NAKAYAMA ALGEBRAS: KUPISCH SERIES AND MORITA TYPE

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Throughout, Λ is assumed to be a finite dimensional k-algebra, defined over an algebraically closed field k. We let J be the (Jacobson) radical of Λ. A Λ-module M of length $\ell(M)$ is called *uniserial* if the following equivalent conditions hold:

- M possesses exactly one composition series.
- $(J^iM)_{i\geq 0}$ is a composition series of M.
- For every $i \in \{0, \dots, \ell(M)\}$, J^iM is the unique submodule of length $\ell(M) - i$.

The algebra Λ is referred to as *pro-uniserial* if all its projective indecomposable modules are uniserial.

Let $\mathcal{S}(\Lambda)$ denote a complete set of representatives of the simple Λ-modules.

Proposition 1 (Thm. 9 of [2]). *The following statements are equivalent:*

(1) Λ *is pro-uniserial*
(2) $\sum_{T\in\mathcal{S}(\Lambda)} \dim_k \operatorname{Ext}^1_\Lambda(S,T) \leq 1$ *for every* $S \in \mathcal{S}(\Lambda)$.

Proof. (1) $\Rightarrow$(2). Let S be an element of $\mathcal{S}(\Lambda)$ with projective cover $P(S)$. There results an exact sequence

$$(*) \qquad (0) \longrightarrow JP(S) \longrightarrow P(S) \longrightarrow S \longrightarrow (0).$$

If $T \in \mathcal{S}(\Lambda)$ is another simple Λ-module, then general theory implies that

$$(**) \qquad \operatorname{Ext}^1_\Lambda(S,T) \cong \operatorname{Hom}_\Lambda(JP(S)/J^2P(S),T).$$

Since $P(S)$ is uniserial, the module $JP(S)/J^2\,P(S)$ is either (0) or simple. Schur's Lemma then yields $\dim_k \operatorname{Ext}^1_\Lambda(S,T) = 1$ for at most one $T \in \mathcal{S}(\Lambda)$.

(2) $\Rightarrow$ (1). Let S be an element of $\mathcal{S}(\Lambda)$ and consider the exact sequence $(*)$. The module $JP(S)/J^2P(S)$ is semi-simple, and condition (2) in conjunction with $(**)$ shows that $JP(S)/J^2P(S)$ is either zero or simple.

Given $n > 1$, suppose that $J^{n-1}P(S)/J^nP(S)$ is simple. If Q is a projective cover of $J^{n-1}P(S)$, then it is also a projective cover of $J^{n-1}P(S)/J^nP(S)$, and the above observation ensures that JQ/J^2Q is zero or simple. The surjective map $\pi : Q \longrightarrow J^{n-1}P(S)$ induces a surjection $\hat{\pi} : JQ/J^2Q \longrightarrow J^nP(S)/J^{n+1}P(S)$, so that the latter module is also either zero or simple. It now follows inductively that the Loewy series of $(J^iP(S))_{0\leq i\leq \ell(P(S))}$ is a composition series. Consequently, $P(S)$ is uniserial. □

Corollary 2. *The algebra* Λ *is pro-uniserial if and only if* Λ/J^2 *is pro-uniserial.*

Proof. Setting $\Lambda' := \Lambda/J^2$, we note that the pullback functor

$$\pi^* : \operatorname{mod}\Lambda' \longrightarrow \operatorname{mod}\Lambda$$

induces a bijection between the simple modules. Moreover, $P(S)/J^2P(S)$ is the projective cover of the simple Λ-module S, considered as a Λ'-module. It readily follows from $(**)$, that

$$\operatorname{Ext}^1_\Lambda(\pi^*(S),\pi^*(T)) \cong \operatorname{Ext}^1_{\Lambda'}(S,T) \qquad \forall\ S,T \in \mathcal{S}(\Lambda').$$

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Our assertion is now a direct consequence of Proposition 1. □

Definition. The algebra Λ is a *Nakayama algebra* if every projective indecomposable and every injective indecomposable Λ-module is uniserial.

Remarks. (1) A self-injective algebra is a Nakayama algebra if and only if it is pro-uniserial.

(2) The algebra $\Lambda = k[1 \to 2 \leftarrow 3]$ is pro-uniserial, but the injective indecomposable Λ-module I_2 belonging to the vertex 2 has a top of length 2, so that Λ is not a Nakayama algebra.

(3) Using duality, we see that Λ is a Nakayama algebra if and only if Λ and Λ^{op} are pro-uniserial. Consequently, Corollary 2 also holds for Nakayama algebras.

(4) An algebra Λ is a Nakayama algebra if and only if Proposition 1 and its dual

$$\sum_{T \in \mathcal{S}(\Lambda)} \dim_k \mathrm{Ext}^1_\Lambda(T, S) \leq 1 \qquad \forall\ S \in \mathcal{S}(\Lambda)$$

hold.

Proposition 3. *Let Λ be a Nakayama algebra. Then every indecomposable Λ-module is uniserial, and Λ has finite representation type.*

Proof. We prove the first assertion by induction on the Loewy length $\ell\ell(\Lambda)$ of Λ, the case $\ell\ell(\Lambda) = 1$ being trivial. Assuming $\ell := \ell\ell(\Lambda) \geq 2$, we consider an indecomposable Λ-module M. If $J^{\ell-1}M = (0)$, then M is an indecomposable module for the Nakayama algebra $\Lambda/J^{\ell-1}$, and the inductive hypothesis yields the assertion. Alternatively, there exists a simple left ideal $S \subset J^{\ell-1}$ with $S.M \neq (0)$. We can therefore find $m \in M \setminus \{0\}$ such that

$$\psi_m : S \longrightarrow M \quad ; \quad s \mapsto s.m$$

is injective. Hence there is a map $\hat{\psi}_m : M \longrightarrow E(S)$ to the injective envelope $E(S)$ of S, whose composite with ψ_m is the canonical inclusion $S \hookrightarrow E(S)$. As $E(S)$ is uniserial, we can find $i \geq 0$ with $\hat{\psi}_m(M) = J^i E(S)$. Consequently, $J^{\ell-i}M \subset \ker \hat{\psi}_m$, while $J^{\ell-1}M \not\subset \ker \hat{\psi}_m$. As a result $i = 0$, so that $\hat{\psi}_m$ is surjective and $J^{\ell-1}E(S) \neq (0)$. Since the uniserial projective cover $\pi : P \longrightarrow E(S)$ of $E(S)$ satisfies $\ell(P) = \ell\ell(P) \leq \ell = \ell\ell(E(S)) = \ell(E(S))$, we have $P \cong E(S)$. As M is indecomposable, it now follows that $\hat{\psi}_m$ is an isomorphism. Thus, M is uniserial.

As an upshot of the above, every indecomposable Λ-module M has a simple top and is thus of the form

$$M \cong P(S)/J^i P(S) \qquad 0 \leq i \leq \ell\ell(\Lambda),$$

for some simple module S. Consequently, Λ has finite representation type. □

Example. The path algebra $k[\tilde{D}_4]$ of the four subspace quiver $\tilde{D}_4$ is pro-uniserial, but not of finite representation type. The same holds of course for any subspace quiver involving at least four subspaces.

We let Q_Λ be the Gabriel quiver of Λ and denote by A_n and $\tilde{A}_n$ the quivers with vertices $\{1, \ldots, n\}$ and $\mathbb{Z}/(n+1)$, respectively and arrows $i \to i+1$.

An analogue of following result, which is an easy consequence of Proposition 1 and its dual, was established by Kupisch prior to the introduction of quivers.

Theorem 4 (cf. Satz 5 of [3]). *Let Λ be a connected Nakayama algebra. Then $Q_\Lambda = A_n,\ \tilde{A}_n$.*

Proof. Let p be a directed path of maximal length in Q_Λ subject to every vertex of Q_Λ occurring at most once. We denote by $V(p)$ the set of vertices of p and claim that $V(p) = (Q_\Lambda)_0$.

Writing $V(p) = \{p_1, \dots, p_n\}$ with arrows $p_i \to p_{i+1}$, we suppose there is a vertex $x \in (Q_\Lambda)_0 \setminus V(p)$ which is connected to some vertex $p_i \in V(p)$. If $x \to p_i$, then the dual of Proposition 1 implies $i = 1$, and the maximality of p gives a contradiction. Alternatively, we have $p_i \to x$, and the above reasoning first shows $i = n$ and then yields a contradiction. Since Q_Λ is connected, our claim follows.

Let $\alpha \in (Q_\Lambda)_1$ be an arrow. If the starting point of α is p_i, then Proposition 1 shows that α belongs to the path whenever $i < n$. For $i = n$, the dual of Proposition 1 implies that α is the unique arrow $p_n \to p_1$. As an upshot of our discussion, we conclude that $Q_\Lambda = A_n$ in case there is no arrow originating in p_n, and $Q_\Lambda = \tilde{A}_{n-1}$ otherwise. □

In view of our Theorem there exists an ordering $S_1, \dots, S_n$ of the simple Λ-modules such that their projective covers $P_i := P(S_i)$ satisfy

$$JP_i/J^2P_i \cong S_{i+1} \qquad 1 \le i \le n-1,$$

with $JP_n/J^2P_n \cong S_1$ if $JP_n \neq (0)$. This ordering is often called the *Kupisch series* of Λ. Note that the foregoing isomorphism also implies

$$\ell(P_{i+1}) \geq \ell(P_i) - 1.$$

It follows from the above, that the Morita equivalence class of Λ is determined by the n-tuple $(\ell(P_1), \dots, \ell(P_n))$.

Example. Suppose that Λ is a connected hereditary Nakayama algebra. Then Λ is Morita equivalent to $k[A_n]$, so that $\ell(P_i) = n + 1 - i$. Note that $k[A_n]$ is isomorphic to the algebra of lower triangular $(n \times n)$-matrices.

We let $k[\tilde{A}_n]^\dagger$ be the space generated by all paths of length ≥ 1.

Corollary 5. *Let Λ be a connected Nakayama algebra. Then Λ is self-injective if and only if Λ is Morita equivalent to $k[\tilde{A}_n]/(k[\tilde{A}_n]^\dagger)^m$ for $n = |\mathcal{S}(\Lambda)| - 1$ and $m = \ell\ell(\Lambda)$.*

Proof. If Λ is Morita equivalent to $k[\tilde{A}_n]/(k[\tilde{A}_n]^\dagger)^m$, then we have $\mathrm{Soc}(P_i) \cong S_{i+m-1}$, where the indices are to be taken $\mathrm{mod}(n+1)$. In view of [1, Theorem], the algebra Λ is self-injective.

For the reverse direction, we pick r such that $\ell(P_r)$ is maximal. If $n \neq 0$, then no simple Λ-module is projective and there is a surjection

$$P_{r+1} \longrightarrow JP_r.$$

Since $\ell(P_r) \geq \ell(P_{r+1}) \geq \ell(P_r) - 1$, the assumption $\ell(P_r) \neq \ell(P_{r+1})$ implies that the above map is in fact an isomorphism. Thus, JP_r is injective and hence a direct summand of P_r. Consequently, $JP_r = (0)$, so that S_r is projective, a contradiction. We obtain $\ell(P_{r+1}) = \ell(P_r)$, and repeat the argument to see that $\ell(P_i) = \ell\ell(\Lambda)$ for $i \in \{1, \dots, n+1\}$.

Since Λ has Loewy length $m = \ell(P_r)$, it follows that Λ is Morita equivalent to $k[\tilde{A}_n]/(k[\tilde{A}_n]^\dagger)^m$. □

References

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