Fix a field $k$.

## Definition

We call a set $X$ of paths in a quiver $Q$ *gently closed* if for all vertices $v$, arrows $a, a', b, b'$ and paths $p, q$ in $Q$ the following properties hold:

(1) $avb, a'vb' \in X \Rightarrow (a = a',\ b = b'$ or $a \neq a', b \neq b')$.

(2) $pa, aq \in X \Rightarrow paq \in X$.

(3) $pq \in X \Rightarrow p, q \in X$.

(4) $a \in X$.

A *gentle quiver* $\mathcal{Q} = (Q, P, F)$ is a triple consisting of

(i) a finite quiver $Q$ with all indegrees and outdegrees $\leq 2$ and without isolated vertices,

(ii) a gently closed set $P$ of paths in $Q$, which are called *permitted*,

(iii) a gently closed set $F$ of paths in $Q$, which are called *forbidden*,

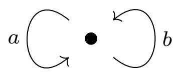
such that each path of length 2 in $Q$ is either permitted or forbidden (but not both).

The *gentle algebra* defined by $\mathcal{Q}$ is $k\mathcal{Q} = kQ/\langle F \cap Q_2 \rangle$.

Note that $\mathcal{Q}$ is completely determined by $(Q, P \cap Q_2)$ or $(Q, F \cap Q_2)$, respectively.

**Lemma.** *$P$ is a basis of $k\mathcal{Q}$. In particular, $k\mathcal{Q}$ is finite-dimensional iff $|P| < \infty$.*

**Example.** The gentle quiver $\mathcal{Q}$ given by

$Q =$ 


with $P = \langle ab, ba \rangle$ and $F = \langle a^2, b^2 \rangle$ yields the gentle algebra $k\mathcal{Q} = k\langle a, b \rangle / (a^2, b^2)$.

## Classification of Modules etc.

Assume in this section $|P| < \infty$ so that $k\mathcal{Q}$ is finite-dimensional.

The indecomposable modules in $\operatorname{mod} k\mathcal{Q}$ can be classified in terms of strings and bands. Recall Bill's talk in our Friday seminar on 15 November 2019 and see also [Cra18].

For a combinatorial classification of morphisms between string and band modules and the Auslander–Reiten quiver of $\operatorname{mod} k\mathcal{Q}$ see [BR87; Cra89; Kra91].

A classification of the indecomposable complexes in $D^b(\operatorname{mod} k\mathcal{Q})$ and of morphisms between them has been worked out, too. See [BM03; ALP16; Ben19] and also [Opp17].

**Example.** The Kronecker quiver is gentle. Hence, its Auslander–Reiten quiver (and much more) can be described using string and band combinatorics. See `https://www.math.uni-bielefeld.de/~jgeuenich/string-applet/?example=Kronecker#string-info`.

## Bardzell Resolution

Bardzell described the minimal projective bimodule resolution for the diagonal bimodule over a monomial algebra, which for gentle algebras assumes the following form:

**Theorem** ([Van94; Bar97])**.** *The complex* $(k\mathcal{Q}\otimes_{kQ_0} kF_\ell \otimes_{kQ_0} k\mathcal{Q}[-\ell], d_\ell)$ *with differentials*

$$d_\ell(1\otimes a_1\cdots a_\ell\otimes 1) \;=\; a_1\otimes a_2\cdots a_\ell\otimes 1 + (-1)^\ell\otimes a_1\cdots a_{\ell-1}\otimes a_\ell$$

*is a minimal projective resolution of* $k\mathcal{Q}$ *as a graded module over* $(k\mathcal{Q})^e = k\mathcal{Q}^{\mathrm{op}}\otimes_k k\mathcal{Q}$.

## Koszul Duality

The *dual* of a gentle quiver $\mathcal{Q} = (Q, P, F)$ is the gentle quiver $\mathcal{Q}^{\mathrm{op}} = (Q^{\mathrm{op}}, F^{\mathrm{op}}, P^{\mathrm{op}})$.

**Example.** With $\mathcal{Q}$ as in the previous example, $k\mathcal{Q}^{\mathrm{op}} \cong k[x,y]/(xy)$.

A result by Green–Zacharia for monomial algebras specializes to the following:

**Theorem** ([GZ94])**.** *Gentle algebras* $k\mathcal{Q}$ *graded by path length are Koszul with dual* $k\mathcal{Q}^{\mathrm{op}}$. *Mapping* $a\in Q_1$ *to the short exact sequence* $0\to \operatorname{soc} M(a)\to M(a)\to \operatorname{top} M(a)\to 0$, *where* $M(a)$ *is the string module defined by* $a$, *induces an isomorphism of graded* $k$*-algebras*

$$k\mathcal{Q}^{\mathrm{op}} \xrightarrow{\cong} \operatorname{Ext}^*_{k\mathcal{Q}}(k\mathcal{Q}/\langle Q_1\rangle, k\mathcal{Q}/\langle Q_1\rangle) = E^*(k\mathcal{Q})\,.$$

*Proof.* Applying $-\otimes_{(k\mathcal{Q})^e} k\mathcal{Q}$ to Bardzell's resolution yields the Koszul complex of $k\mathcal{Q}$. □

In particular, $k\mathcal{Q}^{\mathrm{op}}$ is finite-dimensional iff $k\mathcal{Q}$ has finite global dimension (see [Kva15]).

**Example.** The polynomial ring $k[x]$ and the ring of dual numbers $k[\varepsilon]/(\varepsilon^2)$ are gentle algebras, which are Koszul dual to each other.

## Regularity

Recall that a $k$-algebra is said to be *homologically smooth* if it has a finite projective resolution as a module over its enveloping algebra by finitely generated modules.

The aforementioned results yield:

**Proposition** ([HKK17], [LP19])**.** $k\mathcal{Q}$ *is homologically smooth iff* $|F|<\infty$.

*More precisely,* $\operatorname{gl.dim} k\mathcal{Q} = \operatorname{proj.dim} k\mathcal{Q}_{(k\mathcal{Q})^e} = g(\mathcal{Q})$ *with* $g(\mathcal{Q}) = \sup\{\ell : F_\ell\neq\emptyset\}$.

*Proof.* Use $g(\mathcal{Q}) = \sup\{\ell : E^\ell(k\mathcal{Q})\neq 0\} = \operatorname{gl.dim} k\mathcal{Q} \le \operatorname{proj.dim} k\mathcal{Q}_{(k\mathcal{Q})^e} \le g(\mathcal{Q})$. □

Even though the global dimension of $k\mathcal{Q}$ can be infinite, still the following is true:

**Theorem** ([GR05])**.** *Finite-dimensional gentle algebras* $k\mathcal{Q}$ *are Gorenstein.*

*More precisely,* $\operatorname{inj.dim} k\mathcal{Q}_{k\mathcal{Q}} = \operatorname{inj.dim} {}_{k\mathcal{Q}}k\mathcal{Q} = \ell$, *if* $\ell>0$ *is maximal such that there is a forbidden path* $p$ *of length* $\ell$ *in* $\mathcal{Q}$ *not properly contained in any other forbidden path.*

## Periodicity

According to Marczinzik, a finite-dimensional $k$-algebra $\Lambda$ is said to be *eventually periodic* if for sufficiently large $n$ there exists some $i > 0$ such that we have $\Omega^{n+i}_{\Lambda^e}(\Lambda) \cong \Omega^n_{\Lambda^e}(\Lambda)$. Bardzell's resolution thus yields an answer to a question of Marczinzik:

**Proposition.** *Finite-dimensional gentle algebras are eventually periodic.*

## Gradings

Let $\mathcal{Q}$ be a gentle quiver endowed with a *grading* $Q_1 \to \mathbb{Z}$, $a \mapsto |a|$.

We view $kQ_1$ as a dg vector space with grading induced by $|\cdot|$ and differential $d = 0$.

The path algebra $kQ = \bigoplus_\ell kQ_\ell$ then becomes a dg algebra with induced grading

$$|\prod a_i| \;=\; \textstyle\sum_i |a_i|$$

and induced differential

$$d(\prod a_i) \;=\; \textstyle\sum_i \Big( (\prod_{j<i} a_j) \cdot d(a_i) \cdot (\prod_{j>i} a_j) \Big) \;=\; 0\,.$$

Of course, $kQ$ is also graded by path length and $kQ_+$ is a dg ideal since $d(kQ_\ell) \subseteq kQ_\ell$.

This construction turns $k\mathcal{Q}$ into a dg algebra equipped with an additional length grading.

**Remark** (see also David's talk)**.** Let $A_i$ $(i = 1, 2)$ be two homologically smooth graded gentle algebras and let $D(A_i)$ be their derived categories of perfect dg modules. Lekili–Polishchuk, motivated by the work of Haiden–Katzarkov–Kontsevich, associate with $A_i$ partially wrapped Fukaya categories and deduce a sufficient criterion for $D(A_1) \simeq D(A_2)$.

Note that $D(k\mathcal{Q})$ is equivalent to $K^b(\operatorname{proj} k\mathcal{Q})$ if the grading of $\mathcal{Q}$ is 0.

A significant part of the data appearing in this criterion for derived equivalence is provided by the (graded) Avella-Alaminos–Geiß (AG) invariant, which we will define next.

## Threads

By a *trail* in a quiver we mean a subquiver $t_p$ spanned by a path $p$.

A trail $t_p$ given by a path $p$ in a gently closed set $X$ is called a *thread* in $X$ if it has either of the following two properties:

(1) $t_p$ is not properly contained in any other trail $t_q$ with $q \in X$.

(2) $p$ is the trivial path at a vertex with indegree and outdegree both $\leq 1$.

Denote by $T_X$ the set of all threads in $X$.

**Remark.** Let $X \in \{P, F\}$. Each arrow of $Q$ occurs in exactly one thread in $X$ and the second condition above ensures that each vertex of $Q$ occurs in exactly two threads in $X$.

**Example.** For $\mathcal{Q}$ with underlying quiver

$$Q = 1 \;\begin{smallmatrix} \xleftarrow{\;b\;} \\ \xrightarrow{\;a\;} \\ \xleftarrow[\;c\;]{} \end{smallmatrix}\; 2$$

and $F = \langle ab, ba\rangle$ and thus $P = \langle ac, ca\rangle$ we have $T_F = \{ab = ba, c\}$ and $T_P = \{ac = ca, b\}$.

Reformulating the results from before we may conclude:

**Proposition.** *A gentle algebra $k\mathcal{Q}$ is [ finite dimensional / homologically smooth ] iff its gentle quiver $\mathcal{Q}$ has no [ permitted / forbidden ] cyclic threads of positive length.*

## Boundary Components

Following [LP19], a *boundary component* $B$ of a gentle quiver $\mathcal{Q}$ is a trail $t_p$ in $\overline{Q} = Q \amalg Q^{\mathrm{op}}$ such that, assuming $p$ to be chosen with pairwise distinct arrows, it is cyclic, avoids $aa^{\mathrm{op}}$ with $a \in \overline{Q}_1$ and *alternates between forbidden and permitted threads* in the sense that

$$p = \prod_{i \in \mathbb{Z}/n\mathbb{Z}} p_i \text{ with } t_{p_i} \in T_F \cup T_{P^{\mathrm{op}}} \text{ and, if } n > 1, \text{ also } t_{p_i} \in T_F \Leftrightarrow t_{p_{i+1}} \in T_{P^{\mathrm{op}}}\,.$$

*Degree* and *winding number* of such a boundary component $B$ are defined as

$$n(B) \;:=\; \left\lfloor \frac{n}{2} \right\rfloor \qquad \text{and} \qquad w(B) \;:=\; \sum_i w(t_{p_i})$$

where

$$w(t_{p_i}) \;:=\; \begin{cases} -(\ell(p_i) - \delta_{n>1}) + |p_i| & \text{for } t_{p_i} \in T_F, \\ -|p_i| & \text{for } t_{p_i} \in T_{P^{\mathrm{op}}}. \end{cases}$$

**Example.** For $\mathcal{Q}$ as in the last example we have the following boundary components:

| $B$ | $n(B)$ | $w(B)$ |
|---|---|---|
| $ab$ | 0 | $-2 + \lvert ab\rvert$ |
| $(ac)^{\mathrm{op}}$ | 0 | $-\lvert ac\rvert$ |
| $cb^{\mathrm{op}}$ | 1 | $\lvert c\rvert - \lvert b\rvert$ |

## AG Invariant

**Definition** ([LP19])**.** *The* (graded) AG invariant $\mathbb{N} \times \mathbb{N} \xrightarrow{\phi_{\mathcal{Q}}} \mathbb{N}$ *of $\mathcal{Q}$ is defined as*

$$\phi_{\mathcal{Q}}(n, m) \;:=\; \#\Big\{B \text{ boundary component of } \mathcal{Q} : (n(B), n(B) - w(B)) = (n, m)\Big\}\,.$$

For finite-dimensional ungraded gentle algebras $\mathcal{Q}$, it is clear (essentially by definition) that this AG invariant $\phi_{\mathcal{Q}}$ coincides with the invariant introduced in [AG08].

**Remark** (see [LP19] or wait for David's talk)**.** $\phi_{\mathcal{Q}}$ is a derived invariant for homologically smooth $\mathcal{Q}$. In the ungraded setting this was already proved by Avella-Alaminos–Geiß.

## References


[ALP16] K. K. Arnesen, R. Laking, and D. Pauksztello, "Morphisms between indecomposable complexes in the bounded derived category of a gentle algebra," *J. Algebra*, vol. 467, pp. 1–46, 2016, DOI: 10.1016/j.jalgebra.2016.07.019. arXiv: 1411.7644 [math.RT].

[AG08] D. Avella-Alaminos and C. Geiß, "Combinatorial derived invariants for gentle algebras," *J. Pure Appl. Algebra*, vol. 212, no. 1, pp. 228–243, 2008, DOI: 10.1016/j.jpaa.2007.05.014.

[Bar97] M. J. Bardzell, "The alternating syzygy behavior of monomial algebras," *J. Algebra*, vol. 188, no. 1, pp. 69–89, 1997, DOI: 10.1006/jabr.1996.6813.

[BM03] V. Bekkert and H. A. Merklen, "Indecomposables in derived categories of gentle algebras," *Algebr. Represent. Theory*, vol. 6, no. 3, pp. 285–302, 2003, DOI: 10.1023/A:1025142023594.

[Ben19] R. Bennett-Tennenhaus, "Functorial filtrations for homotopy categories of some generalisations of gentle algebras," version 5, 2019, arXiv: 1608.08514 [math.RT].

[BR87] M. C. R. Butler and C. M. Ringel, "Auslander-Reiten sequences with few middle terms and applications to string algebras," *Comm. Algebra*, vol. 15, no. 1-2, pp. 145–179, 1987, DOI: 10.1080/00927878708823416.

[Cra18] W. Crawley-Boevey, "Classification of modules for infinite-dimensional string algebras," *Trans. Amer. Math. Soc.*, vol. 370, no. 5, pp. 3289–3313, 2018, DOI: 10.1090/tran/7032. arXiv: 1308.6410 [math.RA].

[Cra89] W. Crawley-Boevey, "Maps between representations of zero-relation algebras," *J. Algebra*, vol. 126, no. 2, pp. 259–263, 1989, DOI: 10.1016/0021-8693(89)90304-9.

[GR05] C. Geiß and I. Reiten, "Gentle algebras are Gorenstein," in *Representations of algebras and related topics*, vol. 45, Amer. Math. Soc., Providence, RI, 2005, pp. 129–133.

[GZ94] E. L. Green and D. Zacharia, "The cohomology ring of a monomial algebra," *Manuscripta Math.*, vol. 85, no. 1, pp. 11–23, 1994, DOI: 10.1007/BF02568180.

[HKK17] F. Haiden, L. Katzarkov, and M. Kontsevich, "Flat surfaces and stability structures," *Publ. Math. Inst. Hautes Études Sci.*, vol. 126, pp. 247–318, 2017, DOI: 10.1007/s10240-017-0095-y. arXiv: 1409.8611 [math.AG].

[Kra91] H. Krause, "Maps between tree and band modules," *J. Algebra*, vol. 137, no. 1, pp. 186–194, 1991, DOI: 10.1016/0021-8693(91)90088-P.

[Kva15] S. Kvamme, *Necessary and sufficient conditions*, BIREP Summer School on Koszul Duality, 2015, Online: https://www.math.uni-bielefeld.de/birep/activities/summer2015/notes/Kvamme.pdf.

[LP19] Y. Lekili and A. Polishchuk, "Derived equivalences of gentle algebras via Fukaya categories," version 5, 2019, arXiv: 1801.06370 [math.SG].

[Opp17] S. Opper, *Classification of indecomposable objects in the bounded derived category of a gentle algebra*, BIREP Summer School on Gentle Algebras, 2017, Online: https://www.math.uni-bielefeld.de/birep/activities/summer2017/summer-school-2017.pdf#section.10.

[Van94] M. Van den Bergh, "Noncommutative homology of some three-dimensional quantum spaces," in *Proceedings of Conference on Algebraic Geometry and Ring Theory in honor of Michael Artin, Part III (Antwerp, 1992)*, vol. 8, 1994, pp. 213–230, DOI: 10.1007/BF00960862.