Bijections in τ-tilting theory - a selection

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Bijections in τ-tilting theory

$$\begin{tikzcd}
 & & \text{s}\tau\text{-tilt}\,A^{op} \arrow[dd, leftrightarrow] & & & & & \\
\text{Biref}\,A & & & & & & & \\
 & & \text{s}\tau\text{-tilt}\,A \arrow[ull, dashed] \arrow[dll, dashed] \arrow[rrrr, leftrightarrow] \arrow[ddrr, leftrightarrow] \arrow[dddddd, leftrightarrow] \arrow[dddddl] & & & & \text{s}\tau^{-1}\text{-tilt}\,A \arrow[dddddd, leftrightarrow] \arrow[dddddr] & \\
\text{torf-epi}\,A \arrow[uu] \arrow[ddddddd, dashed] & & & & & & & \\
 & & & & \text{2-silt}\,A \arrow[dl, leftrightarrow] \arrow[dd] \arrow[dr, leftrightarrow] & & & \\
 & & & \text{2-smc}\,A \arrow[ddll] \arrow[dr, leftrightarrow] \arrow[ddrrrr] & & \text{int-co-t-str}\,A \arrow[dl] & & \\
 & & & & \text{int-t-str}\,A \arrow[ddll] \arrow[ddrr] & & & \\
 & \text{f}_{\text{L}}\text{-sbrick}\,A \arrow[dd, leftrightarrow] & & & & & & \text{f}_{\text{R}}\text{-sbrick}\,A \arrow[dd, leftrightarrow] \\
 & & \text{f-tors}\,A \arrow[rrrr, leftrightarrow] \arrow[dl] & & & & \text{f-torf}\,A & \\
 & \text{f}_{\text{L}}\text{-wide}\,A \arrow[drrr, dashed] & & & & & & \text{f}_{\text{R}}\text{-wide}\,A \arrow[dlll, dashed] \\
\text{fd-torf-epi}\,A \arrow[rrrr] & & & & \text{f-wide}\,A & & &
\end{tikzcd}$$

Recall

Let A be a finite dimensional basic algebra over a field $k = \bar{k}$. All modules are considered basic.

Definition

- $M \in \text{mod-}A$ is *τ-rigid* if $\mathrm{Hom}_A(M, \tau M) = 0$.
- M is τ-tilting if additionally $|M| = |A|$.
- M is almost-complete τ-tilting if $|M| = |A| - 1$, instead.
- M is support-τ-tilting if there is an $e = e^2 \in A$ with M τ-tilting over A/AeA.
- $(M, P) \in \text{mod-}A \times \mathrm{proj}\, A$ is *τ-rigid* if $\mathrm{Hom}_A(P, M) = 0$ and M is τ-rigid.
- (M, P) is support-τ-tilting if additionally $|A| = |P| + |M|$.
- (M, P) is almost-complete support-τ-tilting if $|A| = |P| + |M| - 1$, instead.

Theorem ([AIR14, Theorem 2.7])

There is a bijection

$$\begin{aligned} \mathrm{s}\tau\text{-tilt}\, A &\longleftrightarrow \text{f-tors}\, A \\ M &\longmapsto \operatorname{gen} M \\ P(\mathcal{T}) &\longleftarrow\!\shortmid \mathcal{T} \end{aligned}$$

which maps a support τ-tilting module T to a functorially finite torsion class $\operatorname{gen} T$, and conversely, a functorially finite torsion class $\mathcal{T}$ to the basic module $P(\mathcal{T}) = \bigoplus_{i=1}^{k} T_i$, with T_i Ext*-projective indecomposables in $\mathcal{T}$ (i.e. $\mathrm{Ext}_A^1(T_i, \mathcal{T}) = 0$).*

Corollary

There is an induced partial order on $\mathrm{s}\tau\text{-tilt}\, A$, defined by:

$$M \leq N \quad :\Leftrightarrow \quad \operatorname{gen} M \subset \operatorname{gen} N.$$

Theorem ([AIR14, Theorem 2.30])

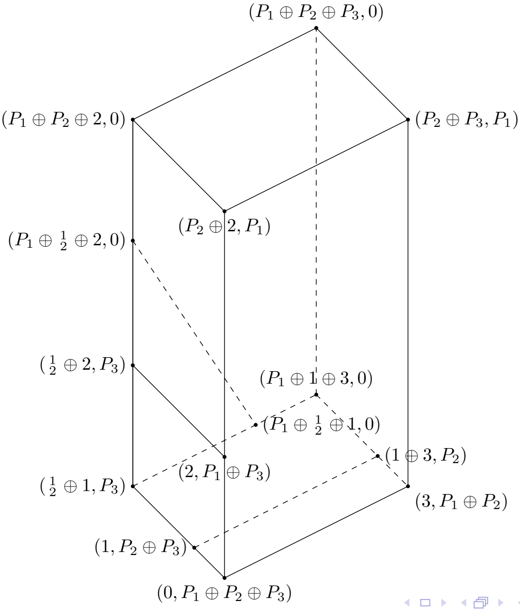
Let $T = X \oplus U$ be a basic τ-tilting A-module which is the Bongartz completion of U with X indecomposable. Let further

$$X \xrightarrow{f} U' \xrightarrow{g} Y \to 0$$

be an exact sequence with f a minimal left $(\operatorname{add} U)$-approximation. Then the following holds:

- *if U is not sincere, then $Y = 0$ and $U = \mu_X^-(T)$. Therefore, the left mutation of T is a basic support τ-tilting module which is not τ-tilting.*
- *if U is sincere, then $Y \in \operatorname{add} Y_1$ for some indecomposable $Y_1 \notin \operatorname{add} T$. In this case $\mu_X^-(T) = Y_1 \oplus U$ is a basic τ-tilting module.*

Example: $Q = 1 \to 2 \to 3$, $A = kQ$.



2-term silting objects

Definition

Let A be a ring. A complex $P \in \mathsf{H}^b(\mathrm{proj}\,A)$ is called

1. *presilting* if $\mathrm{Hom}_{\mathsf{H}^b(\mathrm{proj}\,A)}(P, P[i]) = 0$ for any $i > 0$, and it is called
2. *silting* if it is presilting and if additionally the summands of shifts of P generate $\mathsf{H}^b(\mathrm{proj}\,A)$.
3. $P = (P^i, d^i)$ is called *two-term* if P^i vanishes for all $i \neq 0, -1$ up to chain homotopy equivalence.

Proposition ([AI12, Theorem 2.11])

There is a partial order on silting complexes defined by

$$P \geq Q \quad :\Leftrightarrow \quad \mathrm{Hom}_{\mathsf{H}^b(\mathrm{proj}\, A)}(P, Q[i]) = 0 \textit{ for all } i > 0.$$

Lemma

A complex P is two-term if and only if $A \geq P \geq A[1]$.

Silting mutation [AI12]

Proposition

Let $M = M_1 \oplus ... \oplus M_n$ be an indecomposable decomposition of a silting sequence M. Then for any minimal left $\mathrm{add}(\bigoplus_{j \neq i} M_j)$-approximation sequence $M_i \xrightarrow{f} E \to M_i^* \to M_i[1]$ of M_i there is a left mutation of M at the direct summand M_i:

$$\mu_{M_i}^{-}(M) = M_i^* \oplus \bigoplus_{j \neq i} M_j,$$

.

Proposition

The Hasse quiver of silting sequences coincides with the mutation quiver of silting sequences.

Bijection: $\mathrm{s}\tau\text{-tilt}\, A$ and $2\text{-silt}\, A$

Theorem ([AIR14, Theorem 3.2])

Let A be a finite dimensional k-algebra. Then there is a bijection

$$\begin{aligned} 2\text{-silt}\, A &\longrightarrow \mathrm{s}\tau\text{-tilt}\, A \\ P &\longmapsto H^0(P) \\ (P_1 \oplus P \xrightarrow{(f\ \ 0)^t} P_0) &\longleftarrow\!\shortmid (M, P), \end{aligned}$$

for f a minimal projective presentation of M.

Corollary

This map in is an isomorphism of partially ordered sets. In particular, it induces an isomorphism between the two-term silting quiver $Q(2\text{-silt}\, A)$ and the support τ-tilting quiver $Q(\mathrm{s}\tau\text{-tilt}\, A)$.

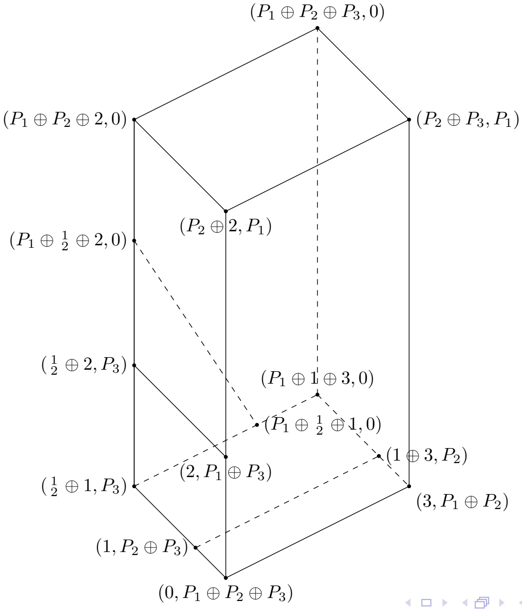
Alternatively:

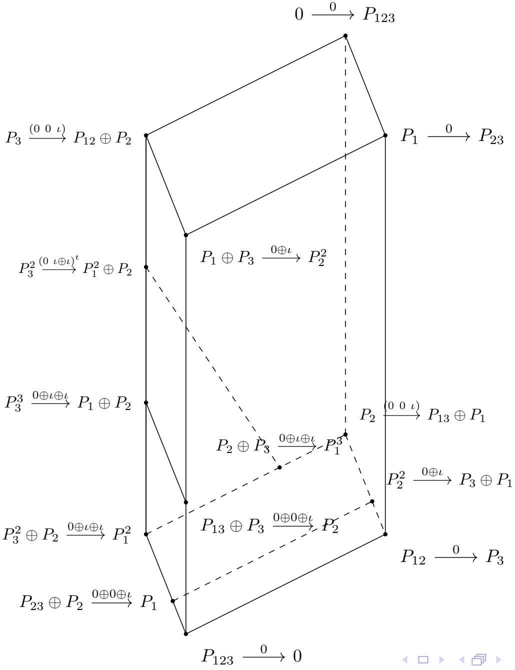
Corollary

This map preserves mutation.

Sketch: Let $M_i \xrightarrow{f} E \to M_i^* \to M_i[1]$ be a mutation sequence for $2\text{-silt}\, A$. Taking 0-th cohomology, we get an $\mathrm{add}(\bigoplus_{i \neq j} H^0 M_j)$-left-approximation sequence of $H^0 M_i$.
$H^0 f$ is left minimal: a morphism $\varphi \in \mathrm{End}_A(H^0 E)$ with $\varphi \circ H^0 f = H^0 f$ extends uniquely to $\tilde{\varphi} \in \mathrm{End}_{\mathsf{H}^b(\mathrm{proj}\, A)}(E)$ such that $\tilde{\varphi} \circ f = f$. Thus, by minimality of f, $\tilde{\varphi}$ is an isomorphism and so is φ.
The converse is shown similarly.

Example: $Q = 1 \to 2 \to 3$, $A = kQ$.




$0 \xrightarrow{0} P_{123}$
$P_3 \xrightarrow{(0\ 0\ \iota)} P_{12} \oplus P_2$
$P_1 \xrightarrow{0} P_{23}$
$P_1 \oplus P_3 \xrightarrow{0\oplus\iota} P_2^2$
$P_3^2 \xrightarrow{(0\ \iota\oplus\iota)^t} P_1^2 \oplus P_2$
$P_3^3 \xrightarrow{0\oplus\iota\oplus\iota} P_1 \oplus P_2$
$P_2 \xrightarrow{(0\ 0\ \iota)} P_{13} \oplus P_1$
$P_2 \oplus P_3 \xrightarrow{0\oplus\iota\oplus\iota} P_1^3$
$P_2^2 \xrightarrow{0\oplus\iota} P_3 \oplus P_1$
$P_3^2 \oplus P_2 \xrightarrow{0\oplus\iota\oplus\iota} P_1^2$
$P_{13} \oplus P_3 \xrightarrow{0\oplus 0\oplus\iota} P_2$
$P_{12} \xrightarrow{0} P_3$
$P_{23} \oplus P_2 \xrightarrow{0\oplus 0\oplus\iota} P_1$
$P_{123} \xrightarrow{0} 0$

Universal localisations and bireflective subcategories

Let A be a ring and Σ a set of morphisms in the category $\mathrm{proj}\, A$.

$$\mathcal{D}_\Sigma := \{X \in \mathrm{Mod}\text{-}A \mid \mathrm{Hom}_A(\sigma, X) \text{ is surjective for all } \sigma \in \Sigma\}.$$

If $\Sigma = \{\sigma\}$, we just write $\mathcal{D}_\sigma$.

Proposition ([AMV16, Proposition 3.15])

Let $T \in \text{mod-}A$. T is

- *τ-rigid iff there are $P, Q \in \mathrm{proj}\, A$ and $P \xrightarrow{\sigma} Q \to T \to 0$ such that $\mathcal{D}_\sigma$ is a torsion class containing T ('silting presentation').*
- *support-τ-tilting iff additionally $\mathcal{D}_\sigma = \mathrm{Gen}(T)$.*

Proof: Let $P^{-1} \xrightarrow{\sigma'} P^0 \to T \to 0$ with σ' minimal.
1. $\mathrm{Hom}_A(M, \tau T) = 0$ iff $\mathrm{Hom}_A(\sigma', M)$ is surjective:
$\mathrm{Hom}_A(M, \nu P) \cong \mathrm{Hom}_k(M \otimes_A P^*, k) = D(M \otimes_A P^*) \cong D\,\mathrm{Hom}_A(P, M)$ $\forall P \in \mathrm{proj}\, A$. Recall the τ-translate $0 \to \tau T \to \nu P^{-1} \to \nu P^0$. We have the commutative diagram:

$$\begin{tikzcd}
0 \arrow[r] & \mathrm{Hom}_A(M, \tau T) \arrow[r] & \mathrm{Hom}_A(M, \nu P^{-1}) \arrow[r] \arrow[d, "\wr"] & \mathrm{Hom}_A(M, \nu P^0) \arrow[d, "\wr"] \\
 & & D\,\mathrm{Hom}_A(P^{-1}, M) \arrow[r, "Dh^{\sigma'}(M)"] & D\,\mathrm{Hom}_A(P^0, M)
\end{tikzcd}$$

2. Hence, $T \in \mathcal{D}_{\sigma'}$ iff it is τ-rigid, which is a torsion class, since σ' is a perfect complex and thus finitely presented. Therefore, Gen $T \subset D_{\sigma'}$.
3. (T, R) is support-τ-tilting iff Gen $T = {}^{\perp}\tau T \cap R^{\perp}$, and this equals $\mathcal{D}_{\sigma}$ with $\sigma = \sigma' \oplus R[1]$.
(cf. [AIR14][Corollary 2.13] and [Mar15][Proposition 7.4.2])

Definition

Two ring epimorphism starting in A, $f_1 : A \to B_1$ and $f_2 : A \to B_2$, are said to be in the same *epiclass* if there is a ring isomorphism $\rho : B_1 \to B_2$ such that $f_2 = \rho \circ f_1$

The class of epiclasses of ring epimorphisms starting in A has an intrinsic partial order given by

$$f_1 \geq f_2 \;\; :\Leftrightarrow \;\; \exists \text{ring epimorphism } \rho : B_1 \to B_2 \text{ such that } f_2 = \rho \circ f_1.$$

Bireflective and wide subcategories

Definition

A full subcategory $\mathcal{X}$ in $\mathrm{Mod}\text{-}A$ is called *bireflective* if the inclusion functor admits both a left and right adjoint.
A full subcategory $\mathcal{W}$ is called *wide* if it is closed under kernels, cokernels and extensions in the ambient category.

A subcategory $\mathcal{X}$ of $\mathrm{Mod}\text{-}A$ is bireflective if and only if it is closed under products, coproducts, kernels and cokernels.

Universal localisations

Definition

Let Σ be a set of morphisms in $\operatorname{proj} A$. Then a ring homomorphism $f_\Sigma : A \to B$ is called *universal localisation of A at Σ* if the following properties hold:

1. f_Σ is *Σ-inverting*, i.e. $\sigma \otimes_A B$ is an isomorphism for every $\sigma \in \Sigma$.
2. f_Σ is *universal Σ-inverting*, i.e. every Σ-inverting ring homomorphism $f : A \to B'$ factors uniquely through f_Σ.

In this case we write $A_\Sigma := B$.

For any ring R and any set of maps in $\operatorname{proj} R$, a universal localisation exists, see [Sch85, Theorem 4.1]. Every universal localisation defines an epiclass of ring epimorphisms.

Theorem ([AMV19])

There is a commutative diagram of injections (bijections if A τ-tilting finite):

$$\begin{tikzcd}
\left\{ \begin{array}{c} \textit{Equivalence classes of} \\ \textit{support-}\tau\textit{-tilting} \\ A\textit{-modules} \end{array} \right\} \arrow[rr, "\alpha"] \arrow[dr, "\beta"] & & \left\{ \begin{array}{c} \textit{Epiclasses of ring} \\ \textit{epimorphisms } A \to B \\ \textit{with } \mathrm{Tor}_1^A(B,B) = 0 \end{array} \right\} \arrow[dl, "\epsilon"] \\
& \left\{ \begin{array}{c} \textit{Bireflective extension-} \\ \textit{closed subcategories} \\ \textit{of } \mathrm{Mod}\text{-}A \end{array} \right\} &
\end{tikzcd}$$

Moreover the maps α, β and ϵ preserve the partial order, where the partial order on bireflective subcategories is given by inclusion. These will be defined in the subsequent steps.

Proposition ([GdlP87, Theorem 1.2])

There is a bijection

$$\left\{\begin{array}{c}\textit{Epiclasses of ring}\\ \textit{epimorphisms}\\ A \to B\end{array}\right\} \xrightarrow{\epsilon} \left\{\begin{array}{c}\textit{Bireflective}\\ \textit{subcategories}\\ \textit{of } \mathrm{Mod}\text{-}A\end{array}\right\}$$

$$(A \to B) \longmapsto \mathrm{essIm}(\mathrm{res}_A^B).$$

Proposition ([Sch85, Theorem 4.8])

$\mathrm{essIm}\left(\mathrm{res}_A^B\right)$ *is closed under extensions iff* $\mathrm{Tor}_1^A(B, B) = 0$.

Therefore we get:

Corollary

The map ϵ *is a bijection.*

A similar result:

Proposition ([Iya03, Theorem 1.6.1])

For a finite dimensional algebra A we have a bijection

$$\left\{ \begin{array}{c} \textit{Epiclasses of ring} \\ \textit{epimorphisms } A \to B \textit{ with} \\ \operatorname{Tor}_1^A(B,B) = 0 \textit{ and } \dim B < \infty \end{array} \right\} \longrightarrow \left\{ \begin{array}{c} \textit{Functorially finite} \\ \textit{wide subcategories} \\ \textit{of } \text{mod-}A \end{array} \right\}$$

$$(A \to B) \longmapsto \operatorname{essIm}(\operatorname{res}_A^B) \cap \text{mod-}A.$$

Bijections in τ-tilting theory

$$\begin{tikzcd}
 & & \text{s}\tau\text{-tilt}\,A^{op} \arrow[dd, leftrightarrow] & & & & & \\
\text{Biref}\,A & & & & & & & \\
 & & \text{s}\tau\text{-tilt}\,A \arrow[ull, dashed] \arrow[dll, dashed] \arrow[rrrr, leftrightarrow] \arrow[ddrr, leftrightarrow] \arrow[dddddl] \arrow[dddddd, leftrightarrow] & & & & \text{s}\tau^{\text{-}1}\text{-tilt}\,A \arrow[dddddd, leftrightarrow] \arrow[dddddr] & \\
\text{torf-epi}\,A \arrow[uu] \arrow[ddddddd, dashed] & & & & & & & \\
 & & & & \text{2-silt}\,A \arrow[dl, leftrightarrow] \arrow[dd] \arrow[dr, leftrightarrow] & & & \\
 & & & \text{2-smc}\,A \arrow[dr, leftrightarrow] \arrow[ddll] \arrow[ddrrrr] & & \text{int-co-t-str}\,A \arrow[dl] & & \\
 & & & & \text{int-t-str}\,A \arrow[ddll] \arrow[ddrr] & & & \\
 & \text{f}_{\text{L}}\text{-sbrick}\,A \arrow[dd, leftrightarrow] & & & & & & \text{f}_{\text{R}}\text{-sbrick}\,A \arrow[dd, leftrightarrow] \\
 & & \text{f-tors}\,A \arrow[rrrr, leftrightarrow] \arrow[dl] & & & & \text{f-torf}\,A & \\
 & \text{f}_{\text{L}}\text{-wide}\,A \arrow[drrr, dashed] & & & & & & \text{f}_{\text{R}}\text{-wide}\,A \arrow[dlll, dashed] \\
\text{fd-torf-epi}\,A \arrow[rrrr] & & & & \text{f-wide}\,A & & &
\end{tikzcd}$$

From $\mathrm{s}\tau$-tilt A to epiclasses

Proposition

For a τ-finite algebra A there is a bijection

$$\left\{ \begin{array}{c} \textit{Equivalence} \\ \textit{classes in} \\ \mathrm{s}\tau\text{-tilt}\, A \end{array} \right\} \xrightarrow{\alpha} \left\{ \begin{array}{c} \textit{Epiclasses of ring} \\ \textit{epimorphisms } A \to B \\ \textit{with } \mathrm{Tor}_1^A(B,B) = 0 \end{array} \right\},$$

$$[T] \xmapsto{\alpha} f_{\sigma_1},$$

for a minimal left $\mathrm{Add}(\sigma)$*-approximation* φ *of* $A[0]$ *in* $\mathrm{D}^b(A)$

$$A[0] \xrightarrow{\varphi} \sigma_0 \to \sigma_1 \to A[1],$$

σ *a silting presentation of* T *and* f_{σ_1} *the universal localisation of* A *at* σ_1*.*

Sketch:

- $A[0] \xrightarrow{\varphi} \sigma_0 \to \sigma_1 \to A[1] \quad \longleftrightarrow \quad A \xrightarrow{f} T_0 \to T_1 \to 0$, with f a minimal left $\mathrm{add}(T)$-approximation of A.
- $T_1 \in \mathrm{add}\, T$, thus τ-rigid.
- $\mathcal{X}_{\sigma_1} := \{X \in \mathrm{Mod}\text{-}A \mid \mathrm{Hom}_A(\sigma, X)$ is bijective $\} = \mathcal{D}_{\sigma_1} \cap \mathsf{Coker}(\sigma_1)^{\perp} = \mathsf{Gen}(T) \cap T_1^{\perp}$.
- $\mathsf{Gen}(T) \cap T_1^{\perp}$ is extension-closed bireflective (see next Prop).
- $\mathcal{X}_{\sigma_1} = \mathrm{essIm}(\mathrm{res}_A^B)$ for some ring epimorphism $A \to B$ with $\mathrm{Tor}_1^A(B, B) = 0$.
- This epi is the universal localisation of A at σ_1 (see e.g. Hennings book vol. 2, chapter 2.3).

From $\mathrm{s}\tau\text{-}\mathrm{tilt}\, A$ to bireflective subcategories

Let $\mathcal{T}$ be a torsion class in an Abelian category $\mathcal{A}$. Then we define the full subcategory

$$\mathfrak{a}(\mathcal{T}) := \{X \in \mathcal{T} \mid \text{if } (g : Y \to X) \in \mathcal{T}, \text{ then } \operatorname{Ker}(g) \in \mathcal{T}\}.$$

For T support τ-tilting, we set

$$\beta([T]) := \mathfrak{a} \circ \mathrm{Gen}(T).$$

Proposition

We have the diagram of maps

$$\begin{tikzcd}
\left\{\begin{array}{c} [T] \subset \text{mod-}A \\ \textit{support-}\tau\textit{-} \\ \textit{tilting class} \end{array}\right\} \arrow[rr, "\beta = \mathfrak{a} \circ \mathrm{Gen}"] \arrow[dr, "\mathrm{Gen}"] & & \left\{\begin{array}{c} \textit{Bireflective extension-} \\ \textit{closed subcategories} \\ \textit{of } \mathrm{Mod}\text{-}A \end{array}\right\} . \arrow[dl, shift left, "\mathrm{FiltGen}"] \\
& \left\{\begin{array}{c} \textit{Torsion classes} \\ \mathcal{T} \subset \mathrm{Mod}\text{-}A \end{array}\right\} \arrow[ur, shift left, "\mathfrak{a}"] &
\end{tikzcd}$$

The map β is bijective if A is τ-tilting finite, and it preserves the partial order. The inverse of $\mathfrak{a}$ is given by $\mathrm{FiltGen}(-)$. Moreover, the diagram in the theorem is commutative.

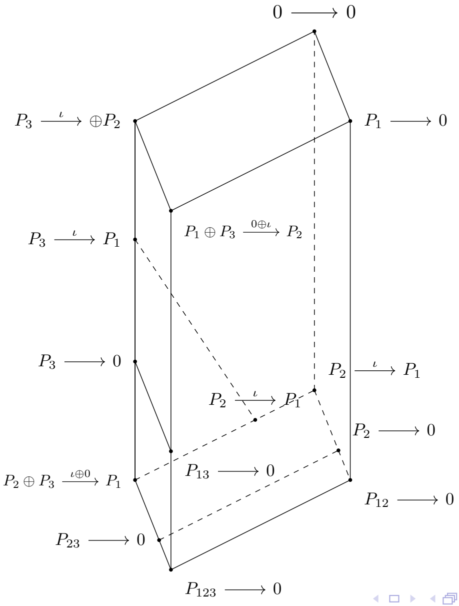
Sketch:

- $\mathfrak{a}(\mathcal{T})$ is a wide subcategory for $\mathcal{T}$ a torsion class.
- $\mathfrak{a}(\mathsf{Gen}\, T) = \mathsf{Gen}(T) \cap T_1^{\perp}$ for T_1 as in the last proposition.
- The RHS is closed under (co)products, thus bireflective.
- $\mathfrak{a}$ is left-inverse of $\mathrm{Filt}\mathsf{Gen}$.
- $\mathrm{Filt}\mathsf{Gen}$ is left inverse of $\mathfrak{a}|_{\mathrm{Im}(\mathsf{Gen}(-))}$.
- If $\mathcal{T} \cap \text{mod-}A \in \text{f-tors}\, A$, then $\mathcal{T} = \mathsf{Gen}\, T$ for $T \in \text{s}\tau\text{-tilt}\, A$.
- A is τ-tilting finite iff all torsion classes are functorially finite.
- Then, Gen is bijective here.
- The order is preserved.

Calculation of ring epis

Given a support-τ-tilting module (T, R), what is the associated ring epi?
Take minimal left $\mathrm{add}(T)$-approximation $A \to T_0 \to T_1 \to 0$.
Localise at direct sum $(P \xrightarrow{\sigma'} Q) \oplus (R \to 0)$ for σ' a minimal projective presentation of T_1

Example



$0 \longrightarrow 0$

$P_3 \xrightarrow{\iota} \oplus P_2$

$P_1 \longrightarrow 0$

$P_1 \oplus P_3 \xrightarrow{0\oplus\iota} P_2$

$P_3 \xrightarrow{\iota} P_1$

$P_3 \longrightarrow 0$

$P_2 \xrightarrow{\iota} P_1$

$P_2 \xrightarrow{\iota} P_1$

$P_2 \longrightarrow 0$

$P_{13} \longrightarrow 0$

$P_2 \oplus P_3 \xrightarrow{\iota\oplus 0} P_1$

$P_{12} \longrightarrow 0$

$P_{23} \longrightarrow 0$

$P_{123} \longrightarrow 0$

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