Prüfer modules of finite type.

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Let Λ be an artin algebra. Recall that a module M is said to be a *Prüfer module* provided there exists a surjective locally nilpotent endomorphism ϕ of M with non-zero kernel of finite length.

A module M is said to be *of finite type* provided it is the direct sum of copies of finitely many indecomposable modules of finite length. A module G is called *generic* provided it is indecomposable, of infinite length, and endo-finite (the latter means: it is of finite length when considered as a module over its endomorphism ring).

Claim. *If there are no generic modules, then all Prüfer modules are of finite type.*

More precisely:

Proposition. *Let M be a Prüfer module. The following conditions are equivalent:*
(i) *M is not of finite type.*
(ii) *There is an infinite index I set such that the product module M^I has a generic direct summand.*
(iii) *For every infinite index I set, the product module M^I has a generic direct summand.*

Proof: The implications (iii) $\implies$ (ii) is trivial. Also (ii) $\implies$ (i) is obvious: If M is of finite type, then also all product modules M^I are of finite type. We only have to show (i) $\implies$ (iii). (It it sufficient to consider $I = \mathbb{N}$ in (iii), since any infinite index set I can be written as the disjoint union of $\mathbb{N}$ and some other index set I', and then $M^I = M^{\mathbb{N}} \oplus M^{I'}$ — however, there is no problem to work in general.)

Now assume that I is an infinite index set and that M^I has no indecomposable direct summand which is endo-finite and of infinite length. Since M is a Prüfer module, there is a surjective, locally nilpotent endomorphism ϕ with kernel $W = W[1]$ non-zero and of finite length. Let $W[n]$ be the kernel of ϕ^n. Thus

$$M[1] \subset M[2] \subset \cdots \subset \bigcup_n M[n] = M$$

is a filtration of M with finite length modules $M[n]$. We obtain a corresponding chain of inclusions

$$M[1]^I \subset M[2]^I \subset \cdots \subset \bigcup_n M[n]^I = M'.$$

It has been shown in [R1] (see also [K]) that M' is isomorphic to a direct sum of copies of M and itself a direct summand of M^I; there is an endo-finite submodule E of M^I such that

$$M^I = M' \oplus E.$$

Any endo-finite module E can be written as a direct sum of copies of finitely many indecomposable endo-finite modules, say $E_1, \dots, E_t$. By assumption, all these modules E_i are of finite length. A well-known lemma of Auslander asserts that any indecomposable direct summand of M^I of finite length is a direct summand of M itself, thus the modules $E_1, \dots, E_t$ occur as direct summands of M.

Since M is artinian as a module over its endomorphism ring, M is Σ-algebraic compact, thus it is a direct sum of indecomposable modules with local endomorphism ring. Write $M = A \oplus B$, where A is a direct sum of copies of the various E_i and B has no direct summand of the form E_i, for any i. We want to show that B is of finite length. This then shows that M is of finite type.

The modules A, B are also filtered, with $A_n = A \cap M[n],\ B_n = B \cap M[n]$ (it is obvious that $A = \bigcup_n A_n,\ B = \bigcup_n B_n$). For any n there is some n' with $M[n] \subseteq A_{n'} \oplus B_{n'}$. (Namely, let $x \in M[n]$, write $x = a + b$ with $a \in A,\ b \in B$. Then there is some n' with $a, b \in M_{n'}$, thus $a \in A_{n'},\ b \in B_{n'}$.)

We write $A' = \bigcup_i A_i^I$ and $B' = \bigcup_i B_i^I$. Then

$$M' = A' \oplus B'$$

(the inclusion $\supseteq$ is obvious, the other follows from $M[n]^I \subseteq (A_{n'} \oplus B_{n'})^I = A_{n'}^I \oplus B_{n'}^I \subseteq A' \oplus B'$.). We see that

$$(A^I/A') \oplus (B^I/B') = M^I/M' = E,$$

thus $A^I/A' = E_A$ and $B^I/B' = E_B$ with $E = E_A \oplus E_B$. In particular, E_A and E_B are direct sums of copies of $E_1, \dots, E_t$. Since the direct sum of the inclusion maps

$$A' \to A^I \quad \text{and} \quad B' \to B^I$$

is a split monomorphism, the maps themselves are split monomorphisms, thus

$$A^I \simeq A' \oplus E_A \quad \text{and} \quad B^I \simeq B' \oplus E_B.$$

Consider the last isomorphism. If E_i is a direct summand of E_B, then it is a direct summand of B (Auslander Lemma), impossible. This shows that $E_B = 0$. But then $B' = B^I$ implies that $B = B_n$ for some n, thus $B \subseteq M[n]$. This shows that B is of finite length.

May-be one should record: *Assume that M^I/M' is the direct sum of copies of indecomposable modules $E_1, \dots, E_t$ of finite length, then M is the direct sum of a finite length module B and of copies of the modules E_i.*

For dealing with Prüfer modules obtained using the ladder construction $M = U_\infty/U_0$, it seems to be of interest to relate the finite type properties of U_∞ and M. More generally, let us consider filtered modules in more generality.

Lemma. *Let $U_0 \subset U_1 \subset \cdots \subset \bigcup_i U_i = U_\infty$ be a filtration of U_∞ using finite length modules U_i and proper inclusions $U_i \subset U_{i+1}$. Consider the following conditions:*

(i) *$\bigoplus_{i\in\mathbb{N}} U_i$ is of finite type.*

(ii) *U_∞ is of finite type.*

(ii′) *U_∞/U_0 is of finite type.*

(iii) *Only finitely many inclusions $U_i \subset U_{i+1}$ are radical morphisms.*

(iv) *Only finitely many modules U_i are indecomposable.*

The following implications hold: (i) $\implies$ (ii) $\implies$ (iii) $\implies$ (iv), *and the conditions* (ii) *and* (ii′) *are equivalent.*

Proof: (i) $\implies$ (ii): Projectify the modules M_i, so that all the modules U_i are projective. Then U_∞ is flat, thus projective $\dots$.

(ii) $\implies$ (ii′): Let $U_\infty = \bigoplus_{i\in I} M_i$ with all M_i indecomposable of finite length, and with only finitely many isomorphism classes of modules involved. Now $U_0 \subseteq \bigoplus_{i\in I'} M_i = M'$ with I' a finite subset of I. Then

$$U_\infty/U_0 = M'/U_0 \oplus \bigoplus_{i\in I\setminus I'} M_i,$$

is a direct sum of indecomposable modules of finite length (one has to decompose M'/U_0) and only finitely many isomorphism classes are involved.

(ii′) $\implies$ (ii): Roiter's extension argument.

(ii) $\implies$ (iii): Let $U_\infty = M \oplus M'$ with M of finite length. Then $M \subseteq U_i$ for some i. But then the inclusion $M \subset U_\infty$ factors as follows $M \subseteq U_i \subset U_{i+1} \subset U_\infty$. This inclusion splits, thus there is a projection $U_\infty \to M$ such that the composition

$$M \to U_i \to U_{i+1} \to U_\infty \to M$$

is the identity. This shows that $U_i \to U_{i+1}$ is not in the radical.

(iii) $\implies$ (iv): If a proper inclusion $V \subset W$ does not belong to the radical, then W cannot be indecomposable: namely, there are indecomposable direct summands V' of V and W' of W which are isomorphic. Then $|W'| = |V'| \le |V| < |W|$, thus W is decomposable.

References

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