Connections for sheaves on weighted projective lines

Andrew Hubery

Universität Bielefeld

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A review of some results of Crawley-Boevey in his/our proof of the Deligne-Simson Problem.

Connections

Given a manifold $\mathcal{M}$ over $\mathbb{R}$, one would like some way to relate the local geometry at different points. There are two approaches to this problem.

Let $E \to \mathcal{M}$ be a vector bundle.

Parallel Transport

Given a curve $\gamma\colon [0,1] \to \mathcal{M}$, prescribe how to translate a vector $v = v_0 \in E_{\gamma(0)}$ along the path γ, obtaining vectors $v_t \in E_{\gamma(t)}$.

Covariant Derivative

Given a tangent vector $X \in T_x\mathcal{M}$, specify the derivative along X, analogous to the directional derivative in Euclidean geometry.

Connections

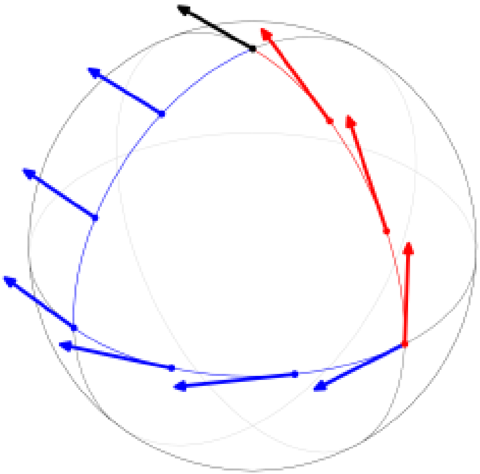


Figure: Parallel transport of tangent vectors around a closed curve on the sphere

Connections

We can go between these two ideas.

Given a notion of parallel transport, we can define the derivative $\nabla_X(\sigma)$ of a section $\sigma \in \Gamma(E, \mathcal{M})$ along the tangent vector X of a curve $\gamma \colon [0,1] \to \mathcal{M}$ at $\gamma(0)$, by first transporting $\sigma(\gamma(t)) \in E_{\gamma(t)}$ back to $E_{\gamma(0)}$ and then taking the usual limit.

Given a covariant derivative and a curve γ, a sequence of vectors $v_t \in E_{\gamma(t)}$ are parallel along γ provided the derivative at each point along the tangent vector is zero.

Connections

Koszul gave the modern formulation in terms of a connection.

This is an $\mathbb{R}$-linear map

$$\nabla \colon E \to T^*\mathcal{M} \otimes E$$

satisfying

$$\nabla(f\sigma) = f\nabla(\sigma) + df \otimes \sigma, \quad f \in \mathcal{C}^\infty(\mathcal{M}), \quad \sigma \in \Gamma(E).$$

The directional derivative is given by evaluation

$$\nabla_X \colon \Gamma(E) \to \Gamma(E), \quad \sigma \mapsto \nabla(\sigma)(X).$$

Sheaves on $\mathbb{P}^1$

We use the affine cover

$$\mathcal{U}^+ = \mathbb{P}^1 - \{\infty\} \quad \text{and} \quad \mathcal{U}^- = \mathbb{P}^1 - \{0\}.$$

A coherent sheaf is then a triple $(M^+, M^-; \theta)$ consisting of

- a finitely generated $k[s]$-module M^+
- a finitely generated $k[s^-]$-module M^-
- a $k[s, s^-]$-isomorphism $\theta\colon k[s, s^-] \otimes M^- \xrightarrow{\sim} k[s, s^-] \otimes M^+$.

We call $M^\pm$ the **charts** and θ the **glue**.

Sheaves on $\mathbb{P}^1$

Examples

- $\mathcal{O}(m)$ has charts $k[s^{\pm}]$ and glue given by multiplication by s^m. These are indecomposable and locally free.
- Given $\sigma \in k[x, y]$ homogeneous, set $\sigma^+ = \sigma(s, 1)$ and $\sigma^- = \sigma(1, s^-)$. Then S_σ has charts $k[s^{\pm}]/\sigma^{\pm}$ and glue the identity map. These are torsion, and indecomposable provided σ is a power of an irreducible polynomial.
- Given a sheaf $M = (M^{\pm}; \theta)$, its d-th shift is $M(d) = (M^{\pm}; s^d\theta)$.

Sheaves on $\mathbb{P}^1$

A morphism $f\colon M \to N$ consists of $k[s^{\pm}]$ linear maps $f^{\pm}\colon M^{\pm} \to N^{\pm}$ compatible with the glue

$$\begin{array}{ccc} k[s, s^{-}] \otimes M^{-} & \xrightarrow{1\otimes f^{-}} & k[s, s^{-}] \otimes N^{-} \\ \downarrow \theta & \circlearrowright & \downarrow \phi \\ k[s, s^{-}] \otimes M^{+} & \xrightarrow[1\otimes f^{+}]{} & k[s, s^{-}] \otimes N^{+} \end{array}$$

The category $\operatorname{coh} \mathbb{P}^1$ is k-linear, hereditary abelian, with finite dimensional hom and ext spaces.

Grothendieck group

The Grothendieck group of $\operatorname{coh}\mathbb{P}^1$ is $\mathbb{Z}^2$, where

$$[\mathcal{O}(m)] = (1, m) \quad \text{and} \quad [S_\sigma] = (0, \deg \sigma).$$

In general we write

$$[M] = (\operatorname{rank} M, \deg M).$$

The Euler form is given by

$$\begin{aligned}\{M, N\} &= \dim \operatorname{Hom}(M, N) - \dim \operatorname{Ext}^1(M, N) \\ &= (\operatorname{rank} M)(\operatorname{rank} N) + (\operatorname{rank} M)(\deg N) - (\deg M)(\operatorname{rank} N).\end{aligned}$$

Serre duality

Let $M, N \in \operatorname{coh} \mathbb{P}^1$ with N locally free. An extension

$$\eta \colon 0 \to M(-2) \to E \to N \to 0$$

is split on both charts, so E has charts $E^{\pm} = N^{\pm} \oplus M^{\pm}$, and glue

$$\begin{pmatrix} \phi & 0 \\ \gamma\phi & \theta \end{pmatrix}, \quad \gamma \colon k[s, s^-] \otimes N^+ \to k[s, s^-] \otimes M^+.$$

If $f \colon M \to N$, then $f^+\gamma \in \operatorname{End}(k[s, s^-] \otimes N^+)$, so has trace $\operatorname{tr}(f^+\gamma) \in k[s, s^-]$. Write $\operatorname{res} \operatorname{tr}(f^+\gamma)$ for the coefficient of s^-.

We define

$$\langle -, - \rangle \colon \operatorname{Hom}(M, N) \times \operatorname{Ext}^1(N, M(-2)) \to k$$

$$\langle f, \eta \rangle := -\operatorname{res} \operatorname{tr}(f^+\gamma).$$

Serre duality

The pairing $\langle -, - \rangle$ extends to a non-degenerate, bifunctorial and shift-invariant pairing on $\operatorname{coh}\mathbb{P}^1$.

Given a pair of sheaves M, N, there exists a locally free N_0 with $\operatorname{Ext}^1(N_0, M(-2)) = 0$ mapping onto N. The kernel N_1 is again locally free, and every $\eta \in \operatorname{Ext}^1(N, M(-2))$ is the pushout along some $g\colon N_1 \to M(-2)$.

$$\begin{array}{lccccccccc}
\varepsilon\colon & 0 & \longrightarrow & N_1 & \longrightarrow & N_0 & \longrightarrow & N & \longrightarrow & 0 \\
 & & & \downarrow{\scriptstyle g} & & \downarrow & & \| & & \\
\eta = g\varepsilon\colon & 0 & \longrightarrow & M(-2) & \longrightarrow & E & \longrightarrow & N & \longrightarrow & 0
\end{array}$$

Given $f\colon M \to N$ we define

$$\langle f, \eta \rangle = \langle f, g\varepsilon \rangle = \langle fg, \varepsilon \rangle = \langle \mathrm{id}_{N_0}, \varepsilon fg \rangle.$$

Serre duality

Example

There is a short exact sequence

$$\eta\colon \quad 0 \longrightarrow \mathcal{O}(-2) \xrightarrow{\binom{x}{y}} \mathcal{O}(-1)^2 \xrightarrow{(y,-x)} \mathcal{O} \longrightarrow 0$$

Taking appropriate splittings on the charts, the glue is given by

$$\begin{pmatrix} 1 & 0 \\ -s^{-} & s^{-2} \end{pmatrix}.$$

Thus

$$\langle 1, \eta \rangle = 1.$$

Connections

A **connection** on M is a k-linear map

$$\nabla\colon M \to M(-2)$$

satisfying the Leibniz rule.

Explicitly, we have k-linear maps on charts

$$\nabla^{\pm}\colon M^{\pm} \to M^{\pm}$$

satisfying

- $\nabla^{+}\theta = s^{-2}\theta\nabla^{-}$
- $\nabla^{+}(fm) = f\nabla^{+}(m) + \frac{df}{ds}m$, $f \in k[s]$
- $\nabla^{-}(fm) = f\nabla^{-}(m) - \frac{df}{ds^{-}}m$, $f \in k[s^{-}]$.

Connections

Atiyah defined $A(M)$ to be the sheaf having charts $M^{\pm} \oplus M^{\pm}$ with twisted $k[s^{\pm}]$-action

$$f \cdot (m, m') = (fm, fm' \pm \tfrac{df}{ds^{\pm}} m)$$

and glue

$$\begin{pmatrix} \theta & 0 \\ 0 & s^{-2}\theta \end{pmatrix}.$$

There is a functorial exact sequence

$$\alpha(M)\colon \quad 0 \longrightarrow M(-2) \xrightarrow{\binom{0}{1}} A(M) \xrightarrow{(1,0)} M \longrightarrow 0$$

and connections on M are in bijection with sections via $\nabla \mapsto \binom{1}{\nabla}$.

Connections

Atiyah's construction does not commute with the shift

$$A(M)(1) \ncong A(M(1)).$$

We are led to the following.

Given scalars λ, μ, define $A_{\lambda,\mu}(M)$ to be the sheaf having charts $M^{\pm} \oplus M^{\pm}$ with twisted $k[s^{\pm}]$-action

$$f \cdot (m, m') = (fm, fm' \pm \lambda \tfrac{df}{ds^{\pm}} m)$$

and glue

$$\begin{pmatrix} \theta & 0 \\ -\mu s^{-}\theta & s^{-2}\theta \end{pmatrix}.$$

Again there is a functorial exact sequence

$$\alpha_{\lambda,\mu}(M): \quad 0 \longrightarrow M(-2) \xrightarrow{\binom{0}{1}} A_{\lambda,\mu}(M) \xrightarrow{(1,0)} M \longrightarrow 0$$

Connections

We have

$$A(M) = A_{1,0}(M) \quad \text{and} \quad A_{\lambda,\mu}(M)(1) = A_{\lambda,\mu-\lambda}(M(1)).$$

Also, $\alpha_{t\lambda,t\mu}(M)$ is the pushout $t\alpha_{\lambda,\mu}(M)$ for $t \in k$.

Let k be algebraically closed and $M \in \operatorname{coh} \mathbb{P}^1$ indecomposable.

Then

$$\langle f, \alpha_{\lambda,\mu}(M)\rangle = (\lambda \deg M + \mu \operatorname{rank} M)\bar{f},$$

where $\bar{f} \in \operatorname{End}(M)/J\operatorname{End}(M) \cong k$.

Thus $\alpha_{\lambda,\mu}(M)$ admits a section if and only if

$$\lambda \deg M + \mu \operatorname{rank} M = 0.$$

In particular, M admits a connection if and only if $\deg M = 0$, so $M \cong \mathcal{O}$.

Sheaves on $\mathbb{X}$

Fix distinct points $p_1, \ldots, p_r \in \mathbb{P}^1$ (of degree 1) with positive integer weights $w_1, \ldots, w_r$.

Geigle and Lenzing introduced a category $\operatorname{coh} \mathbb{X}$, which is again k-linear, hereditary abelian, with finite dimension hom and ext spaces, and Serre duality.

We can describe objects in $\operatorname{coh} \mathbb{X}$ in terms of periodic functors $\mathbb{Z}^r \to \operatorname{coh} \mathbb{P}^1$.

Fix (linear) homogeneous $\sigma_i \in k[x, y]$ representing the p_i.

Sheaves on $\mathbb{X}$

A functor $M\colon \mathbb{Z}^r \to \operatorname{coh}\mathbb{P}^1$ consists of

- sheaves $M_d \in \operatorname{coh}\mathbb{P}^1$ for each $d \in \mathbb{Z}^r$
- a unique morphism $\phi_{d,e}\colon M_d \to M_{d+e}$ for all d, e with $e \geq 0$

We say that M is **periodic** provided

$$M_{d+w_i x_i} = M_d(1), \qquad \phi_{d+w_i x_i, e} = \phi_{d,e}$$

$$\phi_{d,w_i x_i} = \sigma_i\colon M_d \to M_d(1).$$

A morphism (nat. transf.) $\psi\colon M \to N$ is **periodic** provided

$$\psi_{d+w_i x_i} = \psi_d.$$

$\operatorname{coh}\mathbb{X}$ is the category of periodic functors and morphims.

Sheaves on $\mathbb{X}$

This construction is over-specified.

The forgetful functor sending a periodic functor to its restrictions to the co-ordinate axes is fully-faithful and exact.

In other words, it is enough to give a sheaf

$$M_0 \in \operatorname{coh} \mathbb{P}^1$$

and an r-tuple of periodic functors

$$M_i \colon \mathbb{Z} \to \operatorname{coh} \mathbb{P}^1 \quad \text{with } M_{i,0} = M_0.$$

Recollement

The exact functor

$$\pi\colon \operatorname{coh}\mathbb{X} \to \operatorname{coh}\mathbb{P}^1, \quad M \mapsto M_0,$$

determines a recollement

$$\mathcal{C} \; \overset{\longleftarrow}{\underset{\longleftarrow}{\longrightarrow}} \; \operatorname{coh}\mathbb{X} \; \overset{\overset{\pi_!}{\longleftarrow}}{\underset{\underset{\pi_*}{\longleftarrow}}{\longrightarrow}} \; \operatorname{coh}\mathbb{P}^1$$

The category $\mathcal{C}$ is equivalent to modules over the union of linearly oriented $\mathbb{A}_{w_i-1}$.

Moreover, the functors $\pi_!$ and π_* are exact.

Examples of sheaves

The sheaf $\pi_! M$ has M in position ax_i for all $0 \leq a < w_i$, with the equality maps between them.

For $r = 2$ with weights $w_1 = 3, w_2 = 2$ we can draw the restriction of a periodic sheaf to the box $[0, 3x_1] \times [0, 2x_2] \subset \mathbb{Z}^2$. Then $\pi_! M$ can be drawn as

$$
\begin{array}{ccccccc}
M & = & M & = & M & \xrightarrow{\sigma_1} & M(1) \\
\Vert & & \Vert & & \Vert & & \Vert \\
M & = & M & = & M & \xrightarrow{\sigma_1} & M(1) \\
\Big\downarrow{\scriptstyle \sigma_2} & & \Big\downarrow{\scriptstyle \sigma_2} & & \Big\downarrow{\scriptstyle \sigma_2} & & \Big\downarrow{\scriptstyle \sigma_2} \\
M(1) & = & M(1) & = & M(1) & \xrightarrow{\sigma_1} & M(2)
\end{array}
$$

The structure sheaf on $\mathbb{X}$ is $\mathcal{O} = \mathcal{O}_{\mathbb{X}} = \pi_! \mathcal{O}_{\mathbb{P}^1}$.

Examples of sheaves

The shift $M(d)$ is given by $M(d)_e = M_{d+e}$. For example here is $\pi_! M(x_1)$ for $w_1 = 3$ and $w_2 = 2$ as before

$$\begin{array}{ccccccc}
M & = & M & \xrightarrow{\sigma_1} & M(1) & = & M(1) \\
\| & & \| & & \| & & \| \\
M & = & M & \xrightarrow{\sigma_1} & M(1) & = & M(1) \\
\downarrow{\scriptstyle \sigma_2} & & \downarrow{\scriptstyle \sigma_2} & & \downarrow{\scriptstyle \sigma_2} & & \downarrow{\scriptstyle \sigma_2} \\
M(1) & = & M(1) & \xrightarrow{\sigma_1} & M(2) & = & M(2)
\end{array}$$

We have $M(w_i x_i) = M(1)$, so the shift group is

$$\mathbb{L} = (\mathbb{Z}c \oplus \mathbb{Z}^r)/(\{w_i x_i - c\}).$$

Examples of sheaves

Up to scalars there is a unique map $\mathcal{O}(d) \to \mathcal{O}(d + x_i)$.

In particular, we have an essentially unique non-split sequence

$$0 \to \mathcal{O}((a-1)x_i) \to \mathcal{O}(ax_i) \to S_{ia} \to 0,$$

and the S_{ia} are simple torsion sheaves.

For $r = 1$ and $w_1 = 3$ we have

$$\begin{array}{ccccccccccc}
\mathcal{O}_{\mathbb{X}} & : & \mathcal{O} & = & \mathcal{O} & = & \mathcal{O} & \xrightarrow{\sigma_1} & \mathcal{O}(1) \\
\downarrow & & \Vert & & \Vert & & \downarrow{\scriptstyle \sigma_1} & & \Vert \\
\mathcal{O}_{\mathbb{X}}(x_1) & : & \mathcal{O} & = & & \xrightarrow{\sigma_1} & \mathcal{O}(1) & = & \mathcal{O}(1) \\
\downarrow & & \downarrow & & \downarrow & & \downarrow & & \downarrow \\
S_{11} & : & 0 & \longrightarrow & 0 & \longrightarrow & S_{\sigma_1} & \longrightarrow & 0
\end{array}$$

Standard presentation

Every sheaf M is the cokernel of a functorial morphism

$$\bigoplus_{i,a}(\pi_! M_{ax_i})(-(a+1)x_i) \longrightarrow (\pi_! M_0) \oplus \bigoplus_{i,a}(\pi_! M_{ax_i}(-ax_i))$$

(We can also describe the kernel using recollements.)

Grothendieck group

Using the recollement we obtain

$$K_0(\mathrm{coh}\,\mathbb{X}) = K_0(\mathbb{P}^1) \oplus \bigoplus_i K_0(\mathbb{A}_{w_i-1}),$$

having basis

- $[\mathcal{O}]$
- $\partial = [\mathcal{O}(1)] - [\mathcal{O}] = [\pi_! S]$ for any torsion sheaf $S \in \mathrm{coh}\,\mathbb{P}^1$ of degree one
- the simple torsion sheaves $[S_{ia}]$ for $0 < a < w_i$.

Grothendieck group

We have

$$[M] = (\deg M_0)\partial + \underline{\dim}\, M$$

where

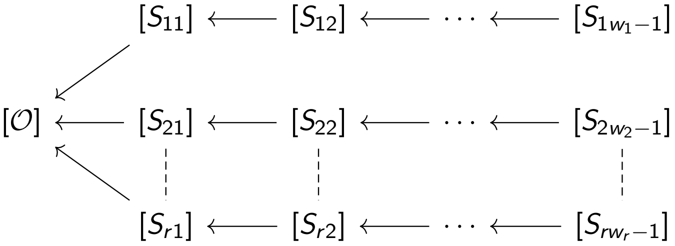
$$\underline{\dim}\, M = (\operatorname{rank} M_0)[\mathcal{O}] + \sum_{i,a} \big(\deg M_{(w_i - a)x_i} - \deg M_0 \big)[S_{ia}].$$

Grothendieck group

With respect to this basis the Euler form is given by

$$\begin{aligned}\{M, N\} &= \dim \operatorname{Hom}(M, N) - \dim \operatorname{Ext}^1(M, N) \\ &= \{\underline{\dim}\, M, \underline{\dim}\, N\} + (\operatorname{rank} M_0)(\deg N_0) - (\deg M_0)(\operatorname{rank} N_0).\end{aligned}$$

Here $\{\underline{\dim}\, M, \underline{\dim}\, N\}$ is the usual Euler form for the star-shaped quiver



Serre duality

We set $\omega = -2c + \sum_i (w_i - 1)x_i$ in $\mathbb{L}$.

Recall the standard epimorphism

$$\pi_! M_0 \oplus \bigoplus_{i,a} \pi_! M_{ax_i}(-ax_i) \twoheadrightarrow M.$$

Applying $\mathrm{Hom}(-, N)$ gives an injection

$$\mathrm{Hom}_{\mathbb{X}}(M, N) \rightarrowtail \mathrm{Hom}_{\mathbb{P}^1}(M_0, N_0) \oplus \bigoplus_{i,a} \mathrm{Hom}_{\mathbb{P}^1}(M_{ax_i}, N_{ax_i}).$$

Similarly, since $\pi_! M(\omega) = \pi_* M(-2)$, we have a surjection

$$\mathrm{Ext}^1_{\mathbb{P}^1}(N_0, M_0(-2)) \oplus \bigoplus_{i,a} \mathrm{Ext}^1_{\mathbb{P}^1}(N_{ax_i}, M_{ax_i}(-2)) \twoheadrightarrow \mathrm{Ext}^1_{\mathbb{X}}(N, M(\omega)).$$

Serre duality

We can now lift Serre duality on $\operatorname{coh}\mathbb{P}^1$ to $\operatorname{coh}\mathbb{X}$.

Given $\eta \in \operatorname{Ext}^1(N, M(\omega))$, write it as the image of (η_0, η_{ia}) under the epimorphism

$$\operatorname{Ext}^1(N_0, M_0(-2)) \oplus \bigoplus_{i,a} \operatorname{Ext}^1(N_{ax_i}, M_{ax_i}(-2)) \twoheadrightarrow \operatorname{Ext}^1(N, M(\omega))$$

Then

$$\langle -, - \rangle_{\mathbb{X}} \colon \operatorname{Hom}(M, N) \times \operatorname{Ext}^1(N, M(\omega)) \to k,$$

$$\langle f, \eta \rangle_{\mathbb{X}} = \langle f_0, \eta_0 \rangle_{\mathbb{P}^1} + \sum_{i,a} \langle f_{ia}, \eta_{ia} \rangle_{\mathbb{P}^1},$$

is a non-degenerate, bifunctorial and shift invariant pairing on $\operatorname{coh}\mathbb{X}$.

Connections

We fix a map $\zeta\colon K_0(\operatorname{coh}\mathbb{X}) \to k$, say with

- $\zeta([S_{ia}]) = \zeta_{ia}$
- $\zeta([\mathcal{O}]) = \mu$
- $\zeta(\partial) = \lambda - \sum_{ia} \zeta_{ia}$.

We can then take the image $\beta_\zeta(M) \in \operatorname{Ext}^1(M, M(\omega))$ of the tuple of extensions

$$(\alpha_{\lambda,\mu}(M_0), \alpha_{\zeta_{ia},0}(M_{ia}))$$

coming from the generalised Atiyah sequences in $\operatorname{coh}\mathbb{P}^1$.

Connections

Let k be algebraically closed and $M \in \operatorname{coh} \mathbb{X}$ indecomposable.

Then

$$\langle f, \beta_\zeta(M) \rangle = \bar{f} \cdot \zeta([M])$$

where $\bar{f} \in \operatorname{End}(M)/J\operatorname{End}(M) \cong k$.

Connections

For $\lambda = 1$ Crawley-Boevey gave an explicit construction of the functorial sequence $\beta_\zeta(M)$.

He then defined a ζ-connection on M to be a k-linear map

$$\nabla\colon M \to M(\omega)$$

yielding a section $\binom{1}{\nabla}$ of $\beta_\zeta(M)$.

Thus an indecomposable $M \in \operatorname{coh} \mathbb{X}$ admits a ζ-connection if and only if

$$\zeta([M]) = 0.$$

Connections

In fact, this is all backwards. Bill constructed $\beta_\zeta(M)$ directly, but using a different language for $\operatorname{coh}\mathbb{X}$. It was my task to rewrite this in the language of periodic functors.

I then showed that $b_\zeta(M)$ is the image of the tuple of generalised Atiyah sequences, and hence could compute the Serre pairing.

Parabolic bundles

Let $M \in \operatorname{coh}\mathbb{X}$ be locally free.

We have an exact commutative diagram

$$\begin{array}{ccccccccc}
0 & \longrightarrow & M_0(-1) & \longrightarrow & M_{-ax_i} & \longrightarrow & M_{i,a} & \longrightarrow & 0 \\
 & & \| & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & M_0(-1) & \xrightarrow{\sigma_i} & M_0 & \longrightarrow & M_{i,0} & \longrightarrow & 0.
\end{array}$$

Thus $M_{i,0}$ is the fibre of the sheaf M_0 at the point p_i.

The quotients $M_{i,a}$ all lie in $\operatorname{add} S_i \cong \operatorname{mod} k$, and we obtain a flag of subspaces

$$M_{i,0} \supseteq M_{i,1} \supseteq \cdots M_{i,w_i-1} \supseteq M_{i,w_i} = 0.$$

Parabolic bundles

Let M be locally free.

A connection ∇_0 on M_0 induces an endomorphism of each fibre $M_{i,0}$.

There is a bijection between ζ-connections on M and connections ∇_0 on M_0 such that

$$(\nabla_0 - \zeta_{ia})(M_{i,a-1}) \subseteq M_{i,a} \quad \text{for all } i, a.$$

This is where the name ζ-connection comes from.