Brauer-Thrall I.

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Let Λ be an artin algebra. The modules to be considered are left Λ-modules, and not necessarily of finite length. A module M is said to be *of finite type,* provided M is the direct sum of copies of a finite number of indecomposable modules of finite length.

Theorem. *A module M which is not of finite type contains indecomposable submodules of arbitrarily large finite length.*

Remarks: (1) A typical application will be as follows: Assume there are infinitely many pairwise non-isomorphic indecomposable modules M_i of a fixed length; take the direct sum $M = \bigoplus M_i$. The (Krull-Remak-Schmidt-)Azumaya theorem shows that M is not of finite type, thus there are indecomposable submodules of arbitrarily large finite length. This yields a proof of the first Brauer-Thrall conjecture: If there are infinitely many indecomposable modules of a fixed length, then there are indecomposable modules of arbitrarily large finite length. (Recall that the conjecture was solved by Roiter [Ro] in 1968.) But even in this special situation, the assertion is more specific: We obtain the large indecomposable modules as **submodules** of the module M. A weaker version has been shown in [R1], Appendix A: there are large indecomposable modules which are **cogenerated** by M (but this means only that the direct sum of countably many copies of M contains indecomposable submodules of arbitrarily large length).

Part of the proof will consist in dealing precisely with the special case of M being the direct sum of infinitely many pairwise non-isomorphic indecomposable modules of equal length.

(2) The result may also be compared with the following well-known assertion: *An artin algebra Λ is of finite representation type if and only if any Λ-module is of finite type* (see [T], Corollary 9.5, [RT], and also [A], Corollary 4.8), since the essential implication asserts: the existence of a module which is not of finite type implies that Λ is of infinite representation type.

Proof of Theorem. We assume that the indecomposable submodules of M of finite length are of bounded length, thus there are only finitely many possible Gabriel-Roiter measures. Assume that the indecomposable submodules of M of finite length have Gabriel-Roiter measure $\gamma_1 < \gamma_2 < \cdots < \gamma_s$. We show by induction on s that M is of finite type. The case $s = 1$ is well-known and easy to see: if any indecomposable submodule of M of finite length is simple, then M has to be semi-simple, thus of finite type.

Assume now that $s \geq 2$. Consider a submodule M' of M which is a direct sum of modules of Gabriel-Roiter measure γ_s, and maximal with this property. If M' is of finite type, then [R2], theorem 4.2 asserts that M' is Σ-pure injective in $\mathcal{D}(\gamma_s)$, and of course M' is a pure submodule of M, thus M' is a direct summand of M, say $M = M' \oplus M''$ for some module M''. However, the indecomposable submodules of M'' of finite length have

Gabriel-Roiter measure $\gamma_1, \dots, \gamma_{s-1}$ (γ_s cannot occur by the maximality of M'), thus by induction M'' is of finite type. Then also $M = M' \oplus M''$ is of finite type.

Thus we can assume that there is a submodule $M^1 = \bigoplus_{i \geq 1} M_i$ of M which is an infinite direct sum of pairwise non-isomorphic indecomposable modules M_i with Gabriel-Roiter measure γ_s, indexed over $\mathbb{N}$. For any $r \in \mathbb{N}$, let $M^r = \bigoplus_{i \geq r} M_i$.

The modules M_i have all the same length, say length t. Let $\mathcal{U}_r$ be the set of isomorphism classes of indecomposable submodules of M^r of length at most $t-1$.

(a) *The set $\mathcal{U}_r$ is finite for almost all r.* Otherwise, we chose inductively pairwise non-isomorphic submodules U_j of M^1 of length at most $t-1$ such that $U = \sum_{j \in \mathbb{N}} U_j$ is the direct sum of the modules U_j. (Namely, assume we have found $U_1, \dots U_s$ with $U' = \bigoplus_{j=1}^{s} U_j \subseteq M^1$, then $U' \subseteq \bigoplus_{i=1}^{r-1} M_i$ for some r. If $\mathcal{U}_r$ is infinite, we find inside M^r an indecomposable submodule U_{s+1} of length at most $t-1$ which is not isomorphic to any of the $U_1, \dots, U_s$. Since $\bigoplus_{i=1}^{r-1} M_i$ and M^r intersect in zero, we see that $\sum_{j=1}^{s+1} U_j$ is a direct sum.) As a submodule of M, all the indecomposable submodules of U of finite length have Gabriel-Roiter measure γ_i with $1 \leq i \leq s$ and actually γ_s does not occur as a Gabriel-Roiter measure (since such a submodule would be a direct summand of U, impossible). By induction, U has to be of finite type — but by construction, $U = \bigoplus_{j \in \mathbb{N}} U_j$ is not of finite type.

Let $\mathcal{U} = \bigcap_r \mathcal{U}_r$. As we have seen, this is a finite set of isomorphism classes, and of course non-empty. There is some r' with $\mathcal{U} = \mathcal{U}_{r'}$ and without loss of generality, we can assume that $r' = 1$ (replacing M^1 by $M^{r'}$). Thus we deal with the following situation: $M^1 = \bigoplus_{i \geq 1} M_i$ is an infinite direct sum of pairwise non-isomorphic indecomposable modules M_i with Gabriel-Roiter measure γ_s, and any indecomposable submodule of M^1 of length at most $t-1$ is also a submodule of $M^r = \bigoplus_{i \geq r} M_i$ for any r.

(b) *Any indecomposable module of length at most $t-1$ and cogenerated by M^1 is isomorphic to a submodule of M^1.* Assume that N is of length at most $t-1$ and cogenerated by M^1, thus there is a finite number of maps $\pi \colon N \to M_i$ such that the kernels of these maps intersect in zero. These maps π cannot be surjective, since N is of length at most $t-1$, whereas M_i is of length t. If we decompose the images $\pi(N)$ of these maps, we obtain indecomposable submodules N_j of M_i of length at most $t-1$, and such submodules N_j occur frequently inside M^1, namely inside M^r, for any r. This shows that N is a submodule of M^1.

(c) In particular, we see that there are only finitely many isomorphism classes of modules which are cogenerated by M^1 and of length at most $t-1$. Let S be the direct sum of all the simple modules. As in [R1], we consider the class $\mathcal{N}$ of all indecomposable modules cogenerated by $M^1 \oplus S$ and not isomorphic to any M_i. Clearly, this class is again closed under cogeneration and still finite. For any module M_i, let $f^{\mathcal{N}} M_i$ be the maximal factor module of M_i which belongs to $\operatorname{add} \mathcal{N}$. Since M_i does not belong to $\mathcal{N}$, we see that $f^{\mathcal{N}} M_i$ is a module of length at most $t-1$ and cogenerated by $M^1 \oplus S$, thus there are only finitely many possibilities. It follows that there is a module Q in $\operatorname{add} N$ such that $f^{\mathcal{N}} M_i = Q$ for infinitely many i. Without loss of generality, we even may assume that $f^{\mathcal{N}} M_i = Q$ for all i (by deleting the remaining factors). For any module M_i, fix a projection $q_i \colon M_i \to Q$ and let K be the kernel of the map $(f_i)_i \colon M^1 \to Q$. Roiter's coamalgamation lemma (see [R1]) asserts that K has no direct summand isomorphic to M_i, thus no submodule of Gabriel-Roiter measure γ_s (since M^1 belongs to $\mathcal{D}(\gamma_s)$ and M_i is relative injective in $\mathcal{D}(\gamma_s)$). By induction we see that K has to be of finite type. But this contradicts the Ext-Lemma [R1]: for any extension of the form

$$0 \to K \to X \to Q \to 0$$

with Q of finite length and K of infinite length and of finite type, the modules K and X will have common indecomposable direct summands. For $X = M^1$, the indecomposable direct summands have Gabriel-Roiter measure γ_s, but K has not even a submodule of measure γ_s.

References

[A] Auslander: Representation theory of artin algebras II. Communications Algebra. 1 (1974), 269-310.

[R1] Ringel: The Gabriel-Roiter measure. Bull. Sci. math. 129 (2005), 726-748.

[R2] Ringel: Foundation of the Representation Theory of Artin Algebras, Using the Gabriel-Roiter Measure. To appear in the Proceedings of the Queretaro Workshop 2004. Contemporary Mathematics. Amer.Math.Soc.

[RT] Ringel, Tachikawa: QF-3 rings. J. Reine Angew. Math. 272 (1975), 49-72.

[Ro] Roiter: Unboundedness of the dimension of the indecomposable representations of an algebra which has infinitely many indecomposable representations. Izv. Akad. Nauk SSSR. Ser. Mat. 32 (1968), 1275-1282.

[T] Tachikawa: Quasi-Frobenius Rings and Generalizations. Springer Lecture Notes in Mathematics 351 (1973).