Infinite-dimensional modules over wild hereditary algebras

(based on a paper by Frank Lukas)

Philipp Fahr, 30th June 2006

1 Introduction

This talk is about infinite-dimensional A-modules, where A is a finite-dimensional connected hereditary k-algebra, and k a field. We will thus work in $\mathrm{Mod}-A$, with a capital "M", since in this category the modules are not necessarily finitely generated. I will present some results and examples of infinite-dimensional modules from a paper by Frank Lukas [L1].[1]

1.1 Background

One should be interested to know in which way the structure of modules of finite length determines the behaviour of arbitrary modules.

Recall that a finite-dimensional algebra is said to be of *finite representation type* if there are only finitely many indecomposable modules of finite length. Then any module is the direct sum of modules of finite length (Ringel-Tachikawa, 1974), and such a decomposition is unique up to isomorphism.

M. Auslander has shown (in "Large modules over artin algebras", 1976) that if A is not of finite representation type, then there exist indecomposable modules which are not of finite length. Auslander gave an existence proof and C. M. Ringel gave a general structure theory for modules of arbitrary length in his "Rome Lectures" (1977, published 1979 [R3]). He showed that there always will be certain important infinite-dimensional representations, and the investigation of these modules also gives some new insight into the behaviour of the modules of finite length. Note also that in general one cannot dualize results for arbitrary modules, since the dual functor $D = \mathrm{Hom}_k(-, k)$ is only an equivalence between the categories $\mathrm{mod}-A$ and $A-\mathrm{mod}$.

Most definitions are motivated by the structure theory of C. M. Ringel for the tame hereditary case, but there is not enough time to present the details for infinite-dimensional A-modules, when A is tame. [R3] is worth reading.

1.2 Aims

Let A be a finite-dimensional connected wild hereditary algebra over a field k. $\mathrm{Mod}-A$ denotes the category of right A-modules and $\mathrm{mod}-A$ the category of finitely generated right A-modules. The finite-dimensional indecomposable modules divide into three classes: preprojective, regular and preinjective modules. These terms always imply finitely generated modules (but not necessarily indecomposable) in contrast to the conventions used in the Rome Lectures [R3].

[1]Frank Lukas was a PhD-student of O. Kerner. Surprisingly he wrote this paper before even handing in his Diploma thesis. Two years later he handed in his PhD thesis on elementary modules [L2].

Definition 1.1. *An arbitrary module M is said to be* divisible *if* $\mathrm{Hom}(M, R) = 0$*, for every regular module R (R finite-dimensional).*

One can show (using Auslander-Reiten theory) that this is equivalent to $\mathrm{Ext}(X, M) = 0$ for all preprojective and all regular[2] modules X or, equivalently, for any module X without indecomposable preinjective direct summand.

Aim 1. The category of divisible modules has enough projective objects, called $\mathcal{D}$-projectives.

Here already one can see that restricting only to finite-dimensional modules hides a part of the structure: a finite-dimensional divisible module M is preinjective (A is hereditary) and then $\mathrm{Ext}(M, \tau M) \neq 0$.

Aim 2. If M is non-zero $\mathcal{D}$-projective, then the class $\mathrm{Add}(M)$ of direct summands of (not necessarily finite) direct sums of copies of M contains all $\mathcal{D}$-projective modules.

The following is due to O. Kerner:

Aim 3. We will construct an example of an infinite-dimensional module, actually a indecomposable divisible module which does not have any preinjective direct summand, but every proper factor of this module is a direct sum of preinjective modules.

2 Definitions

Recall that a subfunctor of the identity functor on $\mathrm{Mod}-A$ is a functor $t : \mathrm{Mod}-A \to \mathrm{Mod}-A$ that assigns to each module M a submodule $tM \subseteq M$ such that each homomorphism $M \to N$ restricts to a homomorphism $tM \to tN$.

Definition 2.1. *A subfunctor t of the identity functor on* $\mathrm{Mod}-A$ *is called an* idempotent radical *if, for every module M, we have $t(tM) = tM$ and $t(M/tM) = 0$.*

Let us define the subfunctor $\mathcal{P}$ of the identity functor as follows:

Let π be a *predecessor closed* set of indecomposable preprojective modules, i.e. for every indecomposable module P with $\mathrm{Hom}(P, P') \neq 0$ for some $P' \in \pi$, the set π contains a module isomorphic to P. Define for every $M \in \mathrm{Mod}-A$ the submodule

$$\mathcal{P}_\pi M := \bigcap_{f:M\to P} \ker f,$$

where the intersection is taken over all $P \in \pi$. So $\mathcal{P}_\pi M$ is the intersection of the kernels of all maps $M \to P$ with $P \in \pi$.

The following theorem was originally stated for modules of finite length, but is also valid for arbitrary modules.

Theorem 2.1 (Ringel). *Let π be a finite predecessor closed set of preprojective modules. Then every module $M \in \mathrm{Mod}-A$ has a decomposition $M = \mathcal{P}_\pi M \oplus M'$ with $M' \in \mathrm{Add}(\pi)$.*

For the dual case we have a *successor closed* set θ of indecomposable preinjective modules, i.e. for every indecomposable module I with $\mathrm{Hom}(I', I) \neq 0$ for some $I' \in \theta$, the set θ contains a module isomorphic to I. We then define the following subfunctor $\mathcal{I}$ of the identity functor:

$$\mathcal{I}_\theta M := \sum_{f:I\to M} \mathrm{Im}\, f,$$

where the sum is taken over all modules $I \in \theta$. So $\mathcal{I}_\theta M$ is the sum of the images of all maps $I \to M$ with $I \in \theta$. We have a similar result:

[2] Note that a module is called *regular*, if it has no indecomposable preprojective or preinjective direct summands.

Theorem 2.2 (Ringel). *Let θ be a finite successor closed set of preinjective modules. Then every module $M \in \mathrm{Mod}-A$ has a decomposition $M = \mathcal{I}_\theta M \oplus M'$ with $\mathcal{I}_\theta M \in \mathrm{Add}(\theta)$.*

We will just write $\mathcal{I}$ instead of $\mathcal{I}_\theta$ if θ is the class of all (isomorphism classes of) indecomposable preinjective modules. Similar for $\mathcal{P}$ and π being the set of all (isomorphism classes of) indecomposable preprojective modules.

Definition 2.2. *The largest submodule U of M with $\mathcal{P}U = U$ is denoted by $\mathcal{P}^\infty M$.*

There is a lot of structure theory for properties of the modules $\mathcal{P}M$ and $\mathcal{I}M$ in the tame case, which C.M. Ringel showed in [R3]. But not everything is true in the wild setting. For example $\mathcal{I}M$ is not always a direct summand of M for A of wild representation type. But there is an important difference between the functors $\mathcal{P}$ and $\mathcal{I}$, which we will explain next.

2.1 Some torsion theory

Recall that a pair $(\mathcal{T}, \mathcal{F})$ of full subcategories of a module category is called a *torsion pair* (or *torsion theory*) if the following conditions are satisfied:

(i) $\mathrm{Hom}(M, N) = 0$ for all $M \in \mathcal{T}$, $N \in \mathcal{F}$.

(ii) $\mathrm{Hom}(M, -)|_\mathcal{F} = 0$ implies $M \in \mathcal{T}$.

(iii) $\mathrm{Hom}(-, N)|_\mathcal{T} = 0$ implies $N \in \mathcal{F}$.

So there is no non-zero homomorphism from an object in $\mathcal{T}$ to an object in $\mathcal{F}$ and the two subcategories are maximal with respect to this property. $\mathcal{T}$ is called the *torsion class*, $\mathcal{F}$ the *torsion-free class*.

Each torsion pair induces an idempotent radical, called *torsion radical*, and conversely: $\mathcal{T}$ is a torsion class of some $(\mathcal{T}, \mathcal{F})$ if and only if there exists an idempotent radical t such that $\mathcal{T} = \{M \,|\, tM = M\}$. So for $M \in \mathrm{Mod}-A$, $tM \in \mathcal{T}$ and $M/tM \in \mathcal{F}$. Also there is always the canonical short exact sequence $0 \to tM \to M \to M/tM \to 0$.

A torsion pair $(\mathcal{T}, \mathcal{F})$ is called *splitting* if each indecomposable module M either lies in $\mathcal{T}$ or in $\mathcal{F}$. Then the canonical sequence above splits.

The functor $\mathcal{I}$ induces a torsion class: M is $\mathcal{I}$-torsion if $\mathcal{I}M = M$. However the functor $\mathcal{P}$ does not induce a torsion-free class, therefore the functor $\mathcal{P}$ does not split. (The problem is that there exits modules M, such that $\mathcal{P}(\mathcal{P}M)$ are proper submodules of $\mathcal{P}M$.) But the functor $\mathcal{P}^\infty$ induces a torsion pair and the torsion modules M are characterised by the property that $\mathrm{Hom}(M, P) = 0$ for all preprojective modules P.

This leads to the torsion class of divisible modules. Recall from above that M is divisible if $\mathrm{Hom}(M, R) = 0$ for every regular module R. This gives us a torsion pair and the radical of this torsion class is denoted by $\mathcal{D}$.

Definition 2.3. *M is* reduced, *if M is torsion-free in this torsion pair; that is, M is reduced if $\mathcal{D}M = 0$.*

Now, given $M \in \mathrm{Mod}-A$ define

$$\mathcal{T}M := \sum_{f:R \to M} f(R),$$

where the sum is taken over all regular modules R.

Definition 2.4. *(i) M is a* torsion module *if $\mathcal{T}M = M$.*

(ii) M is torsion-free *if* $\mathcal{T}M = 0$.

By definition, (ii) is the case when $\mathrm{Hom}(R, M) = 0$ for all regular modules R.

The investigation of the above modules and classes of infinite-dimensional modules was initiated by C.M. Ringel [R3], who carried over ideas from he theory of abelian groups, where the notions of torsion, torsion-free, divisible and reduced come from.

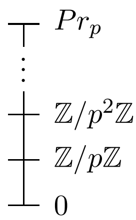
To summarize: We defined the functors $\mathcal{P}_\pi$, $\mathcal{I}_\theta$ and their generalisation to the set of all preprojective resp. preinjective modules $\mathcal{P}$, $\mathcal{I}$. We then denoted the largest submodule $U \subseteq M$ such that $\mathcal{P}U = U$ by $\mathcal{P}^\infty M$. Finally defined similar functors $\mathcal{T}$, $\mathcal{D}$. We will use those in section 3.3 to prove *Aim* 1 and *Aim* 2.

2.2 Prüfer modules

The usual examples of artinian modules which are not of finite length are the so called Prüfer groups: For any prime number p, there are embeddings

$$\mathbb{Z}/p\mathbb{Z} \hookrightarrow \mathbb{Z}/p^2\mathbb{Z} \hookrightarrow \mathbb{Z}/p^3\mathbb{Z} \hookrightarrow \dots$$

and by forming the union (or direct limit) we get such a Prüfer group: $Pr_p = \bigcup_i \mathbb{Z}/p^i\mathbb{Z}$. Pr_p is artinian and its submodule lattice looks as follows:



And one has $\mathbb{Q}/\mathbb{Z} = \bigoplus_p Pr_p$, where the direct sum is taken over all prime numbers p.

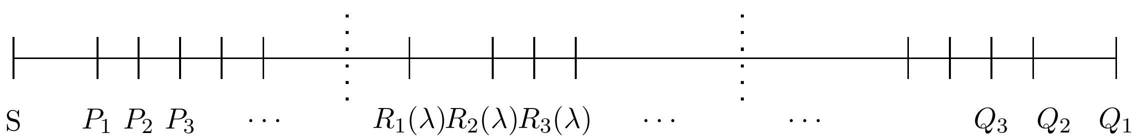
3 Examples

3.1 Tame case

C. M. Ringel showed that for A tame there exists a unique indecomposable torsion-free divisible module $Q \in \mathrm{Mod}-A$ (up to isomorphism). Its endomorphism ring is a field (algebraically closed case), and Q is finite dimensional over $\mathrm{End}(Q)$.[3] In the tame case the divisible modules are direct sums of indecomposable divisible modules, and indecomposable divisible are the indecomposable preinjective modules, Prüfer modules, and Q.

Example. The classical example of an infinite-dimensional module is the following: Let A be the Kronecker algebra, that is, the path algebra of the tame hereditary quiver with two vertices and two arrows in the same direction. Then $Q := (k(X), k(X), \cdot id, \cdot X)$, with $k(X)$ being the field of rational functions in one variable, is the unique indecomposable torsion-free divisible module.

Recall that the totally ordered set of all the Gabriel-Roiter measures can be drawn as follows:



[3]This is also referred to as being *endo-finite*. Indecomposable infinite length modules which are endo-finite have been called *generic* by W. Crawley-Boevey.

There are precisely two accumulation points, which are drawn as dotted vertical lines. They correspond to the only Gabriel-Roiter measures for infinitely generated modules. The first one to the left is the Gabriel-Roiter measure $\{1, 3, 5, 7, \ldots\}$ for all indecomposable torsion-free modules. The second one to the right is $\{1, 2, 4, 6, 8, \ldots\}$ corresponds to the Prüfer modules.

Example. Let A be tame hereditary and $S(1)$ a simple regular module. If

$$S(1) \hookrightarrow S(2) \hookrightarrow S(3) \hookrightarrow \ldots$$

is a chain of irreducible monomorphisms, then the module $S := \bigcup_n S(n)$ is an indecomposable torsion divisible module with local endomorphism ring. This is a Prüfer module[4]. In the tame case every torsion divisible module is a direct sum of indecomposable preinjective modules and Prüfer modules.

3.2 Wild case

In contrast to the tame case, there are no non-zero torsion-free divisible modules if A is a wild hereditary algebra. The following example of an indecomposable divisible module was originally constructed by O. Kerner:

Example. Let $X \neq 0$ be a regular module with $\mathcal{O}(\mathrm{X})$ a regular mono-orbit[5]. Then by Baer's theorem (see [L1], Prop. 1.6) there is a non-zero map $f : X \to \tau^n X$ for some n. Considering the following chain of monomorphisms

$$X \overset{f}{\hookrightarrow} \tau^n X \overset{\tau^n f}{\hookrightarrow} \tau^{2n} X \overset{\tau^{2n} f}{\hookrightarrow} \tau^{3n} X \hookrightarrow \ldots,$$

define $M := \bigcup_r \tau^{rn} X$. Let U be a non-zero finitely generated submodule of M with $U \subset \tau^{rn} X \subset M$ for some $r \in \mathbb{N}$. Since $\mathcal{O}(X)$ is a regular mono-orbit the modules $(\tau^{(r+i)n} X)/U$ are preinjective for all $i \in \mathbb{N}$, otherwise we have the epimorphism and thus an isomorphism from $\tau^{(r+i)n} X$ onto a regular direct summand of $(\tau^{(r+i)n} X)/U$, which is a contradiction. The factor module M/U is an epimorphic image of $\bigoplus (\tau^{(r+i)n} X)/U$ and therefore a direct sum of preinjective modules.

Let I be an indecomposable preinjective module. Look at the short exact sequence $0 \to U \to M \to M/U \to 0$ and apply $\mathrm{Hom}(I, -)$ to get:

$$\ldots \to \mathrm{Hom}(I, M) \to \mathrm{Hom}(I, M/U) \to \mathrm{Ext}(I, U) \to \ldots$$

Since $\mathrm{Hom}(I, M) = 0$ the module M/U has only finitely many direct summands isomorphic to I. So we can write M/U as $\bigoplus_n I_n^{k_n}$ with pairwise non-isomorphic indecomposable preinjective modules I_n and $k_n \in \mathbb{N}$. Since every proper factor of M is a direct sum of preinjective modules, a non-zero map $M \to N$ has to be a monomorphism if $\mathcal{I}M = 0$. This finishes *Aim* 3. Considering *Aim* 1 and *Aim* 2, one can show that this module is not $\mathcal{D}$-projective.

3.3 Divisible module construction

As already mentioned, there are no non-zero torsion-free divisible modules if A is a wild hereditary algebra. Let us now consider *Aim* 1 and *Aim* 2. Thus given the class of divisible modules, we are looking for $\mathcal{D}$-projective modules.

Let $(\mathcal{T}, \mathcal{F})$ be the torsion pair in $\mathrm{Mod}-A$ with $\mathcal{T}$ being the torsion class of divisible modules. Let $\mathcal{S}$ be the class of all submodules of direct sums of regular modules. Then

[4] This is also a Prüfer module in the sense of Ringel, as used in his Topics-Lectures in Bielefeld 2006.

[5] This means, that for all R regular and $n \in \mathbb{N}_0$, all non-zero maps in $\mathrm{Hom}(\tau^n X, R)$ are monomorphisms.

by definition and Auslander-Reiten theory, $\operatorname{Ext}(\mathcal{S}, D) = 0$ for every divisible module D. We will construct divisible modules in the class $\mathcal{E}(\mathcal{S})$, defined as follows: Let $\mathcal{E}(\mathcal{S})$ be the class of all modules M which are the well-ordered union of submodules M_λ, such that $M_0 = 0$, $M_{\lambda+1}/M_\lambda \in \mathcal{S}$ and $M_\lambda = \bigcup_{\mu<\lambda}$, if λ is a limit ordinal[6]. So the modules in $\mathcal{E}(\mathcal{S})$ have a $\mathcal{S}$-filtration. For this class we have the following crucial result by F. Lukas:

Theorem 3.1. *Given a torsion pair $(\mathcal{T}, \mathcal{F})$ in $\operatorname{Mod}-A$ and let $\mathcal{S}$ be a class of modules with $\operatorname{Ext}(\mathcal{S}, \mathcal{T}) = 0$. If there exists a short exact sequence $0 \to A \to T_1 \to T_2 \to 0$ with T_1, T_2 $\mathcal{T}$-projective[7], then $\operatorname{Ext}(M, \mathcal{T}) = 0$ for every $M \in \mathcal{E}(\mathcal{S})$.*

We also need the following important result by D. Baer:

Theorem 3.2 (D. Baer, 1986). *If A is a wild hereditary algebra, then*

(i) there exists a short exact sequence $0 \to A \to R_1 \to R_2 \to 0$ with regular modules R_1, R_2.

(ii) for a regular module R, there exist $k, n \in \mathbb{N}$ and a universal short exact sequence $0 \to A \to X \to \tau^n R^k \to 0$ with $\operatorname{Hom}(X, R) = 0$.

The proofs can be found in [L1].

Let us construct a divisible module $A_\mathcal{D}$ in $\mathcal{E}(\mathcal{S})$: Fix a regular module $R \neq 0$. By Baer's theorem there exists a short exact sequence

$$0 \to A \to A_1 \to A_1/A \to 0$$

with A_1/A regular and $\operatorname{Hom}(A_1, R) = 0$. Recursively taking A_n and $\tau^n R$ (write $A_0 := A$), define A_{n+1} as the middle term of a short exact sequence $0 \to A_n \to A_{n+1} \to A_{n+1}/A_n \to 0$ with A_{n+1}/A_n regular and $\operatorname{Hom}(A_{n+1}, \tau^n R) = 0$.[8]

Considering the monomorphisms as inclusions define

$$A_\mathcal{D} := \bigcup_{n \in \mathbb{N}_0} A_n.$$

By construction $A_\mathcal{D}$, $A_\mathcal{D}/A \in \mathcal{E}(\mathcal{S})$, so by above theorem $A_\mathcal{D}$ and $A_\mathcal{D}/A$ are $\mathcal{T}$-projective, if $A_\mathcal{D}$ is a divisible module. Since $\mathcal{D}$ is the radical of the above torsion pair, the module is also called $\mathcal{D}$-projective.

Proposition 3.3. *$A_\mathcal{D}$ is a divisible module.*

Proof. Let X be a regular module and $f : A_\mathcal{D} \to X$. Since X is finitely generated we can find $N \in \mathbb{N}$ with $f(A_n) = f(A_\mathcal{D})$ for all $n \geq N$. Since $\operatorname{Hom}(A_{n+1}, \tau^n R) = 0$, the module $f(A_\mathcal{D})$ has the property that $\operatorname{Hom}(f(A_\mathcal{D}), \tau^n R) = 0$ for all $n \geq N$. But this is only possible if $f(A_\mathcal{D}) = 0$. We have shown that $\operatorname{Hom}(A_\mathcal{D}, X) = 0$ for all regular modules X, thus $A_\mathcal{D}$ is divisible. □

So we have proven:

Theorem 3.4. *For a wild hereditary algebra A there exists a short exact sequence $0 \to A \to A_\mathcal{D} \to A_\mathcal{D}/A \to 0$ with $\mathcal{D}$-projective modules $A_\mathcal{D}$, $A_\mathcal{D}/A$.*

[6] A limit ordinal is an ordinal number which is neither zero nor a successor ordinal, i.e. has no immediate predecessor. It is equal to the supremum of all the ordinals below it, but not zero.

[7] A torsion module N is called $\mathcal{T}$-projective if $\operatorname{Ext}(N, \mathcal{T}) = 0$.

[8] This is similar to the tower/ladder construction used by C.M. Ringel in his 2006 Topics-Lectures in Bielefeld.

We have $\operatorname{Ext}(M, D) = 0$ for every divisible module D, if $M \in \mathcal{E}(\mathcal{S})$. The converse is also true, i.e. if M satisfies $\operatorname{Ext}(M, D) = 0$ for all divisible D, then $M \in \mathcal{E}(\mathcal{S})$.

Furthermore one can show the following:

Proposition 3.5. *(i) $A_{\mathcal{D}}$ generates all divisible modules D, i.e. there is an I and an epimorphism $A_{\mathcal{D}}^{(I)} \to D$.*

(ii) Every $\mathcal{D}$-projective module is contained in $\operatorname{Add}(A_{\mathcal{D}})$. *Furthermore, if $M \neq 0$ is $\mathcal{D}$-divisible, then every $\mathcal{D}$-projective module is contained in* $\operatorname{Add}(M)$.

This completes *Aim* 1 and *Aim* 2.

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