M_t \|h\|_{\mathbb{H}} \quad \forall x \in E\ \forall h \in \mathbb{H}$$

The first and second partial derivatives are denoted by F_1', F_2', $F''{}_{12}$.

3. $F_1'(t,\cdot) \in Sol\ \forall t \in I$

Remark 5.1.2.
Let $F, F^{\diamondsuit}$ *be twice continuously Frèchet differentiable w.r.t.* $(\mathbb{R} \times \mathbb{H}, \| \, \|_{\mathbb{R}\times\mathbb{E}})$ *and* $(\mathbb{R} \times \mathbb{H}, \| \, \|_{\mathbb{R}\times\mathbb{E}})$ *with bounded derivative. Assume that* $\forall t \in \mathbb{R}, x \in \mathbb{E}$ $D_2F(t,x)(\mathbb{H}) \subset \mathbb{H}$ *and* $\exists M < \infty : \forall h \in \mathbb{H} \ \ \|D_2F(t,x)h\|_{\mathbb{H}} < M \|h\|_{\mathbb{H}}$, $\|D_2F(t,x)h\|_{\mathbb{E}} < M \|h\|_{\mathbb{E}}$, $\left\|D_2F^{\diamondsuit}(t,x)h\right\| \le M \|h\|$ *and* $\|D_1F(t,x)\| < M$. *Then assumption 3 holds, because Theorem 4.2.11 is applicable.*

For this chapter fix $F \in \mathbf{F}_{Sol}(\mathbb{E},\mathbb{H})$ (cf. Definition 5.1.1) and define $\gamma_\tau^t(x) := F(t, F^{-1}(\tau,x))\ \forall t,\tau \in I, x \in \mathbb{E}$.

Similar to [SvW95, Lemma 1] we postulate that

Lemma 5.1.3.
Fix $t \in I$. *Define a vector field* $k_t : \mathbb{E} \to \mathbb{H}$ *by the implicit equation* $k_t(F(t,x)) = F_1'(t,x)$ *and write* $\gamma_\tau^t(x) := F(t, F^{\diamondsuit}(\tau,x))$. *Then for every* $x \in \mathbb{E}$ *we obtain*

$$\lim_{\tau\to t} \frac{\gamma_\tau^t(x) - x}{\tau - t} = -k_t(x).$$

Remark 5.1.4.
The lemma postulates that we define for every $t \in I$ *a vector field* $k_t : \mathbb{E} \to \mathbb{H}$, *which is well defined, because* $F \in \boldsymbol{F}_{Sol}(\mathbb{E},\mathbb{H})$ *is bijective. Differing from [SvW95] we write* k_t *instead of* k. *If* $k_t = k\ \forall t \in I$ *and* $F(t_1+t_2,\cdot) = F(t_1, F(t_2,\cdot))$ $\forall t_1, t_2 \in I : t_1 + t_2 \in I$, *then* $F(t,x)$ *is a local flow for* k *(cf. Definition 4.2.1). In general (e.g.* $F(t,x) = x + th(x)$, $h(x) \in C_b^2$*)* k_t *is not independent of* t.

Proof of Lemma 5.1.3.
The proof is done by the chain rule: For $x \in \mathbb{E}, \tau \in I$ we have

$$\lim_{\tau\to t} \frac{\gamma_\tau^t(x) - x}{\tau - t} = \frac{d}{d\tau} F(t, F^\diamondsuit(\tau, x))\Big|_{\tau=t} = D_2F(t, F^\diamondsuit(\tau, x))D_1F^\diamondsuit(\tau, x)\Big|_{\tau=t}$$
$$= D_2F(t, F^\diamondsuit(t, x))D_1F^\diamondsuit(t, x) \quad (5.1)$$

Now we will calculate the last derivative:
Taking the derivative of $F(t, F^\diamondsuit(t, x)) \overset{F(t,\cdot)\text{ bijective}}{=} x$ w.r.t. t we obtain by the chain rule, which is applicable, because $F^\diamondsuit(t, \cdot) \subset \mathbb{E} = domF(t, \cdot)$:

$$D_1F(t, F^\diamondsuit(t, x)) + D_2F(t, F^\diamondsuit(t, x))D_1F^\diamondsuit(t, x) = 0$$

Thus we gain ($F_2'(t, x)(\mathbb{H}) = \mathbb{H} \supset F_1'(t, x)\ \forall x \in \mathbb{E}$, cf. Definition 5.1.1 no. 2):

$$D_1F^\diamondsuit(t, x) = -(D_2F(t, F^\diamondsuit(t, x)))^{-1}D_1F(t, F^\diamondsuit(t, x))$$

Therefore using this result in (5.1) yields:

$$\begin{aligned}\lim_{t\to s} \frac{\gamma_\tau^t(x) - x}{\tau - t} &= D_2F(t, F^\diamondsuit(t, x))(-(D_2F(t, F^\diamondsuit(t, x)))^{-1}D_1F(t, F^\diamondsuit(t, x))) \\ &= -D_1F(t, F^\diamondsuit(t, x))) \overset{\text{by Def. of } k_t}{=} -k_t(x)\end{aligned}$$

□

The following Lemma summarizes a few properties of γ_τ^t.

Lemma 5.1.5.
We have $\forall t, t + s \in I, x \in \mathbb{E}$:

1. $k_{t.}'(F(t, x)) = F''_{12}(t, x) \circ (F_2'(t, x))^{-1}$
2. $F(t, x) = \gamma_0^t(x)$ *and* $F^\diamondsuit(t, x) = \gamma_t^0(x)$
3. $\gamma_0^{t+s} = \gamma_t^{t+s}\gamma_0^t$ *and* $\gamma_{t+s}^0 = \gamma_t^0\gamma_{t+s}^t$

Proof.
First of all, we know $k_t(F(t, x)) = F_1'(t, x)$. Thus applying the chain rule we obtain $k_{t.}'(F(t, x)) \circ (F_2'(t, x)) = F''_{12}(t, x)$. Secondly, $F(t, x) = F(t, F^\diamondsuit(0, x)) = \gamma_0^t(x)$ and $\gamma_t^0(x) = F(0, F^\diamondsuit(t, x)) = F^\diamondsuit(t, x)$. Last, but not least $\gamma_0^{t+s}(x) \overset{\text{Def}}{=} F(t + s, F^\diamondsuit(0, x)) \overset{F^\diamondsuit(t,\cdot)\circ F(t,\cdot) = \text{id}}{=} F(t + s, F^\diamondsuit(t, F(t, x))) \overset{F(0,\cdot) = id}{=} \gamma_t^{t+s}(F(t, F^\diamondsuit(0, x))) \overset{\text{Def}}{=} \gamma_t^{t+s}\gamma_0^t(x)$ and $\gamma_{t+s}^0 \overset{\text{Def}}{=} F(0, F^\diamondsuit(t + s, x)) = F(0, F^\diamondsuit(t, F(t, F^\diamondsuit(t, x)))) = \gamma_t^0\gamma_{t+s}^t$. □

Corollary 5.1.6.
Let $\phi \in \tilde{C}_b^1$. *We have* $\exists\, \delta > 0$ *and* $\exists M \in L^1(\nu)$, *such that* $\forall\, |s| < \delta$ *and* $\forall x \in \mathbb{E}$*:*

$$\left| \frac{\phi \circ \gamma_{t+s}^0(x) - \phi \circ \gamma_t^0(x)}{s} \right| \leq M(x)$$

Proof.
We use Theorem 4.3.2 for ϕ and $\gamma_t^0 = F^{\diamondsuit}(t, \cdot)$:

$\phi \circ \gamma_t^0$ is differentiable in t. By Definition 5.1.1 no. 2 $\delta_t > 0, M_t \in L^1(\nu)$ such that for $|\xi| < \delta_t$, $\forall x \in \mathbb{E}$

$$\left\| \frac{d}{d\tau}\Big|_{\tau=t+\xi} F^{\diamondsuit}(\tau, x) \right\|_{\mathbb{H}} = \sup_{\xi \in [-\delta_t, \delta_t]} \left\| D_1 F^{\diamondsuit}(t+\xi, x) \right\|_{\mathbb{H}} \leq M_t(x)$$

□

5.2 Key Proposition

After having familiarized with the properties of $\mathbf{F}_{Sol}(\mathbb{E}, \mathbb{H})$ we state and prove the Key Proposition. For this section we fix $h : \mathbb{E} \to \mathbb{H}$. Since we cannot guarantee the existence of the logarithmic gradient for arbitrary vector fields h, i.e. the existence of ${}_C\beta_{\mathbb{H}}^{\nu}(h(\cdot))(\cdot)$, we assume its existence (cf. Definition 2.6.6).

Similar to the core idea of [SvW95, Proposition 2] we have

Proposition 5.2.1 (Key Proposition)**.**
Suppose that

1. $h \in Sol$
2. *the linear operator* ${}_{C_b^1}\beta_{\mathbb{H}}^{\nu} : \mathbb{H} \to L^2(\nu)$ *is continuous and* ${}_{\tilde{C}_b^1}\beta_{\mathbb{H}}^{\nu}(h, \mathrm{id})$ *exists (cf. Definition 2.6.6),*
3. *for every* $x \in \mathbb{E}$ *the restriction of the operator* $h'(x)$ *to* $\mathbb{H}$ *exists and is a trace class operator in* $\mathbb{H}$.
4. *there exists an orthonormal basis* $\{e_i\}_{i\in\mathbb{N}}$ *of* $\mathbb{H}$ *such that* $\forall e_i$ $\|h\|_{\mathbb{H}}$ *is* $\nu_{e_i}^{'C_b^1}$*-integrable ,* $\|h'_{e_i}(\cdot)\|_{\mathbb{H}}$, $\|h(\cdot)\|_{\mathbb{H}}$ *and* $\|h'_{\bullet}(\cdot)\|_{tr}$ *are* ν*-integrable* *and* $\exists M_1 < \infty : \int_{\mathbb{E}} \sum_{e_i} |(h(x), e_i)_{\mathbb{H}}| \nu(dx) \leq M_1$.

Then ν *is* (τ_{tv}*-)differentiable along* h *and*

$$ {}_{\tilde{C}_b^1}\beta_h^{\nu}(x) = {}_{\tilde{C}_b^1}\beta^{\nu}(h(x), x) + tr h'_{\cdot}(x) \tag{5.2}$$

Proof.
Since $h \in Sol$ (cf. assumption 1) we know that there exists $\gamma_t(x) := a(t,x)$, which is the unique differentiable mapping from the definition of differentiable along h (cf. Definition 3.3.7). We define for any $l \in Sol$ with associated flow $a_l : I_b \times \mathbb{E} \to \mathbb{E}$ such that ${}^l\nu^t := a_l(-t,\nu)$, i.e. ${}^l\nu^t(B) := \nu(a_l(t,\cdot)(B)) = \nu(a_l(t,B))$. If we prove that ${}^h\nu^t$ is τ_{tv}-differentiable, we obtain (by Definition 3.3.7) that ν is τ_{tv}-differentiable along h.
We may apply Theorem 3.5.5, i.e. if the assumptions

1. ${}^h\nu^t$ is τ_C-differentiable at 0 and ${}^h\nu^{0'^{\tau_{\tilde{C}_b^1}}} \ll {}^h\nu^0 = \nu$
2. $\phi \circ \gamma_t \in \tilde{C}_b^1 \; \forall t \in I_b,\; \forall \phi \in \tilde{C}_b^1$ and $\tilde{C}_b^1$ is norm defining

are fulfilled we gain that ${}^h\nu^t$ is τ_{tv}-differentiable on I_b. The second assumption is given by Proposition 4.1.6, Definition 4.2.1 no. 5 and Theorem 4.1.15. Thus it remains to prove the first assumption.

$$\exists \tau_{\tilde{C}_b^1} - \lim_{h\to 0} \frac{{}^h\nu^s - {}^h\nu^0}{s} = {}^h\nu^{0'^{\tau_{\tilde{C}_b^1}}}$$

which means that the following limit exists for each $\phi \in \tilde{C}_b^1$

$$\int \phi \; {}^h\nu^{0'^{\tau_{\tilde{C}_b^1}}} = \lim_{s\to 0} \int \phi d\left(\frac{{}^h\nu^s - {}^h\nu^0}{s}\right) = \lim_{s\to 0} \frac{1}{s}\Big(\int \phi d\; {}^h\nu^s - \int \phi d\; {}^h\nu^0\Big)$$

that is in our case

$$\begin{aligned}
&= \lim_{s\to 0} \frac{1}{s}\Big(\int \phi(x)(a(-s,\nu(dx))) - \int \phi(x)(a(-0,\nu(dx)))\Big) \\
&\overset{\text{Cor. 4.3.3}}{=} \int \lim_{s\to 0} \frac{\phi\circ a(-s,x) - \phi\circ a(-0,x)}{s} \nu(dx) \\
&\overset{\text{chain rule}}{=} \int \phi'(a(0,x)) \underbrace{(D_1 a(0,x))}_{\overset{\text{Def. 4.2.1}}{\underset{\text{no. 3}}{=}} h(a(0,x))\in\mathbb{H}} (-1)\nu(dx) \\
&\overset{h\in Sol}{=} -\int \phi'_{h(x)}(x)\nu(dx) = -\int \phi'_h d\nu \qquad (5.3)
\end{aligned}$$

The last integral exists, because for the fixed $\phi \in \tilde{C}_b^1$ exists $M_\phi < \infty$, $\mathbb{H} \subset \mathbb{E}$ continuous (cf. Section 2.1) and by assumption 4 $\exists M_h \in L^1(\nu) : \forall x \in \mathbb{E}$:

$$\left|\phi'_{h(x)}(x)\right| \le M_\phi \left\|h(x)\right\|_{\mathbb{E}} \le M_\phi M_h(x) \in L^1(\nu) \qquad (5.4)$$

Thus we know

$$\begin{aligned}
&{}^h\nu^s = a(-s,\nu) \text{ is } \tau_{\tilde{C}_b^1}\text{-differentiable at } 0 \\
\Leftrightarrow\;& \exists\; {}^h\nu^{0'^{\tau_{\tilde{C}_b^1}}} : \int \phi d\; {}^h\nu^{0'^{\tau_{\tilde{C}_b^1}}} = -\int \phi'_{h(x)}(x)\nu(dx) \qquad (5.5)
\end{aligned}$$

We denote:

$$\phi \otimes h := \phi(\cdot)h(\cdot) \qquad \text{and} \qquad \phi'_{\cdot} \otimes h := \phi'_{\cdot}()h()$$

In the last equation the derivative is meant w.r.t. to $\mathbb{H}$.
By the definition of C-differentiability (Definition 3.2.3) suitable by assumption 2 ($\exists_{\tilde{C}_b^1} \beta_{\mathbb{H}}^{\nu}(h, id)$, and cf. Remark 3.2.4 and Definition 2.6.1) the following functions are pointwise, i.e. for every $\tilde{h} \in \mathbb{H}$, equal (use Theorem 4.4.1):

$$\int_{\mathbb{E}} \underbrace{(\phi \otimes h)'_{\cdot} \nu(dx)}_{=:\phi \otimes h'_{\cdot} \otimes \nu()} = -\int (\phi \otimes h) \; {}^{\cdot}\nu^{0'\tau_{\tilde{C}_b^1}}(dx) \tag{5.6}$$

Furthermore Leibnitz' formula (e.g. [Wer05, p.239]), which is here indeed just the formula for differentiation of a product of two Fréchet-differentiable functions and is proved by adding a 0 (having in mind that $\mathbb{R} \otimes_{\mathbb{R}} \mathbb{R} \cong \mathbb{R}$),

$$(\phi \otimes h)'_{\cdot} = \phi'_{\cdot} \otimes h + \phi \otimes h'_{\cdot} \tag{5.7}$$

yield

$$\int_{\mathbb{E}} \phi'_{\cdot} \otimes h \otimes \nu(dx) \overset{(5.6)}{\underset{(5.7)}{=}} -\int_{\mathbb{E}} \phi \otimes h'_{\cdot} \otimes \nu(dx) - \int_{\mathbb{E}} \phi \otimes h \otimes \; {}^{\cdot}\nu^{0'\tau_{\tilde{C}_b^1}}(dx) \tag{5.8}$$

We know

$$tr(\phi'_{\cdot}(x) \otimes h(x)) = (\phi'_{\cdot}(x), h(x))$$

because for the orthogonal basis $\{e_i\}_{i \in \mathbb{N}} \subset \mathbb{H}$ we obtain

$$\begin{aligned}
tr\,(\phi'_{\cdot}(x) \otimes h(x)) &\overset{\text{[Lax02, 10.2, Thm 3]}}{=} \sum_{e_i} ((\phi'_{\cdot}(x) \otimes h(x))(e_i), e_i)_{\mathbb{H}} \\
&\overset{(\cdot, e_i)\text{ lin.}}{\underset{\exists:(5.11)}{=}} \sum_{e_i} \underbrace{\phi'_{e_i}(x)}_{\in \mathbb{R}} \underbrace{(h(x), e_i)_{\mathbb{H}}}_{\in \mathbb{R}} \overset{\text{Def. 1.0.20}}{=} \phi'_{\sum_{e_i}(h(x),e_i)e_i}(x) \\
&\overset{\text{[Wer05, Satz V.4.9]}}{=} \phi'_{h(x)}(x) \overset{\substack{\text{well-defined!} \\ \text{bilin. and cont}}}{=:} (\phi'_{\cdot}(x), h(x))
\end{aligned} \tag{5.9}$$

Using (5.9) in (5.5) we gain

$$\begin{aligned}
\int_{\mathbb{E}} \phi(x) \; {}^{h}\nu^{0'\tau_{\tilde{C}_b^1}}(dx) &\overset{(5.5)}{\underset{\text{if } \exists\, {}^{h}\nu^{0'\tau_{\tilde{C}_b^1}}}{=}} -\int_{\mathbb{E}} \phi'_{h(x)}(x)\nu(dx) \\
\overset{\text{lin.}}{\underset{(5.9)}{=}} -\int_{\mathbb{E}} \sum_{e_i} ((h \otimes \phi'_{\cdot})(e_i), e_i)\, d\nu &\overset{\text{lin.}}{=} -\int_{\mathbb{E}} \sum_{e_i} (h(x), e_i) \phi'_{e_i}(x)\nu(dx)
\end{aligned} \tag{5.10}$$

We can apply Fubini, because

$$\int_{\mathbb{E}} \sum_{e_i} |(h(x), e_i)_{\mathbb{H}} \phi'_{e_i}(x)| \nu(dx) \overset{(5.4)}{\leq} \int_{\mathbb{E}} \sum_{e_i} |(h(x), e_i)_{\mathbb{H}}| M \underbrace{\|e_i\|_{\mathbb{H}}}_{=1} \nu(dx)$$

$$\leq M \int_{\mathbb{E}} \sum_{e_i} |(h(x), e_i)_{\mathbb{H}}| \nu(dx) \overset{\text{by } 4}{\leq} M M_1 < \infty. \tag{5.11}$$

Using

$$\int_{\mathbb{E}} \left|\phi'_{e_i}(x)\right| \|h(x)\|_{\mathbb{H}} \nu(dx) \overset{(5.4)}{\leq} M_\phi \int_{\mathbb{E}} M_h d|\nu| \overset{\substack{\text{assumption}\\4}}{<} \infty$$

in [PR07, A.2.1] the Bochner integral exists and can be interchanged with $(\cdot, e_i)$, because we use [Coh80, Proposition E.11, p.256] or [PR07, A.2.2] and that $(\cdot, e_i)$ is linear and continuous. Thus (5.10) equals

$$\begin{aligned}
& -\sum_{e_i} \left(\int_{\mathbb{E}} \phi'_{e_i}(x) h(x) \nu(dx), e_i \right) \\
\overset{\text{Def. } \otimes}{=} & -\sum_{e_i} \left(\left(\int_{\mathbb{E}} \phi'_{\cdot} \otimes h \otimes d\nu \right)(e_i), e_i \right) \overset{\text{Def. tr}}{=} -tr \int_{\mathbb{E}} \phi'_{\cdot} \otimes h \otimes d\nu \\
\overset{(5.8)}{=} & \underbrace{tr \int \phi(x) \otimes h'_{\cdot}(x) \otimes \nu(dx)}_{\overset{!}{\underset{(5.13)}{=}} \int_{\mathbb{E}} \phi(x) tr h'(x) d\nu(dx)} + \underbrace{tr \int_{\mathbb{E}} \phi(x) h(x) \otimes {}^{h}\nu^{\cdot\,'\tau_{\tilde{C}_b^1}}(dx)}_{\overset{!}{\underset{(5.14)}{=}} \int_{\mathbb{E}} \phi(x)\, {}_{\tau_{\tilde{C}_b^1}}\beta^{\nu}(h(x), x) \nu(dx)}
\end{aligned} \tag{5.12}$$

Thus it remains to prove the indicated equalities:

$$\begin{aligned}
& tr \int_{\mathbb{E}} \phi(x) \otimes h'_{\cdot}(x) \otimes \nu(dx) = \sum_{e_i} \left(\left(\int_{\mathbb{E}} \phi(x) \otimes h'_{\cdot}(x) \otimes \nu(dx) \right)(e_i), e_i \right) \\
\underset{!1.}{\overset{\text{Def. } \otimes}{=}} & \sum_{e_i} \left(\int_{\mathbb{E}} \phi(x) \underbrace{h'_{e_i}(x)}_{\in \mathbb{H}} \nu(dx), e_i \right) \underset{!2.}{=} \int_{\mathbb{E}} \phi(x) \underbrace{\sum_{e_i} (h'_{e_i}(x), e_i)}_{= tr h'_{\cdot}(x)} \nu(dx)
\end{aligned} \tag{5.13}$$

where

1. the Bochner integral exists, because using [PR07, A.2.1]

$$\int_{\mathbb{E}} |\phi(x)| \left\|h'_{e_i}(x)\right\|_{\mathbb{H}} \nu(dx) \overset{\phi \in \tilde{C}_b^1}{\leq} M_\phi \int_{\mathbb{E}} \left\|h'_{e_i}(\cdot)\right\|_{\mathbb{H}} d|\nu| \overset{\substack{\text{assump.}\\4}}{<} \infty$$

2. and, by [PR07, A.2.2] and $(\cdot, e_i)$ being linear and continuous, we may interchange the Bochner integral with the inner product and justify Fubini

$$\begin{aligned}
& \int_{\mathbb{E}} \sum_{e_i} |\phi(x)| \left|(h'_{e_i}(x), e_i)\right| \nu(dx) \\
\overset{\phi \in \tilde{C}_b^1}{\leq} & M_\phi \int_{\mathbb{E}} \underbrace{\sup_{\{f_n\},\{g_n\} \text{ ONBs}} \sum_n |(h'_{f_n}(x), g_n)| \nu(dx)}_{\overset{[\text{Lax02, p.332 (6)}]}{=} \|h'_{\cdot}(x)\|_{tr} \in L^1(\nu)} \overset{4}{<} \infty
\end{aligned}$$

We obtain for the last summand in (5.12)

$$tr\int \phi(x)h(x)\otimes \;\cdot\nu^{0'\tau_{\tilde{C}_b^1}}(dx) = \sum_{e_i}(\underbrace{\int_{\mathbb{E}} \phi(x)h(x)\;{}^{e_i}\nu^{0'\tau_{\tilde{C}_b^1}}(dx), e_i)}_{\exists \text{ Bochner integral by 4}}$$

$$\overset{[PR07,\ A.2.2]}{=} \sum_{e_i}\int_{\mathbb{E}} \phi(x)(h(x),e_i)\;{}^{e_i}\nu^{0'\tau_{\tilde{C}_b^1}}(dx)$$

$$= \int_{\mathbb{E}} \phi(x)\Big(\sum_{e_i}(h(x),e_i)\;\underbrace{{}^{e_i}\nu^{0'\tau_{\tilde{C}_b^1}}(dx)}_{\tau_{\tilde{C}_b^1}\beta^{\nu}_{\mathbb{H}(e_i)}(x)\nu(dx)}\Big)$$

$$\underset{\text{Prop 4.4.1}}{\overset{2,\text{ Rem 3.3.6}}{=}} \int \phi(x)\;{}_{\tilde{C}_b^1}\beta^{\nu}_{\mathbb{H}}(h(x))(x)\nu(dx) \tag{5.14}$$

Hence the r.h.s. of (5.12) gives a measure ${}^{h}\nu^{0'\tau_{\tilde{C}_b^1}}$:

$${}^{h}\nu^{0'\tau_{\tilde{C}_b^1}} = \Big(trh'_{\bullet} + {}_{\tilde{C}_b^1}\beta^{\nu}(h(\cdot),\cdot)\Big)\,\nu$$

Here the trace is meant w.r.t. $\mathbb{H}$ and is taken in the argument marked by $\bullet$. Therefore

$${}_{\tau_{\tilde{C}_b^1}}\beta^{\nu}_{h}(x) = trh'_{\cdot}(x) + {}_{\tau_{\tilde{C}_b^1}}\beta^{\nu}(h(x),x)$$

All in all we gain that ${}^{h}\nu^{t}$ is $\tau_{\tilde{C}_b^1}$-differentiable at 0 (cf. Definition 3.3.7), iff

$$\exists\;{}_{\tau_{\tilde{C}_b^1}}\beta^{\nu}_{h}(x) = trh'_{\cdot}(x) + {}_{\tilde{C}_b^1}\beta^{\nu}(h(x),x) \text{ a.e. } \forall x\in\mathbb{E}$$

But this exists, because $h'(x)$ exists and its restriction on $\mathbb{H}$ is of trace class (cf. assumption 3). ${}_{\tilde{C}_b^1}\beta^{\nu}(h(x),x)$ exists by assumption (cf. assumption 2 and Remark 3.3.6). Thus we have shown the first assumption, because ${}^{h}\nu^{0'\tau_{\tilde{C}_b^1}} = {}_{\tau_{\tilde{C}_b^1}}\beta^{\nu}_{h}\;{}^{h}\nu^{0}$. Hence we may apply Theorem 3.5.5 (respectively Remark 3.5.6) and ${}^{h}\nu^{0}$ is τ_{tv}-differentiable on I_b and we are done (using Proposition 4.4.1). □

In the last theorem the assumption $\|\phi(x)\| \le M|x|$ would not be sufficient, as the following example shows:

Example 5.2.2.
The assumptions $\phi:\ \mathbb{R}\to\mathbb{R}$, $|\phi(x)|\le M|x|$ and $\phi'(x)\le M$ are not sufficient. (The same is true for h(x)!). We have that the following measure ν is finite

$$\int_{\mathbb{R}} 1\nu(dx) := \int_{\mathbb{R}} \mathbb{1}_{[-1,1]^C}(x)\frac{1}{x^2}\lambda(dx) = 2$$

and $\phi(x) = id$ *is bounded by* $|x|$ *and its derivative by* 1 *in the above sense. But choosing* $h(x) = id$*, then* $h'_\cdot(x) = id$ *and we obtain*

$$\int_{\mathbb{R}} |\phi(x)||h'_{e_i}(x)|_{\mathbb{R}}\nu(dx) \geq \int_1^\infty x\frac{1}{x^2}dx = \infty.$$

Thus the two conditions mentioned above are not sufficient to show the existence of the Bochner integral in the last theorem (cf.(5.13)).

5.3 Main Theorem

So far we have prepared a lot of technical details and therefore it is high time that we treat the Main Theorem (similar to [SvW95, Theorem 1]).

Theorem 5.3.1 (First Main Theorem)**.**
Let $\nu \in M(\mathbb{E}, \mathbb{B}(\mathbb{E}))$ *(cf. Section 2.1). For* $t \in I$ *we set* $k_t(x) := F'_1(t, F^\diamondsuit(t,x))$ $\forall x \in \mathbb{E}$*. Then* $k_{t\cdot}'(x) = F''_{12}(t, F^\diamondsuit(t,x)) \circ (F'_2(t, F^\diamondsuit(t,x)))^{-1}(\cdot)$*, where* $k_{t\cdot}'$ *is the Gâuteaux-derivative in direction* $\cdot$*. Suppose that*

1. *the logarithmic gradient (cf. Definition 2.6.1)* ${}_{\tilde{C}_b^1}\beta^\nu_{\mathbb{H}} : \mathbb{H} \to L^2(\nu)$ *is continuous and* ${}_{\tilde{C}_b^1}\beta^\nu_{\mathbb{H}}\left(F'_1(t, F^\diamondsuit(t,\cdot))\right)(\cdot)$ *exists.*
2. $F : I \times \mathbb{E} \mapsto \mathbb{E}$ *belongs to the class* $\boldsymbol{F}_{Sol}(\mathbb{E}, \mathbb{H})$*.*
3. $\forall x \in \mathbb{E}$ *the restriction of the operator* $k_{t\cdot}'(x) : \mathbb{E} \to \mathbb{H}$ *to* $\mathbb{H}$ *exists and is of trace class in* $\mathbb{H}$ *and* $\|k_{t\cdot}'(x)\|_{\mathrm{tr}} \in L^1(\nu)$*.*
4. $F'_1(t,\cdot) : \mathbb{E} \to \mathbb{H} \subset \mathbb{E}$
5. $k_t(x) \in Sol$ *and there exists an orthonormal base* $\{e_i\}_{i\in\mathbb{N}}$: $\forall e_i$ $\|k_t\|_{\mathbb{H}} \in L^1(\nu_{e_i}^{'C_b^1})$, $\|k_t\|_{\mathbb{H}} \in L^1(\nu)$, $\|k_{t\bullet}'(\cdot)\|_{tr} \in L^1(\nu)$, $\forall e_i : \|k_{te_i}'(\cdot)\|_{\mathbb{H}} \in L^1(\nu)$ *and* $\exists M_t < \infty :$ $\int_{\mathbb{E}} \sum_{e_i} |(k_t(x), e_i)_{\mathbb{H}}|\nu(dx) \leq M_t < \infty$.

Then the measure valued map

$$\begin{aligned} f^\nu_F : \mathbb{R} &\to M(\mathbb{E}, \mathbb{B}(\mathbb{E})) \\ s &\mapsto \nu^s := F^\diamondsuit(s,\nu) := \nu \circ F(s,\cdot) \end{aligned}$$

is τ_{tv}*-differentiable at* t *in* $\mathbb{H}$ *with logarithmic derivative*

$$\begin{aligned} {}^{t}_{\tau_{tv}}\beta^{f^\nu_F}(x) &= \frac{d\nu^{t'\tau_{tv}}}{d\nu^t}(x) \\ &= {}_{\tilde{C}_b^1}\beta^\nu_{\mathbb{H}}(F'_1(t,x))(F(t,x)) + \mathrm{tr}(F''_{12}(t,x) \circ (F'_2(t,x))^{-1}) \qquad (5.15) \end{aligned}$$

Proof.

In Lemma 5.1.3 we set $k_t(x) := F_1'(t, F^{\diamondsuit}(t,x))$. $\nu^{t'^{\tau_{\tilde{C}_b^1}}}$ exists as a $\tau_{\tilde{C}_b^1}$-derivative iff (by Definition 3.3.2) $\forall \phi \in \tilde{C}_b^1$ $\rho_\phi(\nu^{t'^{\tau_{\tilde{C}_b^1}}}) = \lim_{s\to 0} \rho_\phi(\frac{\nu^{t+s}-\nu^t}{s})$ exists, which is

$$\begin{aligned}
& \lim_{s\to 0} \frac{1}{s} \int \phi d(\underbrace{\nu^{t+s} - \nu^t}_{\nu\circ(\gamma_{t+s}^0)^{-1} - \nu\circ(\gamma_t^0)^{-1}}) \\
\overset{\text{def.}}{=} & \lim_{s\to 0} \int \frac{\phi\circ\gamma_t^0 \circ (\gamma_{t+s}^t - \text{id})}{s} d\nu \\
\overset{\substack{\text{! Lebesgue} \\ \text{Thm. 4.3.2}}}{=} & \int \lim_{s\to 0} \frac{\phi\circ\gamma_t^0 \circ (\gamma_{t+s}^t - \gamma_t^t)}{s} d\nu \qquad (5.16)\\
\overset{\substack{\text{chain rule} \\ \text{for } \mathbb{R}}}{=} & \int (\phi\circ\gamma_t^0\circ\gamma_t^t)' \circ (\lim_{s\to 0}\frac{\gamma_{t+s}^t - \gamma_t^t}{s}) d\nu \\
\overset{\text{Lemma 5.1.3}}{=} & \int (\underbrace{\phi\circ\gamma_t^0(x)}_{\in \tilde{C}_b^1 \text{ by Def. 5.1.1 no. 2 and Prop 4.1.6}})'(-k_t(x))\nu(dx) \\
\overset{\substack{\exists \text{ Prop 5.2.1} \\ \text{Def. 3.2.3} \\ \text{assump. 1}}}{=} & \int (\phi\circ\gamma_t^0) \underbrace{{}_{\tilde{C}_b^1}\beta_{k_t}^\nu(\cdot)}_{\in L^1(\nu) \text{ Def. 3.2.3}} d\nu \\
\overset{\text{Prop 5.2.1}}{=} & \int (\phi\circ\gamma_t^0)(x)({}_{\tilde{C}_b^1}\beta_{\mathbb{H}}^\nu(k_t(x))(x) + \text{tr}\, k_{t\cdot}'(x))\nu(dx) \qquad (5.17)
\end{aligned}$$

Applying the Key Proposition(Proposition 5.2.1) is justified by assumptions 1, 2, 3, 4 and 5. This yields

$$\begin{aligned}
\overset{\substack{\text{Lemma 5.1.5} \\ \text{def.}}}{=} & \int \phi(F^{\diamondsuit}(t,x))({}_{\tilde{C}_b^1}\beta_{\mathbb{H}}^\nu (F_1'(t, F^{\diamondsuit}(t,x)), x) \\
& \qquad + \text{tr}\, F''_{12}(t, F^{\diamondsuit}(t,x)) \circ (F_2'(t, F^{\diamondsuit}(t,x)))^{-1})\nu(dx) \\
\overset{\text{def. } \nu^t}{=} & \int \phi(z) \left(\beta^\nu (F_1'(t,z), F(t,z)) + \text{tr}\, F_{12}''(t,z)\circ (F_2'(t,z))^{-1}\right) d\nu^t(dz)
\end{aligned}$$

It remains to prove that we can apply Theorem 4.3.2 in (5.16): $\phi\circ\gamma_t^0 \subset \tilde{C}_b^1$ (cf. Proposition 4.1.6, Definition 5.1.1 no. 2), $\phi\circ\gamma_{t+s}^0$ is differentiable (cf. Proposition 5.1.6), and by Definition 5.1.1 no. 2 there exist $\tilde{\delta}(t) > 0$ and $\tilde{M}_1(t) < L^1(\nu)$, such that $\forall x \in \mathbb{E}$:

$$\begin{aligned}
& \sup_{\xi\in[-\tilde{\delta},\tilde{\delta}]} \left\| \frac{d}{d\tau}\gamma_\tau^t \Big|_{\tau=t+\xi}(x) \right\|_{\mathbb{H}} \\
= & \sup_{\xi\in[-\tilde{\delta},\tilde{\delta}]} \left\| D_2F(t, F^{\diamondsuit}(t+\xi,x)) D_1 F^{\diamondsuit}(t+\xi,x) \right\|_{\mathbb{H}} \\
\leq & \sup_{\xi\in[-\tilde{\delta},\tilde{\delta}]} M_2(t) \left\| D_1F^{\diamondsuit}(t+\xi,x)\right\|_{\mathbb{H}} \leq M_2(t)M_1(t)(x) =: \tilde{M}_1(t)(x) \qquad (5.18)
\end{aligned}$$

□

5.4 A transformation rule for measures

After having shown a condition for a measure to be $\tilde{C}_b^1$-differentiable in the last section, we derive a "nice" representation for the transformation from one measure to another one. The idea of the measure transformation has been done in [Bel90].

Differing from [SvW95], we assume more conditions:

Theorem 5.4.1 (Transformation rule for measures).
Let $\nu \in M(\mathbb{E}, \mathbb{B}(\mathbb{E}))$ (cf. Section 2.1) and $F \in \boldsymbol{F}_{Sol}(\mathbb{E}, \mathbb{H})$. In addition to the assumptions of Theorem 5.3.1 being fulfilled for all $t \in I$, we assume $\forall x \in \mathbb{E}$

1. *For each $t \in I$ $F_2'(t,x)$ is positive, i.e. $(F_2'(t,x)h, h)_{\mathbb{H}} \geq 0$ $\forall h \in \mathbb{H}$, $t \mapsto \ln F_2'(t,x) \mid_{\mathbb{H}}$ is a continuously differentiable map from I into the Banach space of trace class operators on $\mathbb{H}$ equipped with the trace norm,*

2. *$\det F_2'(t,x)$ exists $\forall t \in I$,*

3. *there exist an orthonormal base $\{e_i\}_{i \in \mathbb{N}}$ of eigenvectors of $F_2'(t,x)$ such that*
$$\det F_2'(t,x) = \prod_{i=1}^{\infty} (F_2'(t,x)(e_i), e_i),$$

4. *Choose $0 < T \in I$:*
$$\int_0^T {}_{\tilde{C}_b^1}\beta_{\mathbb{H}}^{\nu}(F_1'(s,x), F(s,x))ds < \infty \ \nu^T + \nu\text{-a.e. and}$$

5. $$[0,T] \times \mathbb{E} \ni (t,x) \mapsto \|k_t'.(x)\|_{tr} \in L^1(\lambda \times \nu) \text{ and}$$
$$\int_0^T \left\| {}_{\tilde{C}_b^1}\beta_{\mathbb{H}}^{\nu}(k_t(\cdot), \cdot) \right\|_{L^1(\nu)} dt < \infty.$$

Then for each $t \in [0,T]$ the measure $\nu^t = f_F^{\nu}(t)$ is equivalent to ν and the Radon-Nikodym density is given by

$$\frac{d\nu^t}{d\nu^0}(x) = \det F_2'(t,x) \exp\{\int_0^t {}_{\tilde{C}_b^1}\beta^{\nu}(F_1'(s,x), F(s,x))ds\} \tag{5.19}$$

Remark 5.4.2.

1. *If $F_2'(t,x) - \mathrm{id}$ is of trace class, $\det F_2'(t,x)$ is defined (cf. [Lax02, 30.4, p.342] or [Sim05, p.33]).*

2. *If $F_2'(t,x)$ is compact and selfadjoint, then there exists an orthonormal base of eigenvectors (cf. Spectral Theorem [Wer05, p.265]).*

3. *If $k_{t\cdot}'$ is independent of t, then the first part of assumption 5 is fulfilled by assumption 3 of Theorem 5.3.1. If k_t is independent of t, then assumption 5 is fulfilled by Definition 2.6.6, i.e.*

$$\int_0^T \left\| {}_{\tilde{C}_b^1}\beta_{\mathbb{H}}^{\nu}(k(\cdot),\cdot) \right\|_{L^1(\nu)} dt$$

$$\overset{C.S.}{\leq} \|\nu\|_{tv}^{\frac{1}{2}} \int_0^T \left\| \sum_{e_i ONB} (k(\cdot),e_i)\, {}_{\tilde{C}_b^1}\beta_{\mathbb{H}}^{\nu}(e_i)(g) \right\|_{L^2(\nu)} dt \overset{Def.\ 2.6.6}{<} \infty$$

This is e.g. the case if $F(t,x) = x + th$ (cf. Chapter 6).

Proof of Theorem 5.4.1.
First we apply Theorem 3.5.4 and then use Theorem 5.3.1 and other properties to rewrite the exponential factor to obtain (5.19).

We will check its assumptions: $\exists a, b \in I$:

1. $(\nu^t)_{t\in I}$ is differentiable for $\tau_{\tilde{C}_b^1}$
2. $\int_a^b \|\nu^{t'^{\tau_{\tilde{C}_b^1}}}\|_{tv} dt < \infty$
3. $$(t,x) \mapsto {}^{t}_{\tau_{\tilde{C}_b^1}}\beta^{\nu^\clubsuit}(x) \text{ is } \mathbb{B}(I)\otimes\mathbb{B}\text{-measurable where } {}^{t}_{\tau_{\tilde{C}_b^1}}\beta^{\nu^\clubsuit} = \frac{d\nu^{t'^{\tau_{\tilde{C}_b^1}}}}{d\nu^t}$$
4. $$\int_a^b |{}^{t}_{\tau_{\tilde{C}_b^1}}\beta^{\nu^\clubsuit}(x)| dt < \infty \ |\nu^a| + |\nu^b| - a.e.$$

in order to gain

$$\nu^t, a \leq t \leq b, \text{ are equivalent and } \frac{d\nu^t}{d\nu^a} = \exp\left(\int_a^t {}^{s}_{\tau_{tv}}\beta^{\nu^\clubsuit}(x) ds \right).$$

We prove the assumptions:

1. By Theorem 5.3.1 $(\nu^t)_{t\in I}$ is differentiable for τ_{tv} and thus (cf. Remark 3.3.10) for $\tau_{\tilde{C}_b^1}$.
2. $$\begin{aligned}\int_0^T \|\nu^{t'^{\tau_{\tilde{C}_b^1}}}\|_{tv} dt &\overset{\text{Def. 3.3.2}}{=} \int_0^T \|{}^{t}_{\tau_{\tilde{C}_b^1}}\beta^{f_F^\nu}\, \nu^t\|_{tv} dt \\ &\underset{\text{Thm. 5.3.1}}{\overset{\text{Lemma 1.0.18}}{=}} \int_0^T \int_{\mathbb{E}} \Big| {}_{\tilde{C}_b^1}\beta_{\mathbb{H}}^{\nu}(F_1'(t,x), F(t,x)) \\ &\qquad + \mathrm{tr}(F_{12}''(t,x) \circ (F_2'(t,x))^{-1}) \Big| d\,|\nu^t|\, dt\end{aligned}$$

$$\overset{\triangle}{\underset{\text{[Bau01, Thm 19.1]}}{\leq}} \int_0^T \int_{\mathbb{E}} \Big| {}_{\tilde{C}_b^1}\beta_{\mathbb{H}}^{\nu}(\underbrace{F_1'(t, F^{\diamondsuit}(t,x))}_{=k_t(x)}, x)\Big|$$

$$+\Big| \operatorname{tr}(\underbrace{F_{12}''(t, F^{\diamondsuit}(t,x)) \circ (F_2'(t, F^{\diamondsuit}(t,x)))^{-1}}_{=k_t{}'(x)})\Big| d\,|\nu|\, dt$$

$$\overset{\triangle}{\leq} \int_0^T \int_{\mathbb{E}} |\ \beta_{\mathbb{H}}^{\nu}(k_t(x), x)|\, d\,|\nu|\, dt + \int_0^T \int_{\mathbb{E}} \|k_t{}'(x)\|_{tr}\, d\,|\nu|\, dt \overset{\text{assu. 5}}{<} \infty$$

3. We know by Theorem 5.3.1 that

$$\begin{aligned} {}^{t}_{\tau_{\tilde{C}_b^1}}\beta^{\nu^{\clubsuit}}(\cdot) &= \frac{d\nu^{t'\tau_{\tilde{C}_b^1}}}{d\nu^t} \\ &= {}_{\tilde{C}_b^1}\beta_{\mathbb{H}}^{\nu}(F_1'(t,x), F(t,x)) + \operatorname{tr}(F_{12}''(t,x) \circ (F_2'(t,x))^{-1}) \end{aligned} \tag{5.20}$$

Since $D_1D_2F(t,x)$ exists and is continuous (cf. Definition 5.1.1 no. 2) we have that F itself, $D_2F(t,x)$ and $D_1D_2F(t,x)$ are continuous w.r.t. t and x. Thus they are $\mathbb{B}(I \times \mathbb{E})$-measurable, where

$$\mathbb{B}(I \times \mathbb{E}) := \{A_I \times A_{\mathbb{E}} \mid\ \forall\ A_I \in \mathbb{B}(I), A_{\mathbb{E}} \in \mathbb{B}(\mathbb{E})\}\,.$$

We know that for a linear operator T being continuous is equivalent to $|Tx| \leq M\,\|x\|$ (cf. [Lax02, Theorem 1, p.160]). Hence we obtain that tr is a continuous operator, because by definition (cf. [Wer05, Definition VI.5.7, p.289]) it is linear and by [Lax02, Theorem 4, p.333] it is bounded. Thus $\operatorname{tr}(F_{12}''(t,x) \circ (F_2'(t,x))^{-1}) \in \mathbb{R}$ is continuous w.r.t. $(t,x) \in I \times \mathbb{E}$ and thus $\mathbb{B}(I \times \mathbb{E})$ measurable.

We know

(a) $I \times \mathbb{E}$ is separable, because $\mathbb{E}$ and I are separable (cf. Section 2.1).

(b) $(t,x) \mapsto {}_{\tilde{C}_b^1}\beta_{\mathbb{H}}^{\nu}(F_1'(t,x), \cdot)$ is continuous, because

$$\begin{array}{ccccc} I \times \mathbb{E} & \to & \mathbb{H} & \to & L^1(\nu) \\ (t,x) & \mapsto & F_1'(t,x) & \mapsto & {}_{\tilde{C}_b^1}\beta_{\mathbb{H}}^{\nu}(F_1'(t,x))(\cdot) \end{array}$$

and by assumption 1 of Theorem 5.3.1, we know that ${}_{\tilde{C}_b^1}\beta_{\mathbb{H}}^{\nu} : (\mathbb{H}, \|\ \|_{\mathbb{H}}) \to (L^2(\nu), \|\ \|_{L^2(\nu)})$ is continuous and $\|\nu\|_{tv} < \infty$.

(c) $(t',x') \mapsto {}_{\tilde{C}_b^1}\beta_{\mathbb{H}}^{\nu}(F_1'(t,x), F(t',x'))$ is $\mathbb{B}(I \times \mathbb{E})$ measurable.

Claim: Then $(t,x) \mapsto {}_{\tilde{C}_b^1}\beta_{\mathbb{H}}^{\nu}(F_1'(t,x), F(t,x))$ is $\mathbb{B}(I \times \mathbb{E})$ measurable

Proof.
A sequence of measurable function will be constructed, whose pointwise

limit is the function above and thus the Claim will be proved (cf. [Röc05a, Satz 5.9, p.34] or [Mal95, 2.5.1 Theorem]). Let $(t_r)_{r\in\mathbb{N}} \subset I$, $(x_{\tilde{r}})_{\tilde{r}\in\mathbb{N}} \subset \mathbb{E}$ be countable sequences, which are separating. Choose

$$\begin{aligned} {}_{\tilde{C}_b^1}\beta_n^\nu \colon I \times \mathbb{E} &\rightarrow \mathbb{R} \\ (t,x) &\mapsto {}_{\tilde{C}_b^1}\beta_{\mathbb{H}}^\nu(F_1'(t^{\frac{1}{n}}, x^{\frac{1}{n}}), F(t,x)) \end{aligned}$$

where

$$\begin{aligned} t^{\frac{1}{n}} &:= t_p : |t_p - t| < \frac{1}{n} \text{ and } |t_l - t| > \frac{1}{n} \\ &\quad \forall l < p, t_p, t_l \in (t_{\tilde{r}})_{\tilde{r}\in\mathbb{N}} \\ x^{\frac{1}{n}} &:= x_p : \|x_p - x\|_{\mathbb{E}} < \frac{1}{n} \text{ and } \|x_l - x\|_{\mathbb{E}} > \frac{1}{n} \\ &\quad \forall l < p, x_p, x_l \in (x_{\tilde{r}})_{\tilde{r}\in\mathbb{N}} \end{aligned}$$

These functions converge in $L^2(\nu)$ for $n \to \infty$ to ${}_{\tilde{C}_b^1}\beta_{\mathbb{H}}^\nu$ by the continuity in the first argument, and thus (by Riesz-Fisher) ν-a.e for a subsequence. W.l.o.g. we choose this subsequence and define for the points $(t,x) \in N \in \mathbb{B}(I \times \mathbb{E})$, where it does not converge, ${}_{\tilde{C}_b^1}\beta_n^\nu(t,x) := {}_{\tilde{C}_b^1}\beta^\nu(F_1'(t,x), F(t,x))$. Thus the sequence converges pointwise everywhere.

The next step is to show the measurability of the functions. Let A be an arbitrary element of the generator of $\mathbb{B}(\mathbb{E})$. We abbreviate for $t' \in I, x' \in \mathbb{E}$

$$\begin{aligned} B^{\frac{1}{n}}(t') &:= \left\{ t \in I \middle| \, |t - t'| < \frac{1}{n} \right\} \in \mathbb{B}(I) \\ B^{\frac{1}{n}}(x') &:= \left\{ x \in \mathbb{E} \middle| \, \|x - x'\|_{\mathbb{E}} < \frac{1}{n} \right\} \in \mathbb{B}(\mathbb{E}) \\ A(t',x') &:= \left\{ (t,x) \in I \times \mathbb{E} \middle| \, {}_{\tilde{C}_b^1}\beta_{\mathbb{H}}^\nu(F_1'(t',x'), F(t,x)) \in A \right\} \overset{(3c)}{\in} \mathbb{B}(I \times \mathbb{E}) \\ N(t',x') &:= \left\{ (t,x) \in N \middle| \, {}_{\tilde{C}_b^1}\beta_{\mathbb{H}}^\nu(F_1'(t',x'), F(t,x)) \in A \right\} \overset{!}{\in} \mathbb{B}(I \times \mathbb{E}) \end{aligned}$$

The last set is a subset of the nullset N and thus measurable, because we assumed $\mathbb{B}(I)$ and $\mathbb{B}(\mathbb{E})$ to be complete w.r.t. Lebesgue measure respectively ν (cf. Definition 1.0.2 respectively Section 2.1). By the separability of $I \times \mathbb{E}$ we obtain

$$({}_C\beta_n^\nu)^{-1}(A) = \bigcup_{p'} \bigcup_{p} B^{\frac{1}{n}}(t_p) \times B^{\frac{1}{n}}(x_{p'}) \cap A(t_p, x_{p'}) \cup N(t_p, x_{p'})$$

Hence the whole set is in $\mathbb{B}(I \times \mathbb{E})$. Moreover this is true for any set A of the generator and therefore (adding a theorem of measure theory (cf. [Röc05a, Theorem 4.3, p.25] or [Mal95, 2.3.4 Proposition])) each ${}_{\tilde{C}_b^1}\beta_n^\nu$ is measurable.

4. The fourth assumption is fulfilled by the assumption 4 and the fact that $[0,T] \in t \mapsto tr(F''_{12}(t,x) \circ (F'_2(t,x))^{-1}) \in \mathbb{R}$ is a continuous map from a compact set and thus bounded.

Therefore Theorem 3.5.4 implies the equivalence of the measures ν^t and ν with Radon-Nikodym derivative

$$\begin{aligned}\frac{d\nu^t}{d\nu^0} &= \exp\{\int_0^t {}_{\tau^s_{\tilde{C}^1_b}}\beta^{f^\nu_F}(x)ds\} \\ &\overset{\text{Thm. 5.3.1}}{=} \exp\{\int_0^t \operatorname{tr}(F''_{12}(s,x) \circ (F'_2(s,x))^{-1})ds\} \\ &\qquad \times \exp\{\int_0^t {}_{\tilde{C}^1_b}\beta^\nu_{\mathbb{H}}(F'_1(s,x), F(s,x))ds\}\end{aligned}$$

The second step is to justify that we can rewrite the first factor

$$\begin{aligned}&\exp\{\int_0^t \operatorname{tr}(F''_{12}(s,x) \circ (F'_2(s,x))^{-1}(\cdot)ds\} \\ \overset{!\,1.}{=}\ &\exp\{\int_0^t \operatorname{tr}(\frac{d}{ds}F'_2(s,x))(F'_2(s,x))^{-1}(\cdot)ds\} \\ \overset{!\,2.}{=}\ &\exp\{\int_0^t \operatorname{tr}(\frac{d}{ds}ln(F'_2(s,x))(\cdot))ds\} \\ \overset{!\,3.}{=}\ &\exp\{\operatorname{tr} ln(F'_2(t,x)(\cdot))\} \\ \overset{!\,4.}{=}\ &\det F'_2(t,x)(\cdot). \qquad (5.21)\end{aligned}$$

to finally obtain

$$\frac{d\nu^t}{d\nu^0} = \det F'_2(t,x)(\cdot) \exp\{\int_0^t {}_C\beta^\nu(F'_1(s,x), F(s,x))ds\}.$$

We conclude

1. By the continuity of the derivative $F''_{12}(s,x) = F''_{21}(s,x)$.

2. By Definition 5.1.1 no. 2 $\exists M \in L^1(\nu)$:

$$\forall h \in \mathbb{H}: \ \|F'_2(t,x)h\|_{\mathbb{H}} \leq M(x)\,\|h\|_{\mathbb{H}} \ \ \forall x \in \mathbb{E}.$$

By assumption 1 the following is well-defined and holds ν-a.e.

$$\begin{aligned}\frac{d}{ds}\ln(F'_2(s,x)) &:= \frac{d}{ds}\sum_{n=1}^\infty \frac{D^n ln(M(x)\,\mathrm{id})}{n!}(F'_2(s,x) - M(x)\,\mathrm{id})^n \\ &= \frac{d}{ds}\sum_{n=1}^\infty \frac{(n-1)!(-1)^{n-1}(M(x)\,\mathrm{id})^{-n}}{n!}(F'_2(s,x) - M(x)\,\mathrm{id})^n \\ &= F''_{21}(s,x)\sum_{n=1}^\infty \frac{(-1)^{n-1}(M(x)\,\mathrm{id})^{-n}}{n} n(F'_2(s,x) - M(x)\,\mathrm{id})^{n-1}\end{aligned}$$

$$= F''_{21}(s,x) \sum_{n=1}^{\infty} (-1)^{n-1} (M(x)\,\mathrm{id})^n (F'_2(s,x) - M(x)\,\mathrm{id})^{n-1}$$
$$= F''_{21}(s,x)(F'_2(s,x))^{-1}$$

and we identify the radius of convergence

$$\frac{1}{\limsup \sqrt[n]{\frac{\|(M(x)\,\mathrm{id})^{-n}\|}{n}}} = \frac{1}{\frac{1}{M(x)}} = M(x)$$

Therefore the ln exists for all positive bounded operators.

3. For any orthonormal base (abbr. ONB) $\{l_i\}_{i\in\mathbb{N}}$ we have (cf. [Lax02, Theorem 3, p.333])

$$\int_0^t \mathrm{tr}\, \frac{d}{ds} \ln F'_2(s,x) ds \overset{\text{def. tr}}{=} \int_0^t \sum_{e_i} \left(\left(\frac{d}{ds} \ln F'_2(s,x) \right)(e_i), e_i \right) ds$$
$$\underset{(3a)}{\overset{!\ \text{Fubini}}{=}} \sum_{e_i} \int_0^t \left(\left(\frac{d}{ds} \ln F'_2(s,x) \right)(e_i), e_i \right) ds$$
$$\underset{(3b)}{\overset{!,(\cdot,e_i)\ \text{linear and cont.}}{=}} \sum_{e_i} \int_0^t \frac{d}{ds} \underbrace{((\ln F'_2(s,x))(e_i), e_i)}_{:\mathbb{R}\to\mathbb{R}} ds$$
$$= \sum_{e_i} (\ln F'_2(t,x)(e_i), e_i) \overset{\text{def. tr}}{=} \mathrm{tr} \ln F'_2(t,x)$$

where we used

(a) the boundedness for Fubini

$$\infty \overset{!}{>} \int_0^t \sum_{e_i} \left| \left((\frac{d}{ds} \ln F'_2(s,x))(e_i), e_i \right) \right| ds$$
$$\leq \int_0^t \underbrace{\sup_{\{e_i\},\{f_i\} ONB} \sum_{e_i,f_i} |((\frac{d}{ds} \ln F'_2(s,x))(e_i), f_i)|}_{\overset{\text{[Lax02, p.332 (6)]}}{=} \|\frac{d}{ds} \ln F'_2(s,x)\|_{tr}} ds$$

Since the argument is continuous (cf. assumption 1), we know by an compactness argument that the last term is even uniformly bounded. Thus we obtain

$$\int_0^t \|\frac{d}{ds} \ln F'_2(s,x)\|_{tr} ds \leq tM(x) < \infty$$

(b) for every differentiable function $f : \mathbb{E} \to \mathbb{R}$, we have

$$\begin{aligned}\left(\frac{d}{ds} f(s), e_i\right) &= \left(\lim_{h\to 0} \frac{f(s+h) - f(s)}{h}, e_i\right) \\ &\overset{\exists \lim}{=} \lim_{h\to 0} \left(\frac{f(s+h) - f(s)}{h}, e_i\right) \\ &\overset{\text{lin}}{=} \lim_{h\to 0} \frac{(f(s+h), e_i) - (f(s), e_i)}{h} \overset{\text{def}}{=} \frac{d}{ds}(f(s), e_i)\end{aligned}$$

4. For a better readability we abbreviate $f = \ln F_2'(t,x)$. Using that e_i being an eigenvector of $F_2'(t,x)$ implies that it is as well an eigenvector of $\ln F_2'(t,x)$ and that $\ln F_2'(t,x)$ is of trace class (assumption 1), we obtain

$$\begin{aligned}\exp(\operatorname{tr} f) &\overset{\text{exp cont.}}{=} \lim_{N\to\infty} \prod_{i=1}^{N} \exp \underbrace{((f(e_i), e_i))}_{\in \mathbb{R}} \\ &\overset{\text{def. exp}}{=} \prod_{i=1}^{\infty} \sum_{n=1}^{\infty} \frac{(f(e_i), e_i)^n}{n!} \\ &\overset{f(e_i) = \lambda_i e_i}{=} \prod_{i=1}^{\infty} \sum_{n=1}^{\infty} \frac{(f^n(e_i), e_i)}{n!} \\ &\overset{\substack{\text{cont. of} \\ (\cdot, e_i)}}{=} \prod_{i=1}^{\infty} \Bigg(\underbrace{\sum_{n=1}^{\infty} \frac{f^n(e_i)}{n!}}_{=\exp f}, e_i\Bigg) \\ &= \prod_{i=1}^{\infty} \big((\underbrace{\exp \circ \ln}_{=\text{id}}(F_2'(t,x)))(e_i), e_i\big) \\ &= \prod_{i=1}^{\infty} (F_2'(t,x)(e_i), e_i) \\ &\overset{\text{assu. 3}}{=} \det F_2'(t,x)\end{aligned}$$

□

Chapter 6

Examples

In this chapter we calculate as an example the case of the Gaussian (Section 6.1) and Wiener measure (Section 6.2). We examine the assertion of Theorem 5.4.1(Transformation rule for measures) for these measures and we recover in the first case Ramer's formula and in the second for an adapted integrand the Maruyama-Girsanov-Cameron-Martin formula.

Along the calculations we give explicit conditions for the existence of the mentioned operators. In order to calculate the Volterra operator, we use the theory of Carleman operators to obtain an integral representation.

Then we consider the adapted case. Inspired by a claim in [SvW95] and the structure of the Carleman operators we give a condition, under which the Carleman operator turns out to have only the eigenvalue zero and in particular trace 0 (Theorem 6.2.7). Using this property we recognize that Theorem 5.4.1 is the Maruyama-Girsanov-Cameron-Martin-formula(Remark 6.2.11).

In [SvW95] neither the explicit conditions for the existence of the operators, nor a remark about how to obtain the different representations were given.

Suppose that $[0,1] \subset I$ and $h\colon \mathbb{E} \to \mathbb{H}$ is a vector field, such that the function

$$F\colon I \times \mathbb{E} \to \mathbb{E},\ (t,x) \mapsto x + th(x)$$

fulfills all assumptions needed to apply Theorem 5.4.1. Then $F_2'(1,x) = id + h'(x)$ and $F_1'(t,x) = h(x)$ for all $t \in I$.

6.1 Gaussian measure

In particular, let $(\mathbb{E}, \mathbb{H}, \nu)$ be an abstract Wiener space, i.e. (cf. [MR92, p.57]) $\mathbb{E}$ a separable real Banach space, $\mathbb{H}$ a separable real Hilbert space continuously and densely embedded into $\mathbb{E}$, i.e.

$$\mathbb{E}' \subset \mathbb{H}' \equiv \mathbb{H} \subset \mathbb{E} \text{ continuously and densely}$$

and ν a Gaussian measure on $\mathbb{B}(\mathbb{E})$ with covariance $<,>_{\mathbb{H}}$.

Lemma 6.1.1.
${}_{C}\beta^{\nu}(h, y)$ is ν-a.s. linear in y and ${}_{\tilde{C}_b^1}\beta^{\nu}(h(x), h(x)) = -\|h(x)\|_{\mathbb{H}}^2$

Proof.
By [MR92, Example II.3 c), p.57] we know that $l \mapsto {}_{\mathbb{E}'} < l, \cdot >_{\mathbb{E}}, l \in \mathbb{E}'$ is an isometry, which extends uniquely to an isometry $h \mapsto X_h, h \in \mathbb{H}$. From [MR92, Theorem II.3.11, p.58] we deduce that this is exactly $-{}_{\tilde{C}_b^1}\beta^{\nu}(h, \cdot)$. They are ν-a.e. pointwise the same, thus
$\beta(h(x), h(x)) = -\sum_{e_i}(h(x), e_i)_{\mathbb{H}} X_{e_i}(h(x)) = -(h(x), h(x))_{\mathbb{H}}$. □

Applying Lemma 6.1.1 and choosing $t = 1$, we deduct

$$\begin{aligned}\int_0^1 \beta^{\nu}(F_1'(\tau, x), F(\tau, x))d\tau &\overset{\substack{\text{Lemma 6.1.1}\\ \text{a.s. linear}}}{=} \beta^{\nu}(h(x), x) + \int_0^1 \tau\beta^{\nu}(h(x), h(x))d\tau \\ &= \beta^{\nu}(h(x), x) - \frac{1}{2}\|h(x)\|_{\mathbb{H}}^2\end{aligned}$$

Thus Theorem 5.4.1 yields Ramer's formula (cf. [Ram74, p.167] or [Bel90]):

$$\begin{aligned}\frac{d\nu^t}{d\nu^0}(x) &\overset{\substack{\text{Theorem}\\ 5.4.1}}{=} \det F_2'(t, x) \exp\{\int_0^t \beta^{\nu}(F_1'(\tau, x), F(\tau, x))d\tau\} \\ &= \det(\mathrm{id} + h'(x)) \exp(-\frac{1}{2}\|h(x)\|_{\mathbb{H}}^2 + \beta^{\nu}(h(x), x))\end{aligned}$$

6.2 Wiener measure

We remind ourselves of (cf. e.g. [DiB02, p.179])

Definition 6.2.1 (f absolutely continuous)**.**
A function $f : [a, b] \to \mathbb{R}$ is absolutely continuous if for every $\varepsilon > 0$, there exists $\delta > 0$ such that for every finite collection of disjoint intervals $(a_j, b_j) \subset [a, b], j = 1, \dots n$ of total length not exceeding δ,

$$\sum_{j=1}^{n} |f(b_j) - f(a_j)| < \varepsilon \left(\sum_{j=1}^{n} b_j - a_j\right). \tag{6.1}$$

The Cameron-Martin space is defined as

$$\mathbb{H} := \{f \mid f \text{ absolutely continuous}, f(0) = 0, \dot{f} \in L^2([0, 1], \lambda)\},$$

where λ denotes the Lebesgue measure, and is densely and continuously embedded in $\mathbb{E} = \{f \in C[0,1] | f(0) = 0\}$. Now let ν be the Wiener measure and h be given by

$$h(x) = \int_0^{\cdot} g(x,t)dt. \tag{6.2}$$

where $g \in L^2(\mathbb{E} \times [0,1], \nu \times \lambda)$ and

$$\exists D_1 g(x,\cdot) : \mathbb{H} \to L^2([0,1],\lambda) : \quad \exists M < \infty : \; \|D_1 g(x,\cdot) f\|_{L^2} \leq M \|f\|_{L^2} \tag{6.3}$$

The operator $h'(x)$ on the Cameron-Martin space $\mathbb{H}$ induces an integral operator K_x on $L^2([0,1],\lambda)$:
This we show in three steps:

1. We calculate $h'(x) : \mathbb{H} \to \mathbb{H}$:

$$\begin{aligned} (h'(x)(f)) &:= \mathbb{H} - \lim_{t\to 0} \frac{h(x+tf) - h(x)}{t} \\ &= \mathbb{H} - \lim_{t\to 0} \int_0^{\cdot} \frac{g(x+tf,\tau) - g(x,\tau)}{t} d\tau \\ &\overset{!}{=} \int_0^{\cdot} D_1 g(x,\tau) f d\tau, \end{aligned} \tag{6.4}$$

where

$$\begin{aligned} & \lim_{t\to 0} \left\| \int_0^{\cdot} \frac{g(x+tf,\tau) - g(x,\tau)}{t} d\tau \right\|_{\mathbb{H}} \\ \overset{\text{def.}}{=} \; & \lim_{t\to 0} \left(\int_0^1 \left| \frac{g(x+tf,\tau) - g(x,\tau)}{t} \right|^2 d\tau \right)^{\frac{1}{2}} = \lim_{t\to 0} \left\| \frac{g(x+tf,\cdot) - g(x,\cdot)}{t} \right\|_{L^2} \\ \overset{(6.3)}{=} \; & \|D_1 g(x,\cdot) f\|_{L^2} = \left\| \int_0^{\cdot} D_1 g(x,\tau) f d\tau \right\|_{\mathbb{H}} \end{aligned}$$

Clearly, $h'(x)(f) \in \mathbb{H}$, because $(h'(x)(f))^{\cdot} = D_1 g(x,\cdot) f \in L^2([0,1],\lambda)$. Furthermore

$$\|h'(x)(f)\|_{\mathbb{H}} = \left\| \int_0^{\cdot} D_1 g(x,\tau) f d\tau \right\|_{\mathbb{H}} = \|D_1 g(x,\cdot) f\|_{L^2} \overset{\text{assump.}}{\leq} M \|f\|_{L^2}$$

and we calculate that for $f \in \mathbb{H}$ using Cauchy-Schwarz (C.S.)

$$\|f\|_{L^2} \leq \|f\|_{\infty} \overset{\triangle}{\leq} \sup_{t\in[0,1]} \int_0^t |f'(\tau)| \, d\tau = \|f'\|_{L_1} \overset{\text{C.S}}{\leq} 1^{\frac{1}{2}} \|f'\|_{L^2} = \|f\|_{\mathbb{H}} \tag{6.5}$$

Therefore $\|h'(x)(f)\|_{\mathbb{H}} \leq M \|f\|_{\mathbb{H}}$.

2. We define $K_x : L^2([0,1],\lambda) \to L^2([0,1],\lambda)$:

Let $f \in L^2([0,1])$, then f is integrable, because $\lambda\mid_{[0,1]}$ is finite. Therefore

$$F(x) := \int_0^x f(\tau)d\tau + 0$$

is well defined and we have

$$F \in \mathbb{H} \subset C([0,1])_0 := \left\{ f \in C([0,1]) \middle| f(0) = 0 \right\},$$

because checking the definition of absolutely continuous with $\delta = \left(\frac{\varepsilon}{\|F\|_{\mathbb{H}}}\right)^2$ we obtain for $a_i \leq b_i$, $\sum_{i=1}^n |b_i - a_i| < \delta$ and $\dot{\bigcup}_{1\leq i\leq n}]a_i, b_i[\subset [a,b]$ that

$$\begin{aligned}
\sum_{j=1}^n \left| \int_{a_j}^{b_j} f(x)dx \right| &\leq \int_0^1 \mathbb{1}_{\bigcup_{j=1}^n [a_j,b_j]}(x)|f(x)|dx \\
&\overset{\text{C.S.}}{\leq} \left(\int_0^1 \mathbb{1}_{\bigcup_{j=1}^n [a_j,b_j]}(x)dx \right)^{\frac{1}{2}} \|f\|_{L^2} \\
&= \left(\sum_{j=1}^n b_j - a_j \right)^{\frac{1}{2}} \|F\|_{\mathbb{H}} < \frac{\varepsilon}{\|F\|_{\mathbb{H}}} \|F\|_{\mathbb{H}} = \varepsilon.
\end{aligned}$$

We have:

$$\begin{aligned}
K_x(f)(\cdot) &:= \frac{d}{d\tau} \underbrace{(h'(x)(F))(\tau)}_{\in \mathbb{H}} \Big|_{\tau=\cdot} \\
&\overset{(6.4)}{=} \frac{d}{d\tau} \int_0^\tau D_1 g(x,r) F dr \Big|_{\tau=\cdot} = D_1 g(x,\cdot) F \qquad (6.6)
\end{aligned}$$

3. Finally we show that K_x is an integral operator:

We show that it is a Carleman operator (cf. [Wei80, p.141,(6.5)]). Then we use that each Carleman Operator starting in L^2 has an integral representation.

Definition 6.2.2 (Carleman operator)**.**
A Carleman operator is a linear operator T from a Hilbert space H into $L^2(M,\lambda)$, where $M \subset \mathbb{R}$, measurable and for which exists a function $k : M \to H$ such that for all $f \in D(T) := \left\{ f \in \mathbb{H} \middle| Tf \text{ is defined} \right\}$

$$Tf(x) = < k(x), f >_H \quad a.e. \text{ in } M.$$

A condition to check that an operator is Carleman (cf. [Wei80, Theorem 6.16, p.144]) is

Theorem 6.2.3.
An operator K from a separable Hilbert space H into $L^2([0,T],\lambda)$ is a Carleman operator iff $Kf^m(s) \to 0$ λ-a.e. in $[0,T]$ for every null-sequence $(f^m)_{m\in\mathbb{N}}$ from $D(K)$.

Since $(L^2([0,1],\lambda), <,>_{L^2})$ is a separable Hilbert space, we check the assumption: For any f_m of a null-sequence $(f^m)_{m\in\mathbb{N}}$ in $L^2([0,1])$, defining $F_m(t) := \int_0^{\cdot} f_m(\tau)d\tau \in \mathbb{H}\ \forall t \in [0,1]$, we have

$$\begin{aligned}\left|\int_0^1 (K_x(f_m))(s)ds\right| &\overset{\text{C.S.}}{\leq} 1^{\frac{1}{2}}\left(\int_0^1 |(K_x(f_m))(s)|^2 ds\right)^{\frac{1}{2}} \\ &\overset{(6.6)}{=} \|D_1 g(x,\cdot)F_m\|_{L^2[0,1]} \overset{(6.3)}{\leq} M\,\|F_m\|_{L^2} \\ &\overset{(6.5)}{\leq} M\,\|F_m\|_{\mathbb{H}} = M\,\|f_m\|_{L^2} \underset{m\to\infty}{\longrightarrow} 0\end{aligned}$$

whence K_x is a Carleman operator from $L^2([0,1],\lambda)$ to $L^2([0,1],\lambda)$.

By [Wei80, Theorem 6.17, p.146] we gain the integral representation of K_x:

Theorem 6.2.4.
An operator K_x from $L^2([0,\tilde{T}],\lambda)$ into $L^2([0,T],\lambda)$ is a Carleman operator iff there exists a measurable function $k_x : [0,\tilde{T}] \times [0,T] \to \mathbb{C}$ such that $k_x(s,\cdot) \in L^2([0,\tilde{T}],\lambda)$ λ-a.e. in $[0,T]$ and

$$K_x f(s) = \int_0^{\tilde{T}} k_x(s,\tau)f(\tau)d\tau$$

λ-a.e. in $[0,T]$, $f \in D(K_x)$.

6.2.1 Wiener measure with an adapted integrand

From now on we assume that the integrand g is adapted (or non-anticipating) to a filtration $(\mathbf{F}_t)_{t\geq 0}$, i.e. $g(\cdot,t)$ is $\mathbf{F}_t$-measurable, where $\mathbf{F}_0$ is complete, i.e.

$$\mathbf{F}_0 = \sigma\left(\mathbf{F}_0 \cup \{N \subset \mathbb{E} \mid \exists N' \in \mathbb{B}(\mathbb{E}) : N \subset N', \nu(N') = 0\}\right)$$

and denoting by $X_t : \mathbb{E} \to \mathbb{R},\ \omega \mapsto \omega(t)$ we consider the usual filtration $F_t := \sigma(\{X_s : 0 \leq s \leq t\})$.
We show that K_x is a Volterra operator.

Definition 6.2.5 (Volterra operator)**.**

$$(K_x(f))(T) = \int_0^1 k(T,\tau)f(\tau)d\tau,\ \textit{where } k(T,\tau) = 0 \textit{ for } T < \tau.$$

If we assume a further condition (cf. Theorem 6.2.7), we derive that all powers of K_x have only the zero eigenvalue and in particular trace 0.

Lemma 6.2.6.
If g is adapted, then the operator K_x is a Volterra operator in the above sense.

Proof.
Let $T \in [0,1]$. First of all we know that then $D_1 g(x,T)\cdot$ is $\mathbf{F}_T$-measurable, i.e.

$$D_1 g(x,T)(F_n(\cdot)) = D_1 g(x,T)(F_n(\cdot \wedge T)),$$

because the L^2-lim exists and thus the pointwise limit exists for ν-a.e. point. For the pointwise derivative we know by the construction and properties of measure theory (i.e. that the limit of measurable functions is measurable) that

$$D_1 g(x,T) f = \lim_{t \to 0} \frac{g(x+tf,T) - g(x,T)}{t}$$

is $\mathbf{F}_T$-measurable. But of course, this property holds for the L^2-limit, because it is equal the pointwise ν-a.e. and all subsets of null sets are already contained in $\mathbf{F}_0$. Therefore, if we consider $K_x(f)(\cdot \wedge T)$ instead of $K_x(f)(\cdot)$ whenever $t \leq T$, we obtain

$$K_x : L^2([0,T],\lambda) \to L^2([0,T],\lambda).$$

We restrict $K_x(f)$ to the interval $[0,T]$ and, denoting this operator by $K_x\Big|_{[0,T]}(f)$ we obtain by the following calculation that it is even a Carleman operator from $[0,T]$ to $[0,T]$: Since $L^2([0,1],\lambda)$ is a separable Hilbert space, we check the assumption of Theorem 6.2.3: For any f_m of a null-sequence $(f_m)_{m\in\mathbb{N}}$ in $L^2([0,1])$ and $F_m(t) := \int_0^t f_m(\tau)d\tau$ $\forall t \in [0,1]$ we have:

$$\begin{aligned}
\left|\int_0^T (K_x(f_m))(s)ds\right| &\overset{\substack{\text{Cauchy}\\ \text{Schwartz}}}{\leq} T^{\frac{1}{2}} \left(\int_0^T |(K_x(f_m))(s)|^2 ds\right)^{\frac{1}{2}} \\
&\underset{(6.6)}{\overset{\text{def. } K_x}{=}} T^{\frac{1}{2}} \left(\int_0^T |D_1 g(x,\tau) F_m|^2 d\tau\right)^{\frac{1}{2}} \\
&\overset{\substack{D_1 g(x,T)\cdot\\ \text{is } F_T\text{-measurable}}}{=} T^{\frac{1}{2}} \left(\int_0^T |D_1 g(x,\tau) F_m(\cdot \wedge T)|^2 d\tau\right)^{\frac{1}{2}} \\
&\underset{\text{like } (6.5)}{\overset{(6.3),}{\leq}} T^{\frac{1}{2}} M \, \|f_m\|_{L^2[0,T]} \underset{m\to\infty}{\longrightarrow} 0
\end{aligned}$$

whence $K_x\Big|_{[0,T]}$ is a Carleman operator from $L^2([0,T],\lambda)$ to $L^2([0,T],\lambda)$.

By Theorem 6.2.4 we know that there exists a measurable function $k_x^T : [0,T] \times [0,T] \to \mathbb{C}$ such that $k_x^T(s,\cdot) \in L^2([0,T],\lambda)$ λ-a.e. in $[0,T]$ and $\forall s \in [0,T]$

$$\int_0^1 k_x(s,\tau)f(\tau)d\tau = K_x(f)(s) = K_x\Big|_{[0,T]}(f)(s) = \int_0^T k_x^T(s,\tau)f(\tau)d\tau \quad (6.7)$$

λ-a.e. in $[0,T]$, $\forall f \in D(K_x) = L^2([0,1],\lambda)$. Hence, taking any linear functional L and using (6.7) we have by the properties of the Lebesgue integral

$$\int_0^1 L\big(k_x(s,\tau)f(\tau)\big)d\tau = \int_0^T L\big(k_x^T(s,\tau)f(\tau)\big)d\tau.$$

Since the set of all linear functionals is point separating and $L^2([0,T],\lambda)$ as well, we obtain

$$k_x^T(s,\tau) = k_x(s,\tau) \ \lambda - a.e. \forall s,\tau \leq T \ \forall T \in [0,1].$$

Thus $k_x(T,\tau) = 0$, if $T < \tau$. □

Theorem 6.2.7.
Let there exist a measurable function $k_x : [0,T] \times [0,T] \to \mathbb{C}$ such that $k_x(x,\cdot) \in L^2([0,T],\lambda)$ a.e. in $[0,T]$ and

$$K_x(f)(T) = \int_0^1 k_x(T,\tau)f(\tau)d\tau$$

a.e. in $[0,T]$, $f \in D(K_x)$, where $k_x(T,s) = 0$, if $T < s$.
If we assume in addition

$$\left(\int_0^1 \left|\int_0^t |k_x(t,\tau)|^2 d\tau\right|^2 dt\right)^{\frac{1}{2}} \leq C < \infty, \quad (6.8)$$

then all powers of K_x have only the zero eigenvalue and in particular trace 0.

Remark 6.2.8.
In the example of an Volterra operator given in [Wer05] the kernel was assumed to be continuous.

Thus the given condition is a reasonable one, because under the mentioned assumption about the kernel (of the Volterra operator), the condition for the kernel (of the Carleman operator) is satisfied.

Proof of Theorem 6.2.7.
We will show that

$$\lim_{n\to\infty} \|{K_x}^n(f)\|_{L^2}^{\frac{1}{n}} \stackrel{!}{=} 0. \quad (6.9)$$

Once deducted this equation we know, using the Gelfand Theorem (cf. [Lax02, Theorem 4, p.195]), that the spectral radius is 0 and, because the spectrum is nonempty, it only has the 0 eigenvalue.
Using the inequality of Cauchy-Schwartz for $t \in [0,1]$ $(a, A \in L^2(\lambda))$

$$\left|\int_0^t a(s)A(s)ds\right|^2 \leq \left(\int_0^t a(s)^2 ds\right)\left(\int_0^t A(s)^2 ds\right) \tag{6.10}$$

we prepare the induction, which we will apply to prove (6.9). Let $n \in \mathbb{N}, t_{n+1} \in [0,1]$ fixed:

$$\begin{aligned}
\|{K_x}^n(f)\|^2_{L^2([0,t_{n+1}])} &= \left(\int_0^{t_{n+1}} \mid {K_x}^n(f)(t_n)|^2\, dt_n\right) \\
&= \left(\int_0^{t_{n+1}} \left|\int_0^{t_n} k_x(t_n,t_{n-1}){K_x}^{n-1}(f)(t_{n-1})dt_{n-1}\right|^2 dt_n\right) \\
&\overset{(6.10)}{\leq} \int_0^{t_{n+1}} \left(\int_0^{t_n} \mid k_x(t_n,t_{n-1})|^2\, dt_{n-1}\right)\left(\int_0^{t_n} \mid {K_x}^{n-1}(f)(t_{n-1})|^2\, dt_{n-1}\right) dt_n \\
&\underset{(6.8)}{\overset{(6.10)}{\leq}} C\left(\int_0^{t_{n+1}} \left|\left(\int_0^{t_n} \mid {K_x}^{n-1}(f)(t_{n-1})|^2\, dt_{n-1}\right)\right|^2 dt_n\right)^{\frac{1}{2}} \\
&= C\left(\int_0^{t_{n+1}} \left|\, \|{K_x}^{n-1}(f)\|^2_{L^2([0,t_n])}\right|^2 dt_n\right)^{\frac{1}{2}} \\
&= C\left(\int_0^{t_{n+1}} \|{K_x}^{n-1}(f)\|^4_{L^2([0,t_n])} dt_n\right)^{\frac{1}{2}} \qquad (6.11)
\end{aligned}$$

Repeating this procedure we obtain

$$\begin{aligned}
\|{K_x}^n(f)\|^4_{L^2([0,t_{n+1}])} &\leq C^2\left(\int_0^{t_{n+1}} \|{K_x}^{n-1}(f)\|^4_{L^2([0,t_n])} dt_n\right) \\
&\leq (C^2)^n\left(\int_0^{t_{n+1}}\left(\int_0^{t_n} \cdots \int_0^{t_1} \|K_x^0(f)\|^4_{L^2([0,t_1])} dt_0 \ldots dt_{n-1}\right) dt_n\right) \\
&\leq C^{2n}\|f\|^4_{L^2([0,1])}\left(\int_0^{t_{n+1}} \cdots \int_0^{t_1} 1 dt_0 \ldots dt_n\right) \\
&\leq C^{2n}\|f\|^4_{L^2([0,1])} \frac{1}{(n+1)!}(t_{n+1})^{n+1} \\
&\overset{t_{n+1} \in [0,1]}{\leq} C^{2n}\|f\|^4_{L^2([0,1])} \frac{1}{(n+1)!} \qquad (6.12)
\end{aligned}$$

and from this we obtain (6.9). □

Corollary 6.2.9.
If all powers of K_x have only the eigenvalue 0, then so have all powers of $h'(x)\cdot$ and their traces are 0 as well.

Proof.
Let $f \in \mathbb{H}$ be an eigenvector of $(h'(x))^n$, such that $(h'(x))^n(f) = \lambda f$, then

$$\begin{aligned}(K_x)^n(\dot{f}) &= \left(\frac{d}{d\tau}\left(h'(x)(\int_0^{\cdot} \bullet dt)\right)(\tau)\Big|_{\tau=\cdot}\right)^{n-1} \\ &\quad \circ \left(\frac{d}{d\tau}\left(h'(x)(\int_0^{\cdot} \dot{f} ds)\right)(\tau)\Big|_{\tau=\cdot}\right) \\ &\overset{\substack{\text{Main}\\ \text{Thm. of}\\ \text{Calculus}}}{=} \frac{d}{d\tau}\left((h'(x))^n f\right)(\tau)\Big|_{\tau=\cdot} = \lambda \frac{d}{d\tau} f(\tau)\Big|_{\tau=\cdot} = \lambda \dot{f}\end{aligned}$$

which using in addition that $trh'(x) \overset{\substack{\text{Lidski}\\ \text{[Lax02, p.334, 30.3]}}}{=} \sum \lambda_j(h'(x))$, where λ_j are the eigenvalues of $h'(x)$, implies the first part. □

Lemma 6.2.10.
The logarithmic derivative along the vector field $h : \mathbb{E} \to \mathbb{H}$ is the (Skorokhod or Ito) stochastic integral $\left(\int gdW\right)(x)$

Proof.
We know by [Nua06, Proposition 1.3.4] that if g is adapted that then the Ito- and Skorokhod-integral coincide. By [NZ86, p.266] and [GT82, Theorem 2, p.236] or Bismut's integration by parts formula ([Nua06, p.35, (1.41)]) we deduce that for every $g \in L^2(\mathbb{E} \times I, \nu \times \lambda)$, $\phi \in \tilde{C}_b^1$ we have, denoting the Malliavin derivative by D,

$$\begin{aligned}E(\phi \int_0^1 g(\cdot, t) dW_t) &= E(< D\phi, g >_{L^2([0,1],ds)}) \\ &\overset{\text{[Nua06, p.25]}}{=} E(\phi'_{\int_0^1 g(\cdot,s)ds}) \qquad (6.13)\end{aligned}$$

□

Using this fact and $trh'(x) = 0$, we conclude with Proposition 5.2.1

$$\begin{aligned}\left(\int gdW\right)(x) &= {}_{\tilde{C}_b^1}\beta_h^{\nu}(x) \\ &= {}_{\tilde{C}_b^1}\beta^{\nu}(h(x), x) + trh'(x) = {}_{\tilde{C}_b^1}\beta^{\nu}(h(x), x) \qquad (6.14)\end{aligned}$$

Furthermore, choosing any eigenvector f_m of $h'(x)$ we have

$$\begin{aligned}&(\ln F_2'(t, x))(f_m) = (ln(id + h'(x)))(f_m) \\ &\overset{\text{Taylor}}{=} \left(\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n}(h'(x))^{n+1}\right)(f_m)\end{aligned}$$

$$= \sum_{n=1}^{\infty} \frac{\overbrace{(-h'(x))^{n+1}(f_m)}^{=0}}{n} = 0$$

Thus by Lidski we obtain $\operatorname{tr} \ln F_2'(t,x) = 0$, and that the determinant in equation (5.19) in Theorem 5.4.1 is 1.

Since $\operatorname{tr} h'(x) = 0$ and since there exists an orthonormal base of eigenvectors of $h'(x)$ (implied by the assumption that exists an orthonormal base of eigenvectors of $F_2'(t,x)$, i.e. in this case each orthonormal base is an orthonormal base of eigenvectors of $h'(x) = 0$), we know that $h(x) = h \in \mathbb{H}$ constant. Furthermore

$$h = h(x) = \left(\int g dW\right)(x) = \left(\int g dW\right) = \left(\int_0^1 \dot{h}(t) dW_t\right)$$

Therefore (5.19) in Theorem 5.4.1 is the classical formula of Maruyama-Girsanov-Cameron-Martin (e.g. [Röc07, Theorem 4.3.6, p.74] or [Mal97, VII 8.3 Theorem]):

$$\frac{d(\nu(id + h(\cdot))}{d\nu} = \exp\left(\int_0^1 \dot{h}(t) dW_t - \frac{1}{2}\int_0^1 (\dot{h}(s))^2 ds\right)$$

Remark 6.2.11.
In general, for the Maruyama-Girsanov-Cameron-Martin formula one only assumes that g is measurable, adapted and square integrable.

Throughout this paper we have demanded a lot of conditions for $F(t,x)$ in order to apply the transformation rule (Theorem 5.4.1). But then this formula is only a special case of Theorem 5.4.1 for the Wiener measure and our assumptions seem to be stronger.

We take a closer look at the assumptions in the case that the vector field is constant. By Lemma 6.1.1 we know that the measure is differentiable along the constant vector field. We see that all other additional assumptions of the used theorems (Theorem 5.3.1, Theorem 5.4.1 and Theorem 6.2.7) are obviously fulfilled, e.g. we use that (by definition) $K_x = 0$ and find that the constant in Theorem 6.2.7 equals 0. Thus the proposed assumptions are the same as the usual ones.

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