Green's Main Theorem

Seminar on Ringel-Hall Algebras
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June 8, 2016

Abstract

In this talk, we wish to relate the Ringel-Hall algebra of a quiver with the Drinfel'd-Jimbo quantum group. As an application, we prove Kac's theorem which connects indecomposable quiver representations with positive roots.

Contents

1 Example: The special linear group of size 2

1.1 The interplay of Lie theory and representation theory

We consider the simple Lie algebra of traceless matrices:

$$\mathfrak{sl}_3(\mathbb{C}) = \{A \in \mathrm{Mat}_{3\times 3}(\mathbb{C}) \mid \mathrm{tr}(A) = 0\}.$$

It is an eight-dimensional Lie algebra spanned by the weight spaces $V_{\alpha_1} = \mathbb{C}e_1$, $V_{\alpha_2} = \mathbb{C}e_2$, $V_{\alpha_1+\alpha_2} = \mathbb{C}e_{12}$, $V_{-\alpha_1} = \mathbb{C}f_1$, $V_{-\alpha_2} = \mathbb{C}f_2$, $V_{-\alpha_1-\alpha_2} = \mathbb{C}f_{12}$, and the Cartan subalgebra $\mathfrak{h} = \mathbb{C}h_1 \oplus \mathbb{C}h_2$ with

$$e_1 = \begin{pmatrix} 0&1&0\\0&0&0\\0&0&0 \end{pmatrix}, \quad e_2 = \begin{pmatrix} 0&0&0\\0&0&1\\0&0&0 \end{pmatrix}, \quad e_{12} = \begin{pmatrix} 0&0&1\\0&0&0\\0&0&0 \end{pmatrix}, \quad h_1 = \begin{pmatrix} 1&0&0\\0&-1&0\\0&0&0 \end{pmatrix},$$

$$f_1 = \begin{pmatrix} 0&0&0\\1&0&0\\0&0&0 \end{pmatrix}, \quad f_2 = \begin{pmatrix} 0&0&0\\0&0&0\\0&1&0 \end{pmatrix}, \quad f_{12} = \begin{pmatrix} 0&0&0\\0&0&0\\1&0&0 \end{pmatrix}, \quad h_2 = \begin{pmatrix} 0&0&0\\0&1&0\\0&0&-1 \end{pmatrix}.$$

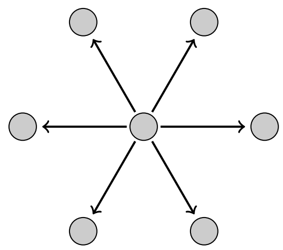

Figure 1: The root system of type A_2

The root system is shown in Figure 1. Note that every root is either positive or negative. The set $\Phi^+ = \{\alpha_1, \alpha_1 + \alpha_2, \alpha_2\}$ of positive roots is linked to the representation theory of the quiver $1 \to 2$ of type A_2. More precisely, the expansion of the three positive roots as linear combinations of the simple roots α_1, α_2, namely the vectors $(1,0)$, $(1,1)$ and $(0,1)$, are the dimension vectors of the three indecomposable representations $S_1 = (k \to 0)$, $P_1 = (k \to k)$ and $S_2 = (0 \to k)$, where k is any field.

Representations of $\mathfrak{g}$ correspond to modules over its *universal enveloping algebra* $U(g)$. It is generated by elements $E_1, E_2, F_1, F_2, H_1, H_2$ (corresponding to $e_1, e_2, f_1, f_2, h_1, h_1$) subject to certain relations such as the *Serre relation* $E_1^2 E_2 - 2E_1E_2E_1 + E_2E_1^2 = 0$ (corresponding to $[e_1, [e_1, e_2]] = 0$).

The Lie algebra $\mathfrak{g} = \mathfrak{sl}_3$ admits a triangular decomposition $\mathfrak{g} = \mathfrak{n}_+ \oplus \mathfrak{h} \oplus \mathfrak{n}_-$ into strictly *upper triangular*, *diagonal* and strictly *lower triangular* matrices. The universal enveloping algebras $U(g)$ and $U(\mathfrak{n}_+)$ admit several *Poincaré-Birkhoff-Witt bases*: every ordered basis of $\mathfrak{g}$ or $\mathfrak{n}$ gives rise to a PBW basis of its universal enveloping algebra. For example, the basis (e_1, e_{12}, e_2) of $\mathfrak{n}_+$ from above yields the basis $\mathcal{P} = \{E_1^a(E_1E_2 - E_2E_1)^b E_2^c \mid (a,b,c) \in \mathbb{N}^3\}$ of $U(\mathfrak{n}_+)$.

The *quantized universal enveloping algebra* U^+ is generated by elements E_1, E_2 subject to the *quantum Serre relations*, e. g. $E_1^2E_2 - (v + v^{-1})E_1E_2E_1 + E_2E_1^2 = 0$, over a suitable ground ring. The quantization has the following advantages:

(1) Lusztig [9, Theorem 42.1.10] uses the quantization to construct another basis of $U(\mathfrak{n}_+)$, the *canonical basis* $\mathcal{B}$, so that application of $\mathcal{B}$ to lowest vectors v_0 yields bases of the irreducible representations. In this case Lusztig's canonical basis is: $\mathcal{B} = \{E_1^aE_2^bE_1^c \mid a + c \geq b\} \cup \{E_2^aE_1^bE_2^c \mid a + c \geq b\}$.

(2) A theorem of Green-Rosso [4, 13] asserts that we can embed $U^+ \subseteq \mathcal{F}_v$ in the quantum shuffle algebra. Here $\mathcal{F}_v$ is spanned by all symbols $w[i_1, i_2, \ldots, i_k]$ for all finite sequences $\underline{i} = (i_1, i_2, \ldots, i_k) \in I^k$ of symbols from the set $\{1, 2, \ldots, n\}$ of length $k \geq 0$. The product of two basis elements is defined as a linear combination of *shuffles*. Green-Rosso's embedding enables us to do computer calculation in U^+ efficiently.

(3) We get a deeper connection between Lie theory and quiver representations. We saw in the first talk of the seminar that we have a short exact sequence $0 \to S_2 \to P_1 \to S_1$ but $\mathrm{Ext}_Q^1(S_2, S_1) = 0$, so that in the Ringel-Hall algebra $\mathcal{H}(Q)$ we have $u_{S_1}u_{S_2} = u_{S_1 \oplus S_2} + u_{P_1}$ but $u_{S_2}u_{S_1} = u_{S_1 \oplus S_2}$. It follows that $u_{P_1} = u_{S_1}u_{S_2} - u_{S_2}u_{S_1}$. Moreover, $u_{S_1}u_{P_1} = qu_{S_1 \oplus P_1} = qu_{P_1}u_{S_1}$. It follows that $u_{S_1}^2u_{S_2} - (q+1)u_{S_1}u_{S_2}u_{S_1} + qu_{S_2}u_{S_1}^2 = 0$. The relation becomes the quantum Serre relation in the twisted Ringel-Hall algebra.

In this talk, we wish to understand the first two lines of the following table. The talk is based on notes by Hubery [5].

	Quiver Q	Lie algebra $\mathfrak{g}$
Gabriel/Kac	indecomposable representations	Φ^+ positive roots
Ringel/Green	twisted Ringel-Hall algebra $H(Q)$	positive quantum group U^+
Lusztig	constructible sheaves on $\operatorname{rep}_k(Q, \underline{d})$	(dual) canonical basis of U^+

2 Ringel's theorem

2.1 The twisted Ringel-Hall algebra and Green's formula

Let Q be a quiver with n vertices, and let k be a finite field with q elements. The *Euler characteristic* of two representations $M, N \in \operatorname{rep}_k(Q)$ is defined to be $\langle M, N\rangle = \dim_k(\operatorname{Hom}_Q(M,N)) - \dim_k(\operatorname{Ext}_Q(M,N))$. Because the category $\operatorname{rep}_k(Q)$ is hereditary, it induces a bilinear form $K_0(Q) \times K_0(Q) \to \mathbb{R}$ where $K_0(Q)$ denotes the Grothendieck group of $\operatorname{rep}_k(Q)$. We consider the symmetrized Euler form defined by $(M,N) = \langle M,N\rangle + \langle N,M\rangle$ for all M,N. Moreover, let $v = \sqrt{q} \in \mathbb{R}$ be a square root of q.

The (twisted) Ringel-Hall algebra $\mathcal{H}(Q)$ is the $\mathbb{R}$-vector space with basis elements u_M for all isoclasses of representations M. We define the (twisted) multiplication and comultiplication on basis elements by the formulae:

$$u_M u_N = v^{\langle M,N\rangle} \sum_X F^X_{M,N} u_X, \qquad \Delta(u_X) = \sum_{M,N} v^{\langle M,N\rangle} \frac{a_M a_N}{a_X} F^X_{M,N} u_M \otimes u_N,$$

where $F^X_{M,N}$ is the number of subrepresentations $U \subseteq X$ such that $U \cong N$ and $X/U \cong M$, and a_X is the cardinality of $\operatorname{Aut}_Q(X)$. Then $\mathcal{H}(Q)$ becomes a (twisted) bialgebra, i. e. we have $\Delta(u_M u_N) = \Delta(u_M) \cdot \Delta(u_N)$ for all N, N when we define a multiplication on $\mathcal{H}(Q) \otimes_{\mathbb{R}} \mathcal{H}(Q)$ by the formula $(u_A \otimes u_B) \cdot (u_C \otimes u_D) = v^{-\langle A,D\rangle}(u_A u_C \otimes u_B u_D)$ or by the formula $(u_A \otimes u_B) \cdot (u_C \otimes u_D) = v^{(B,C)}(u_A u_C \otimes u_B u_D)$ respectively. It becomes a (twisted) Hopf algebra by defining a suitable antipode, see Xiao [14, Theorem 4.5 (c)].

The (twisted) Hopf algebra $\mathcal{H}(Q)$ admits a Hopf pairing $\{\cdot,\cdot\}\colon \mathcal{H}(Q) \times \mathcal{H}(Q) \to \mathbb{R}$, e. g. we have $\{u, vw\} = \sum \{u_{(1)}, v\}\{u_{(2)}, w\}$ whenever we have $\Delta(u) = \sum u_{(1)} \otimes u_{(2)}$ in Sweedler's notation. It is determined by the values on basis elements

$$\{u_M, u_N\} = \delta_{M,N} \frac{v^{\dim(M)}}{a_M}. \tag{1}$$

2.2 Ringel's theorem and the quantum Serre relations

Example 2.1. The quantum Serre relations in Farnsteiner's talk [2] become the following relations in the twisted Ringel-Hall algebra. Here we abbreviate $u_i = u_{S_i}$ for all i, and we denote by $[n] = (v^n - v^{-n})/(v - v^{-1})$ the quantum integer.

Quiver Q	Untwisted relation	Twisted relation
$1 \to 2$	$u_1^2 u_2 - (q+1) u_1 u_2 u_1 + u_2 u_1^2 = 0$	$u_1^2 u_2 - [2] u_1 u_2 u_1 + u_2 u_1^2 = 0$
$1 \rightrightarrows 2$	$u_1^3 u_2 - (q^2+q+1) u_1^2 u_2 u_1$ $+ q(q^2+q+1) u_1 u_2 u_1^2 - q^3 u_2 u_1^3 = 0$	$u_1^3 u_2 - [3] u_1^2 u_2 u_1 + [3] u_1 u_2 u_1^2 - u_2 u_1^3 = 0$
$1 \rightleftarrows 2$	$u_1^3 u_2 - (q+1+q^{-1}) u_1^2 u_2 u_1$ $+ (q+1+q^{-1}) u_1 u_2 u_1^2 - u_2 u_1^3 = 0$	$u_1^3 u_2 - [3] u_1^2 u_2 u_1 + [3] u_1 u_2 u_1^2 - u_2 u_1^3 = 0$

The theorem of Ringel [11] asserts that the examples generalize. The proof is a rank 2 calculation similar to the derivation of the quantum Serre relations for the n-Kronecker quiver in Farnsteiner's talk [2].

Theorem 2.2 (Ringel)**.** Assume that $e_i, e_j \in K_0(Q)$ correspond to distinct simple representations S_i, S_j. As before, we abbreviate $u_i = u_{S_i}$ and $u_j = u_{S_j}$. If $c_{ij} = (e_i, e_j)$, then

$$\sum_{r+s=1-c_{ij}} (-1)^r \begin{bmatrix} r+s \\ r \end{bmatrix} u_i^r u_j u_i^s = 0.$$

Especially, the quantum Serre relations in the twisted Ringel-Hall algebra are invariant under changing the orientation of arrows, i. e. they depend only on the underlying undirected diagram of the quiver. The *composition algebra* $\mathcal{C}(Q) \subseteq \mathcal{H}(Q)$ is the Hopf subalgebra generated by the elements $u_i = u_{S_i}$ with $i \in Q_0$.

3 Green's theorem

3.1 The positive quantum group and its Hopf structure

Recall the construction of the quantum group from the last talk. Let $n \geq 1$. An integer $n \times n$ matrix $C = (c_{ij})$ is called *symmetric generalized Cartan matrix* if $c_{ii} = 2$ for all $1 \leq i \leq n$ and $c_{ij} = c_{ji} \leq 0$ for all $i \neq j$. Such a matrix defines a Kac-Moody Lie algebra $\mathfrak{g} = \mathfrak{n}_+ \oplus \mathfrak{h} \oplus \mathfrak{n}_-$. Drinfel'd-Jimbo replace universal enveloping algebra $U(\mathfrak{n}_+)$, cocommutative Hopf algebra, with the quantized enveloping algebra U^+, a not necessarily cocommutative Hopf algebra, c.f. Drinfel'd [1]. The process is called quantization, because in quantum mechanics classical commuting observables are replaced with not necessarily commutating operators. Let us briefly outline the construction of U^+; the reader can find more details in the work of Kassel [8], Lusztig [9], and Majid [10].

We fix a field k of characteristic 0. The symmetric generalized Cartan matrix C defines a symmetric bilinear form $(\cdot,\cdot)$ on $\Lambda = \mathbb{Z}^n$ with respect to the standard basis whose elements we denote by e_i with $1 \leq i \leq n$. The group algebra U^0 of Λ over k can be endowed with the structure of a Hopf algebra. For every $v \in k^*$, that is not be a root of unity, we can define a Hopf pairing $\{\cdot,\cdot\}\colon U^0 \times U^0 \to k$. Let f^+ be the free k-algebra with generators E_i for $1 \leq i \leq n$. We can extend the Hopf structure on U^0 to a Hopf structure on the tensor product $f^{+,0} = f^+ \otimes U^0$. The Hopf algebra is graded by Λ with $\deg(E_i) = e_i$. The Hopf pairing extends to a Hopf pairing $\{\cdot,\cdot\}\colon f^{+,0} \times f^{+,0} \to k$ in such a way that

$$\{E_i, E_j\} = \delta_{ij} \frac{1}{v - v^{-1}} \tag{2}$$

for all $1 \leq i \leq n$. The ideal $I^+ \subseteq f^+$ is chosen in such a way that the restriction of the Hopf pairing to $U^+ = f^+/I^+$ is non-degenerate. Note that U^+ is not a Hopf subalgebra because $\Delta(E_i) = E_i \otimes 1 + K_i \otimes E_i \notin U^+$.

Put $\mathcal{Z} = \mathbb{Q}[v, v^{-1}, [m]^{-1} \text{ for } m \geq 2]$. Define $U^+_{\mathcal{Z}}$ to be the $\mathcal{Z}$-algebra generated by E_i with $1 \leq i \leq n$ subject to the quantum Serre relations

$$\sum_{r+s=1-c_{ij}} (-1)^r \begin{bmatrix} r+s \\ r \end{bmatrix} E_i^r E_j E_i^s = 0$$

for all $i \neq j$. A theorem asserts that we have an isomorphism $U^+ \cong k \otimes_{\mathcal{Z}} U^+_{\mathcal{Z}}$, and another theorem asserts that the algebra $U^+_{\mathcal{Z}}$ is a free $\mathcal{Z}$-module.

3.2 Green's theorem

Define a comultiplication $\Delta\colon U^+_{\mathcal{Z}} \to U^+_{\mathcal{Z}} \otimes U^+_{\mathcal{Z}}$ by $\Delta(E_i) = 1 \otimes E_i + E_i \otimes 1$. The comultiplication becomes an algebra homomorphism if we define the multiplication $(U^+_{\mathcal{Z}} \otimes U^+_{\mathcal{Z}}) \times (U^+_{\mathcal{Z}} \otimes U^+_{\mathcal{Z}}) \to$

$U^+_\mathcal{Z} \otimes U^+_\mathcal{Z}$ by $(a \otimes b) \cdot (c \otimes d) = v^{(\deg(b),\deg(c))}(ac \otimes bd)$ for all homogeneous elements $a, b, c, d \in U^+$. We say that $U^+_\mathcal{Z}$ is a *twisted bialgebra*. Together with the antipode $S\colon U^+_\mathcal{Z} \to U^+_\mathcal{Z}$ defined by $S(E_i) = -E_i$ it becomes a *twisted Hopf algebra*. Lusztig shows that the twisted Hopf algebra $U^+_\mathcal{Z}$ admits a Hopf pairing $U^+_\mathcal{Z} \times U^+_\mathcal{Z} \to \mathcal{Z}$, which is uniquely determined by formula (2).

Let Q be a quiver with n vertices such that its symmetrized Euler form equals the symmetric bilinear form attached to the generalized Cartan matrix C. Ringel's theorem 2.2 implies that there is homomorphism $\Psi\colon \mathbb{R} \otimes_\mathcal{Z} U^+_\mathcal{Z} \to \mathcal{C}(Q)$ with $\psi(E_i) = u_i$. Green's main theorem [3, Theorem 3] asserts:

Theorem 3.1 (Green)**.** The map ψ is an isomorphism of twisted Hopf algebras and it respects the grading and the Hopf pairing.

Proof. The homomorphism ψ respects the Hopf pairing, because $\{E_i, E_j\} = \frac{\delta_{i,j}}{v - v^{-1}} = \{u_i, u_j\}$ for all i, j. Suppose that $x \in \mathbb{R} \otimes_\mathcal{Z} U^+_\mathcal{Z}$ lies in the kernel of ψ. Then $\{x, y\} = \{\psi(x), \psi(y)\} = 0$ for all $y \in \mathbb{R} \otimes_\mathbb{R} U^+_\mathcal{Z}$ which implies $x = 0$ since the Hopf pairing is non-degenerate. □

4 Kac's theorem

4.1 Characters and the denominator formula

Let n denote the number of vertices of the quiver Q. For every i we define the reflection $r_i(x) = x - (x, e_i)e_i$. The formula implies $r_i(e_i) = -e_i$ for all i. The reflections r_i with $i \in Q_0$ generate a subgroup $W = W(C) \subseteq \mathrm{Aut}(\mathbb{Z}^n)$, which is called the *Weyl group* of C. The root system of the Kac-Moody Lie algebra decomposes in *real* and *imaginary* roots with

$$\begin{aligned}\Phi_{re} &= \{w(e_i)\colon w \in W, 1 \leq i \leq n\} = \{\alpha \in \Phi\colon 0 < (\alpha, \alpha) \leq 2\},\\ \Phi_{im} &= \Phi \backslash \Phi_{re} \subseteq \{\alpha \in \Phi\colon (\alpha, \alpha) \leq 0\}.\end{aligned}$$

A theorem asserts that every root is either positive or negative and has connected support. The multiplicity $\mathrm{mult}(\alpha)$ is defined as the dimension of the corresponding root space. It can be shown that $\mathrm{mult}(\alpha) = 1$ if α is real, and that $\mathrm{mult}(\alpha) = \mathrm{mult}(w(\alpha))$ for all roots α and all Weyl group elements w. For every $\alpha = (a_1, a_2, \ldots, a_n) \in \Lambda$ we define a monomial $e(\alpha) = t_1^{a_1} t_2^{a_2} \cdot \ldots \cdot t_n^{a_n}$ in the power series ring $\mathbb{Z}[[t_1, t_2, \ldots, t_n]]$. Especially, the elements satisfy the exponential rule $e(\alpha)e(\beta) = e(\alpha + \beta)$. Define the *characters* of the composition algebra and the Hall algebra by

$$\mathrm{ch}(\mathcal{C}(Q)) = \sum_{\alpha \geq 0} \dim_\mathbb{R}(\mathcal{C}(Q)_\alpha) e(\alpha), \qquad \mathrm{ch}(\mathcal{H}(Q)) = \sum_{\alpha \geq 0} \dim_\mathbb{R}(\mathcal{H}(Q)_\alpha) e(\alpha).$$

Proposition 4.1. For every dimension vector $\alpha = (a_1, a_2, \ldots, a_n)$ we denote by $\mathrm{ind}(\alpha, k)$ the number of isoclasses of indecomposable representations of Q over k of dimension α. We have the following product formula:

$$\mathrm{ch}(\mathcal{H}(Q)) = \prod_{\alpha > 0} (1 - e(\alpha))^{-\mathrm{ind}(\alpha, k)}$$

Proof. By the theorem of Krull-Remak-Schmidt every representation is a direct sum of indecomposable representations. The indecomposable representation are unique up to permutation and isomorphisms. Hence $\dim_\mathbb{R}(\mathcal{H}(Q)_\alpha)$, the number of isoclasses of representations of Q over k of dimension α, equals the coefficient corresponding to $e(\alpha)$ in the geometric series expansion. □

Green's theorem implies that the character of the composition algebra equals the character of the positive quantum group. For the positive quantum group, Kac's denominator formula [6, Chapter 10.2] holds:

Theorem 4.2 (Kac)**.** The character of the positive quantum is

$$\operatorname{ch}(U^+) = \prod_{\alpha \in \phi^+} (1 - e(\alpha))^{-\operatorname{mult}(\alpha)}$$

4.2 Indecomposable representations and Kac's theorem

Note that formula (1) implies that the Hopf pairing on $\mathcal{H}(Q)$ is positive definite. Following Sevenhant-Van Den Bergh [12, Chapter 3] we choose for every $0 \neq \alpha \in \mathbb{N}^n$ an orthonormal basis $\{\theta_i\}_{i \in I(\alpha)}$ of the subspace

$$\mathcal{G}(Q)_\alpha = \left(\sum_{\substack{\beta+\gamma=\alpha \\ \beta,\gamma>0}} \mathcal{H}(Q)_\beta \mathcal{H}(Q)_\gamma \right)^\perp \subseteq \mathcal{H}(Q)_\alpha.$$

Let I be the union of the $I(\alpha)$ for all α. Note that $\mathcal{G}(Q)_{e_i}$ is a 1-dimensional vector space spanned by u_i so that $I(e_i)$ is a singleton. We define a bilinear form on $\mathbb{Z}I$ by $(i,j) = (\deg(\theta_i), \deg(\theta_j))$. Assume that the matrix $\widetilde{C}$ represents the bilinear form with respect to the basis I.

Lemma 4.3. Every θ_i is primitive, i. e. $\Delta(\theta_i) = \theta_i \otimes 1 + 1 \otimes \theta_i$.

Proof. Let us extend $\{\theta_i\}_{i \in I} \subseteq (\zeta_j)_{j \in J}$ to a homogeneous orthonormal basis of $\mathcal{H}(Q)$. Then we can write $\Delta(\theta_i) = \sum_{j,k \in J} c_{jk} \zeta_j \otimes \zeta_k$. As $\{\cdot, \cdot\}$ is a Hopf pairing, we have

$$\{\theta_i, \zeta_l \zeta_m\} = \sum_{j,k \in J} c_{jk} \{\zeta_j, \zeta_l\}\{\zeta_k, \zeta_m\} = c_{lm}.$$

for all $l, m \in J$. By definition $\{\theta_i, xy\} = 0$ for all homogeneous elements $x, y \in \mathcal{H}(Q)$ of degree β, γ with $\alpha = \beta + \gamma$, hence $c_{l,m} = 0$ unless $(\zeta_l, \zeta_m) = (\theta_i, 1)$ or $(\zeta_l, \zeta_m) = (1, \theta_i)$ in which case $c_{l,m} = 1$. □

A *symmetric Borcherds matrix* is a generalization of a generalized Cartan matrix where diagonal entries are allowed to be in the set $\{2, 0, -2, -4, \ldots\}$. Borcherds introduced these in the context of moonshine.

Lemma 4.4. The matrix $\widetilde{C}$ is a Borcherds matrix. Especially we have $(i,j) \leq 0$ whenever $i \neq j$.

Proof. Let $i, j \in I$ be distinct. By the previous lemma we have

$$\begin{aligned}\Delta(\theta_i \theta_j) &= (\theta_i \otimes 1 + 1 \otimes \theta_i)(\theta_j \otimes 1 + 1 \otimes \theta_j) \\ &= \theta_i \theta_j \otimes 1 + \theta_i \otimes \theta_j + v^{(i,j)} \theta_j \otimes \theta_i + 1 \otimes \theta_i \theta_j.\end{aligned}$$

From this equation we can follow that $\{\theta_i\theta_j, \theta_i\theta_j\} = 1$ and $\{\theta_i\theta_j, \theta_j\theta_i\} = v^{(i,j)}$. The positive definiteness of the Hopf pairing implies for all $x, y \in \mathbb{R}$ the inequality

$$0 \leq \{x\theta_i\theta_j + y\theta_j\theta_i, x\theta_i\theta_j + y\theta_j\theta_i\} = x^2 + 2v^{(i,j)}xy + y^2.$$

Thus, the discriminant of the quadratic form must satisfy $4(1 - v^{(ij)}) \geq 0$. Hence $1 \geq v^{(i,j)}$, so that $(i,j) \leq 0$. For the proof of the other properties of Borcherds matrices we refer the reader to Sevenhant-Van Den Bergh [12, Proposition 3.2]. □

Note that the matrix $\widetilde{C}$ is an infinite matrix with C in the top left corner. It follows that the root systems $\widetilde{\Phi}$ and Φ associated with $\widetilde{C}$ and C have the same real roots and the same Weyl group. With the Borcherds matrix $\widetilde{C}$ we can associate a generalized Kac-Moody Lie algebra. Let $\widetilde{U}^+$ be its positive quantum group, which we construct similarly. As the Hopf pairing is non-degenerate on $\mathcal{H}(Q)$, a Sevenhant-Van Den Bergh's generalization [12] of Green's main theorem to generalized Kac-Moody algebras implies that we have an isomorphism of twisted Hopf algebras $\mathcal{H}(Q) \cong \mathbb{R} \otimes_{\mathbb{Z}} \widetilde{U}^+$. We obtain a different proof of a theorem of Kac [7] for indecomposable quiver representations over finite fields.

Theorem 4.5 (Kac)**.** We have $\widetilde{\Phi}^+ = \{\dim(M) \colon M \text{ indecomposable}\}$. Moreover, the number of isoclasses of indecomposable representations of Q over k with dimension vector $\alpha \in \widetilde{\Phi}^+$ is equal to the multiplicity of α in $\widetilde{\Phi}^+$. Especially, up to isomorphism there exists only one indecomposable representation with dimension vector α for every real root $\alpha \in \Phi_{re}$.

Proof. Compare the product formula (4.1) for $\mathrm{ch}(\mathcal{H}(Q))$ with the product formula (4.2) for the positive quantum group of $\widetilde{C}$. ☐

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