Tilting, cluster-tilting and τ-tilting

A brief introduction

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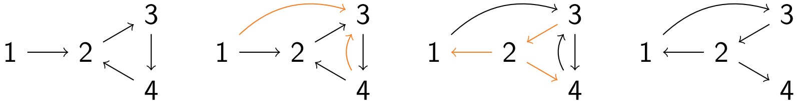
Mutation of quivers

Let Q be a finite, connected quiver without loops.
A mutation on vertex i is done in three steps:

- For every $j \to i \to k$, add $j \to k$.
- Reverse every arrow starting or ending at i.
- Remove 2-cycles.

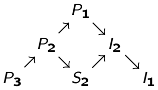
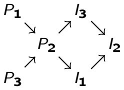
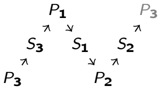
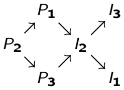


Q $\mu_2(Q)$

Note that $\mu_i(\mu_i(Q)) = Q$.

Mutation and algebras

Let R_Q be the set of relations on Q generated by compositions of two and two arrows in the 3-cycles of Q. Let Q' be obtained by a (series of) mutation(s) on Q.
It turns out that $\operatorname{mod} kQ/R_Q$ and $\operatorname{mod} kQ'/R_{Q'}$ are related.

$1 \longrightarrow 2 \longrightarrow 3$	$1 \longleftarrow 2 \longrightarrow 3$	$\begin{matrix} & 1 & \\ \nearrow & & \searrow \\ 2 & \longleftarrow & 3 \end{matrix}$	$1 \longrightarrow 2 \longleftarrow 3$
			

Setup

- Λ is a finite-dimensional algebra over $k = \overline{k}$.
- $\operatorname{mod} \Lambda$ is the category of finitely generated left Λ-modules.
- $\operatorname{proj} \Lambda \subseteq \operatorname{mod} \Lambda$ is the full subcategory of projectives.
- For a module $M \in \operatorname{mod} \Lambda$:
  - $\operatorname{add} M \subseteq \operatorname{mod} \Lambda$ is the smallest subcategory closed under sums and summands containing M,
  - $\operatorname{sub} M \subseteq \operatorname{mod} \Lambda$ is the full subcategory of submodules of M^n.
  - $\operatorname{fac} M \subseteq \operatorname{mod} \Lambda$ is the full subcategory of factor modules of M^n.
  - $|M|$ is the number indecomposable summands of M up to isomorphism.
- The Auslander-Reiten translation in $\operatorname{mod} M$ is denoted τ.
- All modules (that we talk about) are basic.

This talk will mainly follow [IR]

(classical) Tilting Theory

Idea

- Start with a nice algebra Λ
- Look at a not-so-nice algebra Γ, related to Λ
- Use Λ to understand Γ.

A (very brief) History

1973 Berenstein, Gelfand, Ponomarev: BGP-reflections [BGP]

1979 Auslander, Platzek, Reiten: APR-tilting [APR]

1980 Bongartz: Mutation, completion [Bon]

1980 Brenner, Butler: Tilting theorem [BB]

1982 Happel, Ringel: Further important theorems [HR]

Tilting modules

Definition

A module $T \in \operatorname{mod} \Lambda$ is called *partial tilting* if:

1. $\operatorname{pdim} T \leq 1$,
2. $\operatorname{Ext}^1_\Lambda(T, T) = 0$.

It is called *tilting* if in addition $|T| = |\Lambda|$.

Theorem

T is tilting if it is partial tilting and there exists a short exact sequence

$$0 \to \Lambda \to T_0 \to T_1 \to 0$$

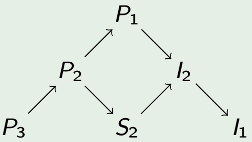
with $T_0, T_1 \in \operatorname{add} T$.

Completions

Theorem [Bon]

Any partial tilting module U is a direct summand of a tilting module $T = U \oplus X$.
T is called the *completion* of U.
X is called the *complement* of U.

Example



The partial tilting module $S_2 \oplus I_2$ has completion $S_2 \oplus I_2 \oplus P_1$.
The partial tilting module $P_3 \oplus P_1$ has completions $P_3 \oplus P_2 \oplus P_1$ and $P_3 \oplus P_1 \oplus I_1$.

Completions II

A partial tilting module U with $|U| = |\Lambda| - 1$ is called an *almost complete tilting module*.

Theorem [HU],+

An almost complete tilting module has one or two completions.

Theorem [HU],+

Let U be an almost complete tilting module with complements X and Y. There exists a short exact sequence

$$0 \to X \xrightarrow{f} U' \xrightarrow{g} Y \to 0$$

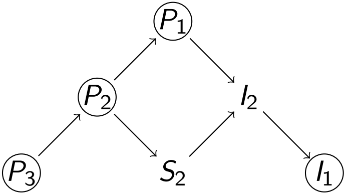
(up to interchanging X and Y) such that:
f is a minimal left add $U-$ *approximation*.
g is a minimal right add $U-$ *approximation*.

"Mutation": $T \to U \to T'$

Example

$\Lambda = kQ$ with

$Q = 1 \xrightarrow{\alpha} 2 \xrightarrow{\beta} 3$



$T = P_1 \oplus P_2 \oplus P_3$ is a tilting object.
$\mathrm{End}(T)^{\mathrm{op}} = kQ$

$U = P_1 \oplus P_3$ is an almost complete tilting object.

$T' = P_1 \oplus P_3 \oplus I_1$ is a tilting object.
$\mathrm{End}(T)^{\mathrm{op}} = kQ/\langle \alpha\beta \rangle$

Torsion pairs

Definition

A *torsion pair* in $\operatorname{mod}\Lambda$ is a pair of full subcategories $(\mathscr{T},\mathscr{F})$ such that

- $\operatorname{Hom}(\mathscr{T},\mathscr{F}) = 0$
- For all $X \in \operatorname{mod}\Lambda$ there is a short exact sequence $0 \to T \to X \to F \to 0$ with $T \in \mathscr{T}$ and $F \in \mathscr{F}$.

$\mathscr{T}$ is called a *torsion class* and $\mathscr{F}$ a *torsion-free class*

- $\mathscr{T} = {}^{\perp}\mathscr{F}$ and $\mathscr{F} = \mathscr{T}^{\perp}$ determine each other
- A torsion pair "determines" the module category

Brenner-Butler Tilting Theorem

The Brenner-Butler tilting theorem [BB]

Let T be a tilting module in $\operatorname{mod}\Lambda$ with $\Gamma = \operatorname{End}(T)^{\mathrm{op}}$.
Let $\mathscr{T} = \operatorname{fac} T$ and $\mathscr{F} = \mathscr{T}^{\perp}$.
In $\operatorname{mod}\Gamma$, let $\mathscr{Y} = \operatorname{sub} DT$ and $\mathscr{X} = {}^{\perp}\mathscr{Y}$.

- $(\mathscr{T}, \mathscr{F})$ is a torsion pair in $\operatorname{mod}\Lambda$
- $(\mathscr{X}, \mathscr{Y})$ is a torsion pair in $\operatorname{mod}\Gamma$
- There are mutually inverse equivalences:

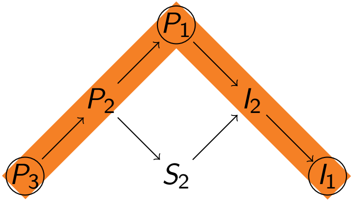
$$\mathscr{T} \underset{- \otimes_{\Gamma} DT}{\overset{\operatorname{Hom}_{\Lambda}(T,-)}{\rightleftarrows}} \mathscr{Y} \qquad \mathscr{F} \underset{\operatorname{Tor}^{1}_{\Gamma}(-,DT)}{\overset{\operatorname{Ext}^{1}_{\Lambda}(T,-)}{\rightleftarrows}} \mathscr{X}$$

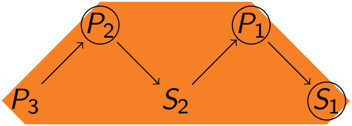
Example

$$\Lambda = kQ$$

$$Q = 1 \xrightarrow{\alpha} 2 \xrightarrow{\beta} 3$$

$$\Gamma = kQ/\langle \alpha\beta \rangle$$





$$T = P_1 \oplus P_3 \oplus I_1$$
$$\mathscr{T} = \text{fac}\, T$$
$$\mathscr{F} = \mathscr{T}^{\perp} = \emptyset$$

$$DT = P_1 \oplus P_2 \oplus S_1$$
$$\mathscr{Y} = \text{sub}\, DT$$
$$\mathscr{X} = {}^{\perp}\mathscr{Y} = \emptyset$$

Cluster tilting theory

- We want to mimic mutation on quivers closer.
- We would like to be able to mutate on every summand of tilting objects.
- Something called cluster algebras that looks interesting.

One way to view the "missing" mutations is that we don't have enough objects.
Solution: Add more objects!

The cluster category

Definition [BMRRT]

Let H be a hereditary algebra. The Verdier quotient category

$$\mathscr{C}_H = \mathscr{D}^b(\operatorname{mod} H)/\tau^{-1}[1]$$

is the *cluster category* of H.

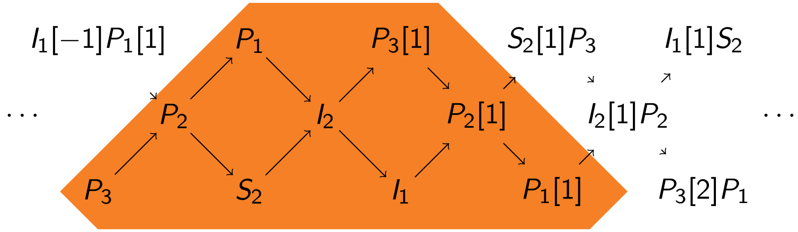
Generalizations for non-hereditary algebras exist.
The objects of $\mathscr{C}_H$ are the $\tau^{-1}[1]$-orbits in $\mathscr{D}^b(\operatorname{mod} H)$. By abuse of notation we write X for the $\tau^{-1}[1]$-orbit containing X.
In $\mathscr{C}_H$, we have $X[1] \cong \tau X$
Morphisms in $\mathscr{C}_H$ are given by

$$\operatorname{Hom}_{\mathscr{C}_H}(X, Y) = \coprod_{i \in \mathbb{Z}} \operatorname{Hom}_{\mathscr{D}^b(\operatorname{mod} H)}(\tau^{-i} X[i], Y)$$

Example

$\Lambda = kQ$

$Q = 1 \xrightarrow{\alpha} 2 \xrightarrow{\beta} 3$



$$\text{mod}\,\Lambda \to \mathscr{D}^b(\text{mod}\,\Lambda) \xrightarrow{-/\tau^{-1}[1]} \mathscr{C}_\Lambda$$

Important properties

- $\operatorname{Ob} \mathscr{C}_\Lambda \cong \operatorname{mod} \Lambda \cup \operatorname{proj} \Lambda[1]$.
- $\mathscr{C}_\Lambda$ is triangulated [Kel]
- $\mathscr{C}_\Lambda$ is 2-Calabi-Yau, Krull-Remak-Schmidt and Hom-finite

All in all this makes it an attractive source of examples.

Cluster-tilting objects

Definition [BMRRT]

An object $T \in \mathscr{C}_\Lambda$ is a *cluster-tilting object* if

- T is rigid, i. e. $\mathrm{Ext}_{\mathscr{C}_\Lambda}(T, T) = 0$.
- $\mathrm{add}\, T = \{X | \, \mathrm{Ext}_{\mathscr{C}_\Lambda}(T, X) = 0\}$

Theorem [BMRRT]

The following are equivalent

- T is cluster-tilting
- T is *maximal rigid*. That is, if $T \oplus X$ is rigid, then $X \in \mathrm{add}\, T$.

Mutation

Definition [BMRRT]

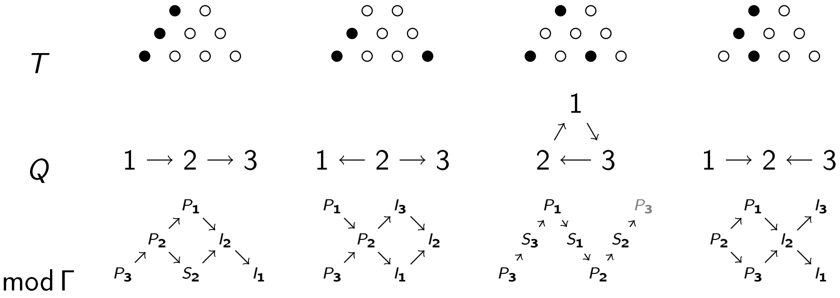
An object $U \in \mathscr{C}_\Lambda$ is called *almost cluster-tilting* if there exists some indecomposable $X \in \mathscr{C}_\Lambda$ such that $U \oplus X$ is cluster-tilting. We call $U \oplus X$ the *completion* of U.

Theorem [BMRRT]

Any almost cluster-tilting object has exactly two completions up to isomorphism.

Example of mutation

Let T be a cluster-tilting object and let $\Gamma = \mathrm{End}_{\mathscr{C}_\Lambda}(T)^{\mathrm{op}} = kQ/R$.



Cluster-tilting

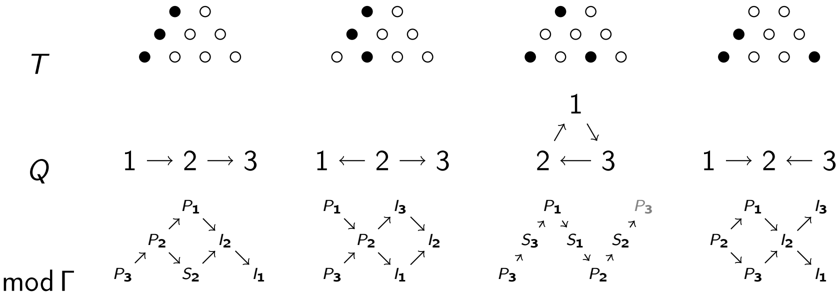
Theorem [BMRRT]

Let Λ be a hereditary algebra with cluster category $\mathscr{C}_\Lambda$. Let $T \in \mathscr{C}_\Lambda$ be cluster-tilting. Let $\mathscr{C}_\Lambda / T[1]$ be the quotient category obtained by factoring out any morphism which factors through add $T[1]$. The following diagram commutes:

$$\begin{array}{ccc} \mathscr{C}_\Lambda & \xrightarrow{\operatorname{Hom}_{\mathscr{C}_\Lambda}(T,-)} & \operatorname{mod} \operatorname{End}_{\mathscr{C}_\Lambda}(T)^{\mathrm{op}} \\ \downarrow{\scriptstyle \pi} & \nearrow{\scriptstyle F} & \\ \mathscr{C}_\Lambda / T[1] & & \end{array}$$

and F is an equivalence of categories.

Example



Notice how we get $\operatorname{mod}\Gamma$ by "removing" $T[1]$ from $\mathscr{C}_\Lambda$.

Why "clusters"?

Because Cluster Algebras!

How to make a cluster algebra[FZ1]

Fix

- A positive integer n,
- A function field $F = \mathbb{Q}(x_1, \cdots, x_n)$,
- A quiver Q with n vertices.

The pair $(\{x_1, \cdots, x_n\}, Q)$ is the *initial seed*.
For $1 \leq i \leq n$, define a mutation of the seed at i to be $(\{x'_1, \cdots, x'_n\}, \mu_i(Q))$, where $x'_j = x_j$ for $j \neq i$ and $x'_i = \frac{m_1+m_2}{x_i}$, where m_1 and m_2 are monomials determined by Q.
Iterate this process!

Mutation is an idempotent operation!

Cluster Algebras II

How to make a cluster algebra, continued

By iteration we obtain a collection of pairs (S, G)
Each S is a generating set of $F = \mathbb{Q}(x_1, \cdots, x_n)$, called a *cluster*.
Its elements are called *cluster variables*.
Each G is a quiver.
We can now define the *cluster algebra*:

$$A = \langle \text{all cluster variables} \rangle \subseteq F$$

Why cluster algebras?

Because they turned out to be very useful! [FZ2]

- Discrete dynamical systems based on rational recurrences.
- Quantum cluster algebras, Poisson geometry and Teichmüller theory.
- Grassmannians, projective configurations and their tropical analogues.
- Generalized associahedra associated with finite root systems.

...and cluster categories provide a categorification of cluster algebras.

τ-tilting theory

- The combinatorics of cluster-tilting are very satisfying!
- Can we find a way to do them in the module category?
- Yes, thanks to [AIR]!

Definition

A module $T \in \operatorname{mod} \Lambda$ is

τ-rigid if $\operatorname{Hom}_\Lambda(T, \tau T) = 0$.

τ-tilting if it is τ-rigid and $|T| = |\Lambda|$.

support τ-tilting if there exists an idempotent $e \in \Lambda$ such that T is is τ-tilting in $\operatorname{mod} \Lambda/\langle e \rangle$.

Important properties

A module M is *faithful* if $\operatorname{ann} M = \{\lambda \in \Lambda | \lambda M = 0\} = 0$

Theorem [AIR]

Tilting modules are precisely faithful (support) τ-tilting modules.

Theorem [AIR]

For a τ-rigid module, the following are equivalent:

- T is τ-tilting.
- T is maximal τ-rigid.
- If $\operatorname{Hom}_\Lambda(T, \tau X) = 0 = \operatorname{Hom}_\Lambda(X, \tau T)$, then $X \in \operatorname{add} T$.

τ-tilting pairs

We got mutation at all summands in $\mathscr{C}_\Lambda$ by adding an "extra copy" of the projectives.

Definition [AIR]

Let (X, P) be a pair of modules in $\operatorname{mod}\Lambda$.
We call (X, P) a *τ-rigid pair* if

- X is τ-rigid
- $P \in \operatorname{proj}\Lambda$
- $\operatorname{Hom}_\Lambda(P, X) = 0$

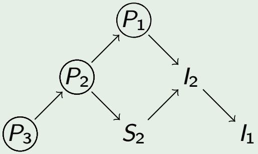
It is *support τ-tilting* if we have $|X| + |P| = |\Lambda|$.
It is *almost complete support τ-tilting* if we have $|X| + |P| = |\Lambda| - 1$.

Mutation

Theorem [AIR]

Any almost complete support τ-tilting pair has exactly two complements.

Example



$(P_1 \oplus P_2 \oplus P_3, 0)$ is support τ-tilting.
$(P_2 \oplus P_3, 0)$ is almost complete support τ-tilting.
$(P_2 \oplus P_3, P_1)$ is support τ-tilting.

Bijections

Theorem [AIR]

There exists a bijection between

- Basic support τ-tilting Λ modules (up to isomorphism)
- Functorially finite torsion classes in $\operatorname{mod}\Lambda$ (up to isomorphism)

Theorem [AIR]

Let $T \in \mathscr{C}_\Lambda$ be cluster-tilting. There exists a one-to-one correspondence between cluster-tilting objects in $\mathscr{C}_\Lambda$ and support τ-tilting pairs in $\operatorname{mod}\operatorname{End}_{\mathscr{C}_\Lambda} T$.

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