# Infinite-dimensional modules over wild hereditary algebras II

(based on a paper by Frank Lukas)

Philipp Fahr, 12th July 2006

## 1 Introduction

In the construction of an infinite dimensional torsion divisible module, which was a projective object in the category of all divisible modules over a wild hereditary algebra, a theorem of F. Lukas [L1] was used. We will now give the detailed proof in its full generality. We will also show some proofs for properties of the constructed divisible module, mainly that if $M$ is non-zero $\mathcal{D}$-projective, then the class $\mathrm{Add}(M)$ of direct summands of (not necessarily finite) direct sums of copies of $M$ contains all $\mathcal{D}$-projective modules.

Let $A$ be a finite-dimensional connected wild hereditary algebra over a field $k$. $\mathrm{Mod}-A$ denotes the category of right $A$-modules and $\mathrm{mod}-A$ the category of finitely generated right $A$-modules. The finite-dimensional indecomposable modules divide into three classes: preprojective, regular and preinjective modules. These terms always imply finitely generated modules (but not necessarily indecomposable). Recall the following definition:

**Definition 1.1.** *An arbitrary module $M$ is said to be* divisible *if* $\mathrm{Hom}(M, R) = 0$*, for every regular module $R$ ($R$ finite-dimensional).*

One can show (using Auslander-Reiten theory) that this is equivalent to $\mathrm{Ext}(X, M) = 0$ for all preprojective and all regular[1] modules $X$ or, equivalently, for any module $X$ without indecomposable preinjective direct summand.

### 1.1 Torsion Theory

Recall that a pair $(\mathcal{T}, \mathcal{F})$ of full subcategories of a module category is called a *torsion pair* (or *torsion theory*) if the following conditions are satisfied:

(i) $\mathrm{Hom}(M, N) = 0$ for all $M \in \mathcal{T}$, $N \in \mathcal{F}$.

(ii) $\mathrm{Hom}(M, -)|_{\mathcal{F}} = 0$ implies $M \in \mathcal{T}$.

(iii) $\mathrm{Hom}(-, N)|_{\mathcal{T}} = 0$ implies $N \in \mathcal{F}$.

So there is no non-zero homomorphism from an object in $\mathcal{T}$ to an object in $\mathcal{F}$ and the two subcategories are maximal with respect to this property. $\mathcal{T}$ is called the *torsion class*, $\mathcal{F}$ the *torsion-free class.*

**Definition 1.2.** *A torsion module $M$ is called* $\mathcal{T}$-projective*, if* $\mathrm{Ext}^i(M, \mathcal{T}) = 0$*, for all* $i \geq 1$.

We write $\mathrm{Ext}^i(M, \mathcal{T}) = 0$ for $\mathrm{Ext}^i(M, T) = 0$, for all $T \in \mathcal{T}$. Let us look at the following question: *What are the $\mathcal{T}$-projective modules?* Since we are in the hereditary case, we are only interested in $\mathrm{Ext}^1$ vanishing.

Suppose $\mathcal{S}$ is a class of modules with $\mathrm{Ext}^1(S, \mathcal{T}) = 0$ for all $S \in \mathcal{S}$. Define the class $\mathcal{E}(\mathcal{S})$ as follows: $\mathcal{E}(\mathcal{S})$ is the class of all modules $M$ which are unions (or direct limits) of submodules $(M_\lambda)_{\lambda \in \Lambda}$ with the properties:

(i) The set $\Lambda$ is well-ordered[2], and for all $\mu < \lambda$, there exists a mono $M_\mu \hookrightarrow M_\lambda$.

[1] Note that a module is called *regular*, if it has no indecomposable preprojective or preinjective direct summands.

[2] This means every non-empty subset has a least element, or, equivalently, it is totally ordered and there is no infinite descending sequence.

(ii) $M_0 := (0)$, and for every $\lambda$, $M_{\lambda+1}/M_\lambda \in \mathcal{S}$.[3]

(iii) For every limit ordinal $\lambda \in \Lambda$, i.e. $\lambda = \sup\{\mu \,|\, \mu < \lambda\}$ and $\lambda \neq 0$, $M_\mu = \bigcup_{\mu<\lambda} M_\mu$.

For the proof of Lukas Theorem we need more on ordinal numbers, discussed next.

### 1.2 Ordinals

A linearly ordered set is *well-ordered* if every non-empty subset has a least element. Recall that every finite totally ordered set is well-ordered. A set $T$ is *transitive* if every element of $T$ is a subset of $T$.

**Definition 1.3.** *A set is an* ordinal number *(or an* ordinal*) if it is transitive and well-ordered by* $\in$.

Transfinite ordinal numbers were introduced by Georg Cantor in 1897, to accommodate infinite sequences and to classify sets with certain kinds of order structures on them. Ordinals are an extension of the natural numbers different from integers and from cardinals.

For ordinals $\alpha$ and $\beta$, define $\alpha < \beta$ if $\alpha \in \beta$. We have the following: $0 = \emptyset$ is an ordinal. If $\alpha$ is an ordinal and $\beta \in \alpha$, then $\beta$ is an ordinal. If $\alpha$ and $\beta$ are distinct ordinals and $\alpha \subset \beta$, then $\alpha \in \beta$. If $\alpha$ and $\beta$ are ordinals, then either $\alpha \subset \beta$ or $\beta \subset \alpha$.

The class of ordinals, **Ord**, is linearly ordered by $<$ and well-ordered. For each $\alpha \in \mathbf{Ord}, \alpha = \{\beta \,|\, \beta < \alpha\}$. Also $\alpha \cup \{\alpha\}$ is an ordinal, and $\alpha \cup \{\alpha\} = inf\{\beta \,|\, \beta > \alpha\}$.

**Definition 1.4.** *Let* $\alpha{+}1 := \alpha \cup \{\alpha\}$*. This is called the* successor ordinal*. If* $\alpha$ *is not a successor ordinal, the* $\alpha = sup\{\beta \,|\, \beta < \alpha\}$*, and* $\alpha$ *is called a* limit ordinal*.*

**Definition 1.5.** *An ordinal number* $\alpha > 0$ *is called a* limit ordinal *if and only if it has no immediate predecessor, i.e., if there is no ordinal number* $\beta$ *such that* $\beta{+}1 = \alpha$*.*

So a limit ordinal is an ordinal number which is neither zero nor a successor ordinal. Phrased in yet another way, an ordinal is a limit ordinal if and only if it is equal to the supremum of all the ordinals below it, but is not zero.

Because the class of ordinal numbers is well-ordered, there is a smallest infinite limit ordinal, denoted by $\omega$. This ordinal $\omega$ is also the smallest infinite ordinal, as it is the least upper bound of the natural numbers. Hence $\omega$ represents the order type of the natural numbers. The set of all ordinal numbers does not exist. In order of increasing size, the ordinal numbers are $0, 1, 2, \ldots, \omega, \omega + 1, \omega + 2, \ldots, \omega + \omega, \omega + \omega + 1, \ldots$. The notation of ordinal numbers is sometimes counterintuitive, e.g., even though $1 + \omega = \omega, \omega + 1 > \omega$. The cardinality of the set of countable ordinal numbers is denoted $\aleph_1$.

We will use ordinal numbers for transfinite induction. Transfinite induction holds in any well-ordered set. Any property which passes from the set of ordinals smaller than a given ordinal $\alpha$ to $\alpha$ itself, is true of all ordinals.

## 2 Lukas Theorem

**Theorem 2.1 (Lukas, 1990).** *Let* $(\mathcal{T}, \mathcal{F})$ *be a torsion pair and* $\mathcal{S}$ *a class of modules with* $\operatorname{Ext}(\mathcal{S}, \mathcal{T}) = 0$*. Then*

*(a) For every torsion module* $M \in \mathcal{E}(\mathcal{S})$*,* $\operatorname{Ext}(M, \mathcal{T}) = 0$*, i.e.* $M$ *is* $\mathcal{T}$*-projective.*

*(b) If there exists a short exact sequence* $0 \to A \to T_1 \to T_2 \to 0$ *with* $T_1, T_2$ $\mathcal{T}$*-projective, then* $\operatorname{Ext}(M, \mathcal{T}) = 0$ *for every* $M \in \mathcal{E}(\mathcal{S})$*.*

If $\mathcal{T}$ is the class of divisible modules, then the converse of $(b)$ holds too, i.e. evey $\mathcal{T}$-projective module is contained in $\mathcal{E}(\mathcal{S})$.

[3] The modules in $\mathcal{E}(\mathcal{S})$ are $\mathcal{S}$-filtered.

*Proof.* We begin with part $(a)$. For $M$ torsion in $\mathcal{E}(\mathcal{S})$ we want $\operatorname{Ext}(M,\mathcal{T})=0$. Let $N$ be a torsion module and

$$(*) \qquad 0 \to N \to Y \to M \to 0$$

be a short exact sequence. We need to show that this sequence splits, i.e. $\operatorname{Ext}(M,N)=0$.

$M \in \mathcal{E}(\mathcal{S})$, so take an $\mathcal{S}$-filtration of $M$, $M=\bigcup_\lambda M_\lambda$. $M_0=(0)$, $M_1\in\mathcal{S}$. $\mathcal{S}$ satisfies $\operatorname{Ext}(\mathcal{S},\mathcal{T})=0$, so $M_1$ is $\mathcal{T}$-projective, i.e. $\operatorname{Ext}(M_1,N)=0$. So we can find a homomorphism $f_1: M_1\to Y$, such that $g\circ f_1=\iota_{M_1}$.

For every ordinal number $\lambda$, the following holds using transfinite induction: If $f_\mu: M_\mu\to Y$ is defined for all $\mu<\lambda$ with $f_{\mu_2}(m)=f_{\mu_1}(m)$ for all $m\in M_{\mu_1}$, whenever $\mu_1\le\mu_2$, and $g\circ f_\mu=\iota_{M_\mu}$ for all $\mu<\lambda$, then there exists a homomorphism $f_\lambda: M_\lambda\to Y$ with $f_\lambda(m)=f_\mu(m)$ for all $m\in M_\mu$, $\mu<\lambda$ and $g\circ f_\lambda=\iota_{M_\lambda}$. If $\lambda$ is a limit ordinal, define $f_\lambda(m)=f_\mu(m)$ for $m\in M_\mu$.

Otherwise we have a short exact sequence

$$0\to M_{\lambda-1}\to M_\lambda\to S\to 0,$$

with $S\in\mathcal{S}$. Combining this sequence with the one above $(*)$, while applying covariant and contravariant Hom-functors, we get a commutative diagramm as follows:

$$\begin{array}{ccccccccc}
 & & 0 & & 0 & & 0 & & \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & \operatorname{Hom}(S,N) & \longrightarrow & \operatorname{Hom}(S,Y) & \longrightarrow & \operatorname{Hom}(S,M) & \longrightarrow & \operatorname{Ext}(S,N) \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & \operatorname{Hom}(M_\lambda,N) & \longrightarrow & \operatorname{Hom}(M_\lambda,Y) & \longrightarrow & \operatorname{Hom}(M_\lambda,M) & \longrightarrow & \operatorname{Ext}(M_\lambda,N) \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & \operatorname{Hom}(M_{\lambda-1},N) & \longrightarrow & \operatorname{Hom}(M_{\lambda-1},Y) & \longrightarrow & \operatorname{Hom}(M_{\lambda-1},M) & \longrightarrow & \operatorname{Ext}(M_{\lambda-1},N) \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
 & & \operatorname{Ext}(S,N) & & \operatorname{Ext}(S,Y) & & \operatorname{Ext}(S,M) & &
\end{array}$$

First note that the torsion class $\mathcal{T}$ is closed under extension, and since $N,M\in\mathcal{T}$, we have that $Y\in\mathcal{T}$ because of the short exact sequence $(*)$. Since $S\in\mathcal{S}$ satisfies $\operatorname{Ext}(S,\mathcal{T})=0$, $\operatorname{Ext}(S,Y)=0=\operatorname{Ext}(S,N)$, since both $N,Y$ are $\mathcal{T}$-projective. So we can start a diagram chase:

From the vanishing of the lower Ext-groups we have the existence of a lifting $f'_\lambda: M_\lambda\to Y$ with $f'_\lambda(m)=f_{\lambda-1}(m)$ for all $m\in M_{\lambda-1}$. Define the deviation $\alpha := g\circ f'_\lambda-\iota_{M_\lambda}$ of this lifting. The restriction of $\alpha: M_\lambda\to M$ to $M_{\lambda-1}$ is zero. So there exists $\alpha'\in\operatorname{Hom}(S,M)$, which maps to $\alpha$ by $\operatorname{Hom}(S,M)\to\operatorname{Hom}(M_\lambda,M)$.

Since $\operatorname{Ext}(S,N)=0$, $\alpha'$ is the image of $\alpha''\in\operatorname{Hom}(S,Y)$. Denote the image of $\alpha''$ in $\operatorname{Hom}(M_\lambda,Y)$ by $f''_\lambda$. Since the diagram above commutes we have $\alpha=g\circ f''_\lambda$.

Now $f_\lambda := f'_\lambda - f''_\lambda$ has the properties: $f_\lambda(m)=f'_\lambda(m)=f_{\lambda-1}(m)$ for all $m\in M_{\lambda-1}$, an we get by applying $g$:

$$\begin{aligned}
g\circ f_\lambda &= g\circ f'_\lambda - g\circ f''_\lambda\\
&= g\circ f'_\lambda-\alpha\\
&= g\circ f'_\lambda-(g\circ f'_\lambda-\iota_{M_\lambda})=\iota_{M_\lambda},
\end{aligned}$$

which is what we wanted to show for part $(a)$, since then the short exact sequence $(*)$ splits, i.e. $\operatorname{Ext}(M,N)=0$, for all $N\in\mathcal{T}$ and torsion module $M\in\mathcal{E}(\mathcal{S})$.

Let us now look at part $(b)$. For this we need the following existence of a so-called universal short exact sequence. For modules $M,T\in\operatorname{Mod}-A$, there exists a *universal short exact sequence* in $\operatorname{Ext}(T,M)$, $0\to M\to M'\to T^{(I)}\to 0$, such that the induced map $\operatorname{Hom}(T,T^{(I)})\to\operatorname{Ext}(T,M)$ is surjective.[4]

[4]For the proof of the existence we refer to [L1], which uses a generator set for $\operatorname{Ext}(T,M)$ and looks at pushout diagrams.

Thus given $M \in \mathcal{E}(\mathcal{S})$, define $T = T_1 \oplus T_2$ and consider the universal short exact sequence:

$$(**) \qquad 0 \to M \to M' \to T^{(I)} \to 0.$$

Apply the functor $\mathrm{Hom}(T, \_\_)$ to it:

$$\mathrm{Hom}(T, M) \to \mathrm{Hom}(T, M') \to \mathrm{Hom}(T, T^{(I)}) \to \mathrm{Ext}(T, M) \to \mathrm{Ext}(T, M') \to \dots$$

Because of the surjectivity property from the universal short exact sequence, we have $\mathrm{Ext}(T, M') = \mathrm{Ext}(T_1 \oplus T_2, M') = 0$. Now apply the contravariant functor $\mathrm{Hom}(\_\_, M')$ to the given short exact sequence $0 \to 0 \to T_1 \to T_2 \to 0$, with $T_1, T_2$ $\mathcal{T}$-projective to get:

$$\mathrm{Hom}(T_2, M') \to \mathrm{Hom}(T_1, M') \to \mathrm{Hom}(A, M') \to \mathrm{Ext}(T_2, M') \to \mathrm{Ext}(T_1, M')$$

Since the last two terms vanish, $M'$ is a torsion module, which lies in $\mathcal{E}(\mathcal{S} \cup \mathrm{Add}(T_1 \oplus T_2))$. Hence we can use part $(a)$ to get $\mathrm{Ext}(M', \mathcal{T}) = 0$, so $M'$ is $\mathcal{T}$-projective and we are left to show this property for $M$:

For $R \in \mathcal{T}$ apply $\mathrm{Hom}(\_\_, R)$ to the universal short exact sequence $(**)$ to get the long exact sequence:

$$\mathrm{Hom}(M', R) \to \mathrm{Hom}(M, R) \to \mathrm{Ext}(T^{(I)}, R) \to \mathrm{Ext}(M', R) \to \mathrm{Ext}(M, R) \to 0$$

Since $\mathrm{Ext}(M', R) = 0$ by above, we have $\mathrm{Ext}(M, R) = 0$, i.e. $M \in \mathcal{E}(\mathcal{S})$ us $\mathcal{T}$-projective.

□

## 3 Divisible modules

In the first part of this sequence of talks on infinite dimensional modules, it was mentioned that there are no non-zero torsion-free divisible modules if $A$ is a wild hereditary algebra. For the class of divisible modules, we constructed already a so-called $\mathcal{D}$-projective module $\mathcal{A}_{\mathcal{D}}, which was torsion.$ For this we took as torsion class $\mathcal{T}$ the class of all divisible modules, where $\mathcal{D}$ was the torsion radical of the torsion pair $(\mathcal{T}, \mathcal{F})$.[5] For $\mathcal{S}$ we took the class of all submodules of direct sums of regular modules. Then by definition, for every divisible module $D \in \mathcal{T}$, we have $\mathrm{Ext}(\mathcal{S}, D) = 0$. The module $\mathcal{A}_{\mathcal{D}}$ had the following properties, which were already shown:

(i) $A_{\mathcal{D}} = \bigcup_{n \in \mathbb{N}_0} A_n$.

(ii) $A_{\mathcal{D}} \in \mathcal{E}(\mathcal{S})$.

(iii) $A_{\mathcal{D}}$ is a divisible module.

And we had:

**Theorem 3.1.** *For a wild hereditary algebra $A$ there exists a short exact sequence $0 \to A \to A_{\mathcal{D}} \to A_{\mathcal{D}}/A \to 0$ with $\mathcal{D}$-projective modules $A_{\mathcal{D}}$, $A_{\mathcal{D}}/A$.*

We have $\mathrm{Ext}(M, D) = 0$ for every divisible module $D$, if $M \in \mathcal{E}(\mathcal{S})$. The converse is also true for divisible modules, i.e. if $M$ satisfies $\mathrm{Ext}(M, D) = 0$ for all divisible $D$, then $M \in \mathcal{E}(\mathcal{S})$ and we also have the following:

**Proposition 3.2.** *(i) $A_{\mathcal{D}}$ generates all divisible modules $D$.*

*(ii) For every divisible module $M$, there exists a short exact sequence*

$$0 \to D \to A_{\mathcal{D}}^{(I)} \to M \to 0,$$

*where $D$ is $\mathcal{D}$-projective. So every $\mathcal{D}$-projective module is contained in* $\mathrm{Add}(A_{\mathcal{D}})$.

[5] So $\mathcal{D}$-projectives are the $\mathcal{T}$-projective modules.

*Proof.* For $(i)$: Let $M$ be a divisible module. Apply $\operatorname{Hom}(\_\_, M)$ to the short exact sequence $0 \to A \to A_{\mathcal{D}} \to A_{\mathcal{D}}/A \to 0$ to get:

$$\operatorname{Hom}(A_{\mathcal{D}}/A, M) \to \operatorname{Hom}(A_{\mathcal{D}}, M) \to \operatorname{Hom}(A, M) \to \operatorname{Ext}(A_{\mathcal{D}}/A, M) \to \dots$$

Since $A_{\mathcal{D}}/A$ is $\mathcal{D}$-projective, $\operatorname{Ext}(A_{\mathcal{D}}/A, M) = 0$, so every homomorphism $f' : A \to M$ can be extended to $f : A_{\mathcal{D}} \to M$, i.e. any divisible module $M$ is generated by $A_{\mathcal{D}}$ which completes part $(i)$.

For $(ii)$: let $(f_i)_{i \in I}$ be a $k$-basis of $\operatorname{Hom}(A_{\mathcal{D}}, M)$. Then by $(i)$ we get a short exact sequence

$$0 \to D \to A_{\mathcal{D}}^{(I)} \to M \to 0.$$

Apply $\operatorname{Hom}(A_{\mathcal{D}}, \_\_)$ to get:

$$\operatorname{Hom}(A_{\mathcal{D}}, D) \to \operatorname{Hom}(A_{\mathcal{D}}, A_{\mathcal{D}}^{(I)}) \to \operatorname{Hom}(A_{\mathcal{D}}, M) \to \operatorname{Ext}(A_{\mathcal{D}}, D) \to \operatorname{Ext}(A_{\mathcal{D}}, A_{\mathcal{D}}^{(I)}) \to \dots$$

Since $(f_i)$ is a $k$-basis of $\operatorname{Hom}(A_{\mathcal{D}}, M)$, the second map is surjective, so $\operatorname{Ext}(A_{\mathcal{D}}, D) = 0$. We know that for every regular module $R$, there exists a short exact sequence $0 \to R \to A_{\mathcal{D}} \to A_{\mathcal{D}}/R \to 0$. To this short exact sequence we finally apply $\operatorname{Hom}(\_\_, D)$ to get:

$$\dots \to \operatorname{Ext}(A_{\mathcal{D}}/R, D) \to \operatorname{Ext}(A_{\mathcal{D}}, D) \to \operatorname{Ext}(R, D) \to 0.$$

And since $\operatorname{Ext}(A_{\mathcal{D}}, D) = 0$ we get $\operatorname{Ext}(R, D) = 0$, which means $D$ is divisible and contained in $\operatorname{Add} A_{\mathcal{D}}$. □

# References


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