Deligne-Lusztig characters

These notes are meant to be a – very brief – introduction to Deligne-Lusztig characters. As an application, the generic character table of $GL_2(q)$ will be constructed. A proper treatment should contain Lusztig induction as well. However, I did not want to introduce Levi subgroups and parabolic subgroups, so this is not done. Also, only the general linear group is considered, although everything holds for any finite reductive group. For this, and for proofs, see [DM91].

Let $\mathbb{F} = \overline{\mathbb{F}_q}$ be an algebraic closure of the field with q elements, let $\mathbf{G} := GL_n(\mathbb{F})$, and let F be the endomorphism of $\mathbf{G}$ which takes every matrix component to its q^{th} power. Then $\mathbf{G}^F$, the group of fixed points of $\mathbf{G}$, is $GL_n(q)$. A (closed) subgroup of $\mathbf{G}$ is called rational iff it is F-stable. Clearly, $\mathbf{T}_1$, the group of diagonal matrices is rational, and so is the group $\mathbf{B}_1$ of upper triangular matrices. Any conjugate of $\mathbf{T}_1$ is called a maximal torus of $\mathbf{G}$, and any conjugate of $\mathbf{B}_1$ is called a Borel subgroup of $\mathbf{G}$. Let N be the normaliser of $\mathbf{T}_1$ in $\mathbf{G}$. Then $N = \mathbf{T}_1 \rtimes W$, where W is the subgroup of permutation matrices. In particular, $W \cong N/\mathbf{T}_1$ is independent of q. The group W is called the Weyl group of $\mathbf{G}$; it is isomorphic to the symmetric group on n letters. A maximal torus ${}^g\mathbf{T}_1$ is rational iff $g^{-1}F(g) \in N$. If $g^{-1}F(g)\mathbf{T}_1 = w\mathbf{T}_1$, the torus ${}^g\mathbf{T}_1$ is said to have type $w \in W$. A torus of type w is denoted by $\mathbf{T}_w$. Using the theorem of Lang-Steinberg, one can show that two rational tori $\mathbf{T}_v$ and $\mathbf{T}_w$ are $\mathbf{G}^F$-conjugate iff v and w are W-conjugate.[1] In particular, $\mathbf{T}_w$ is unique up to $\mathbf{G}^F$-conjugacy.

Let $\mathbf{B}_w$ be a Borel subgroup that contains $\mathbf{T}_w$. Then $\mathbf{B}_w$ has a unique normal complement to $\mathbf{T}_w$, which will be denoted by $\mathbf{U}_w$.[2] In general, $\mathbf{U}_w$, and hence $\mathbf{B}_w = \mathbf{T}_w \ltimes \mathbf{U}_w$, are not rational. In fact, one can choose a rational Borel subgroup $\mathbf{B}_w$ iff $w = 1$. The variety

$$Y_{\mathbf{T}_w \subseteq \mathbf{B}_w} := \{g\mathbf{U}_w \in \mathbf{G}/\mathbf{U}_w \mid g^{-1}F(g) \in \mathbf{U}_w F(\mathbf{U}_w)\} \subseteq \mathbf{G}/\mathbf{U}_w$$

is called a Deligne-Lusztig variety associated to $\mathbf{T}_w$. It is a pure smooth variety whose dimension is equal to the length of w. Obviously, $\mathbf{G}^F$ acts on $Y_{\mathbf{T}_w \subseteq \mathbf{B}_w}$ on the left by translations. Also, $\mathbf{T}_w^F$ acts on the right via $(g\mathbf{U}_w).t := gt\mathbf{U}_w$. Let ℓ be a prime not dividing q. Then $H_c^i(Y_{\mathbf{T}_w \subseteq \mathbf{B}_w}, \overline{\mathbb{Q}}_\ell)$ is a $\mathbf{G}^F$-$\mathbf{T}_w^F$-bimodule.

Definition 1
Let $\mathbf{T}_w \leq \mathbf{G}$ be a rational torus of type w, and let $\mathbf{B}_w$ be a Borel subgroup that contains $\mathbf{T}_w$. For an irreducible character ϑ of $\mathbf{T}_w^F$, the virtual character $R_{\mathbf{T}_w}^{\mathbf{G}}(\vartheta)$, which is afforded by the virtual $\mathbf{G}^F$-module $\bigoplus_{i\geq 0}(-1)^i H_c^i(Y_{\mathbf{T}_w \subseteq \mathbf{B}_w}, \overline{\mathbb{Q}}_\ell) \otimes_{\overline{\mathbb{Q}}_\ell} \vartheta$, is called a Deligne-Lusztig character. □

The Borel subgroup $\mathbf{B}_w$ does not appear in the notation, since Deligne-Lusztig characters do not depend on the chosen Borel subgroup. If $w = 1$ and $\mathbf{B}_1$ is the group of upper triangular matrices, it is easy to see that $Y_{\mathbf{T}_1 \subseteq \mathbf{B}_1} = \mathbf{G}^F\mathbf{U}_1/\mathbf{U}_1 \cong \mathbf{G}^F/\mathbf{U}_1^F$. Thus $R_{\mathbf{T}_1}^{\mathbf{G}} : \mathrm{Irr}(\mathbf{T}_1^F) \to K_0(\mathbf{G}^F)$ coincides with inflating to $\mathbf{B}_1^F$ and inducing to $\mathbf{G}^F$. This is also called Harish-Chandra induction.[3]

Recall that there is a scalar product $(\cdot,\cdot)$ on the space $\mathcal{C}(\mathbf{G}^F)$ of class functions on $\mathbf{G}$, for which the irreducible characters form an orthonormal basis. An irreducible character χ is said to be a constituent of a virtual character ψ, if $(\chi, \psi) \neq 0$.

[1] In general, the Frobenius endomorphism does not act trivially on the Weyl group, and one has to take F-conjugation in W.

[2] $\mathbf{U}_w$ is the unipotent radical of $\mathbf{B}_w$. It is the maximal closed connected normal unipotent subgroup of $\mathbf{B}_w$.

[3] Harish-Chandra induction can be defined for any rational Levi subgroup which is a Levi complement of a rational parabolic subgroup. There is also a generalisation of this for all rational Levi subgroups. It is called Lusztig induction.

Theorem 2
Every irreducible character of $\mathbf{G}^F$ is a constituent of some Deligne-Lusztig character. $\square$

The constituents of the Deligne-Lusztig characters $R_{\mathbf{T}_w}^{\mathbf{G}} 1$ are called unipotent characters. They have the remarkable property that they do not depend on q. They depend only on the rational type of G. In particular, knowledge of the Weyl group W and the action of the Frobenius endomorphism F on W completely determines the unipotent characters.

Theorem 3
Let $\mathbf{T}$ and $\mathbf{T}'$ be rational maximal tori of $\mathbf{G}$, and let ϑ and ϑ' be characters of $\mathbf{T}^F$ and $(\mathbf{T}')^F$, respectively. Then

$$(R_{\mathbf{T}}^{\mathbf{G}}\vartheta, R_{\mathbf{T}'}^{\mathbf{G}}\vartheta') = \frac{|\{x \in \mathbf{G}^F \mid {}^x\mathbf{T} = \mathbf{T}' \text{ and } {}^x\vartheta = \vartheta'\}|}{|\mathbf{T}^F|}$$

$\square$

Corollary 4

$$(R_{\mathbf{T}_w}^{\mathbf{G}} 1, R_{\mathbf{T}_v}^{\mathbf{G}} 1) = \begin{cases} |C_W(w)| & w, v \text{ conjugate in } W^4 \\ 0 & \text{otherwise} \end{cases}$$

Proof: The scalar product can only be non-zero, if $\mathbf{T}_w$ and $\mathbf{T}_v$ are $\mathbf{G}^F$-conjugate, which is equivalent to w and v being conjugate in W. Thus we may assume $w = v$. Then the scalar product is equal to $|\mathbf{T}_w^F|^{-1} \cdot |N_{\mathbf{G}}(\mathbf{T}_w)^F|$. The action of F on $N_{\mathbf{G}}(\mathbf{T}_w)$ can be identified with the action of ${}^wF$ on $N_{\mathbf{G}}(\mathbf{T})$, where w acts by conjugation. Now the claim follows easily. $\square$

Let $\mathbf{T}$ be a rational maximal torus of $\mathbf{G}$, let $u \in \mathbf{G}^F$ be unipotent, and define $Q_{\mathbf{T}}^{\mathbf{G}}(u) := (R_{\mathbf{T}}^{\mathbf{G}} 1)(u)$. The function $Q_{\mathbf{T}}^{\mathbf{G}}$ is called a Green function. Knowledge of the values of the Green function suffices to compute the values of all Deligne-Lusztig characters, as the next theorem shows.

Theorem 5 (The character formula)
Let $\mathbf{T} \leq \mathbf{G}$ be a rational maximal torus, and let ϑ be an irreducible character of $\mathbf{T}^F$. Let $g = su$ be the Jordan decomposition of $g \in \mathbf{G}^F$, where s is semisimple and u is unipotent. Then

$$(R_{\mathbf{T}}^{\mathbf{G}}\vartheta)(g) = \frac{1}{|C_{\mathbf{G}}(s)^F|} \sum_{x \in \mathbf{G}^F, s \in {}^x\mathbf{T}^F} {}^x\vartheta(s) Q_{{}^x\mathbf{T}}^{C_{\mathbf{G}}(s)}(u).^5$$

$\square$

Corollary 6
$(R_{\mathbf{T}}^{\mathbf{G}}\vartheta)(u) = Q_{\mathbf{T}}^{\mathbf{G}}(u)$ *is independent of ϑ if u is unipotent.* $\square$

[4] If F acts non-trivially on W, the elements w and v have to be F-conjugate, and $C_W(w)$ is replaced by W^{wF}.
[5] In general, $C_{\mathbf{G}}(s)$ has to be replaced by its connected component $C_{\mathbf{G}}^0(s)$.

Unipotent characters of the general linear group

Although I have stated everything only for the general linear group, the results hold for all finite reductive groups. This will change now: I will describe the unipotent characters of $GL_n(q)$, following Lusztig and Srinivasan. This method fails in general, because usually not all unipotent characters are linear combinations of Deligne-Lusztig characters.

For $w \in W$, let π_w be the class function on W defined by $\pi_w(v) = |C_W(w)|$, if w and v are conjugate in W, and $\pi_w(v) = 0$ otherwise. Also, let $R_w := R_{\mathbf{T}_w}^{\mathbf{G}} 1$. By corollary 4, the map $\mathcal{C}(W) \to \mathcal{C}(\mathbf{G}^F)$ which is defined by the linear extension of $\pi_w \mapsto R_w$, is an isometry.[6] Since $\chi = \frac{1}{|W|} \sum_{w \in W} \chi(w)\pi_w$ for every irreducible character χ of W, the set $\{R_\chi \mid \chi \in \mathrm{Irr}(W)\}$, where $R_\chi := \frac{1}{|W|} \sum_{w \in W} \chi(w) R_w$, is orthonormal.

Theorem 7
The unipotent characters of $\mathbf{G}^F$ *are* $\{R_\chi \mid \chi \in \mathrm{Irr}(W)\}$.

Sketch of **proof:** One can show that the R_χ are virtual characters, i.e. integral linear combinations of characters. Since $(R_\chi, R_\chi) = 1$, each R_χ is an irreducible character up to a sign. These characters are unipotent by definition. By the second orthogonality relation, $R_w = \sum_{\chi \in \mathrm{Irr}(W)} \chi(w) R_\chi$, hence every unipotent character appears in some R_χ. To prove the claim, it remains to show that the R_χ are actual characters. But, as mentioned above, $R_1 = \mathrm{Ind}_{\mathbf{B}_1^F}^{\mathbf{G}^F} 1$ is an actual character. Using a similar – but simpler – argument as in the proof that the R_χ are virtual characters, one can show that $R_1 = \sum_{\chi \in \mathrm{Irr}(W)} \chi(1) R_\chi$. Since all $\chi(1)$ are greater then 0, the claim follows. □

The unipotent characters are a model for all characters: there is a partition of the irreducible characters into rational series, $\mathrm{Irr}(\mathbf{G}^F) = \coprod_s \mathcal{E}(\mathbf{G}^F, s)$, where s runs through the semisimple conjugacy classes of $\mathbf{G}^F$.[7] The unipotent characters are the elements of $\mathcal{E}(\mathbf{G}^F, 1)$. For any s, there is a bijection from $\mathcal{E}(\mathbf{G}^F, s)$ to $\mathcal{E}(C_{\mathbf{G}}(s)^F, 1)$. For the general linear group over $\mathbb{F}_q$, each centraliser of a semisimple element is a direct product of general linear groups over some extension fields of $\mathbb{F}_q$.

The characters of $GL_2(q)$

Let $\mathbf{G} = GL_2(\mathbb{F})$. Then $W = \{1, s\}$ is the symmetric group on two letters. Thus $R_{\mathbf{T}_1}^{\mathbf{G}} 1 =: 1_{\mathbf{G}^F} + St_{\mathbf{G}}$ has two constituents, which are the unipotent characters of $\mathbf{G}^F$. The character $St_{\mathbf{G}}$ is called the Steinberg character. Let ϑ be an irreducible character of $\mathbb{F}_q^*$. Then $(\vartheta \circ \det).1_{\mathbf{G}^F}$ and $(\vartheta \circ \det).St_{\mathbf{G}}$ are irreducible characters of $\mathbf{G}^F$.

The finite rational tori are $\mathbf{T}_1^F \cong \mathbb{F}_q^* \times \mathbb{F}_q^*$ and $\mathbf{T}_s^F \cong \mathbf{T}_1^{sF} = \{\mathrm{diag}(\lambda, \lambda^q) \mid \lambda \in \mathbb{F}_{q^2}^*\} \cong \mathbb{F}_{q^2}^*$. Let $(\vartheta_1, \vartheta_2)$ be an irreducible character of $\mathbf{T}_1^F$. By theorem 3, the Deligne-Lusztig character $R_{\mathbf{T}_1}^{\mathbf{G}}(\vartheta_1, \vartheta_2)$ is irreducible iff $\vartheta_1 \neq \vartheta_2$. For $\vartheta \in \mathrm{Irr}(\mathbb{F}_q^*)$, one can show that $R_{\mathbf{T}_1}^{\mathbf{G}}(\vartheta, \vartheta) = (\vartheta \circ \det).R_{\mathbf{T}_1}^{\mathbf{G}} 1$. Now let ϑ be an irreducible character of $\mathbb{F}_{q^2}^*$. Then $R_{\mathbf{T}_s}^{\mathbf{G}} \vartheta$ is irreducible iff $\vartheta^q \neq \vartheta$. Otherwise, $\vartheta = \overline{\vartheta} \circ N_{\mathbb{F}_{q^2}/\mathbb{F}_q}$ for some character $\overline{\vartheta}$ of $\mathbb{F}_q^*$. As above, one can show $R_{\mathbf{T}_s}^{\mathbf{G}} \vartheta = (\overline{\vartheta} \circ \det).R_{\mathbf{T}_s}^{\mathbf{G}} 1$ in this case. Also, the $R_{\mathbf{T}_1}^{\mathbf{G}}(\vartheta_1, \vartheta_2)$ are actual characters, and so are the $-R_{\mathbf{T}_s}^{\mathbf{G}} \vartheta$.

In summary, the irreducible characters of $GL_2(q)$ are:

(i) $(\vartheta \circ \det).1_{\mathbf{G}^F}$ and $(\vartheta \circ \det).St_{\mathbf{G}}$, where $\vartheta \in \mathrm{Irr}(\mathbb{F}_q^*)$,

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[6] It is not onto, of course.

[7] In general, s runs through the semisimple classes of the dual group.

(ii) $R_{\mathbf{T}_1}^{\mathbf{G}}(\vartheta_1, \vartheta_2)$, where $(\vartheta_1, \vartheta_2)$ runs through the unordered pairs of distinct irreducible characters of $\mathbb{F}_q^*$, and

(iii) $-R_{\mathbf{T}_s}^{\mathbf{G}}\vartheta$, where ϑ runs the irreducible characters of $\mathbb{F}_{q^2}^*$ up to $\vartheta \sim \vartheta^q$ with $\vartheta^{q-1} \neq 1$.

Using the rational series mentioned above, one can easily show that the characters in this list are all distinct. Also, it is not hard to compute the conjugacy classes. One can show that the Green functions are $Q_{\mathbf{T}_1}^{\mathbf{G}} = 1_{\mathbf{G}^F} + St_{\mathbf{G}}$ and $Q_{\mathbf{T}_s}^{\mathbf{G}} = 1_{\mathbf{G}^F} - St_{\mathbf{G}}$, and the Steinberg character vanishes on non-semisimple elements. Using the character formula and the – known – values of the Steinberg character, one can compute the values of characters just constructed. The generic character table of $GL_2(q)$ is:

classes	$\begin{pmatrix} a & 0 \\ 0 & a \end{pmatrix}$, $a \in \mathbb{F}_q^*$	$\begin{pmatrix} a & 0 \\ 0 & b \end{pmatrix}$, $a, b \in \mathbb{F}_q^*, a \neq b$	$\begin{pmatrix} a & 0 \\ 0 & F(a) \end{pmatrix}$, $a \in \mathbb{F}_{q^2}^*, a \neq F(a)$	$\begin{pmatrix} a & 1 \\ 0 & a \end{pmatrix}$, $a \in \mathbb{F}_q^*$
# of classes	$q-1$	$\frac{(q-1)(q-2)}{2}$	$\frac{q(q-1)}{2}$	q-1
size of class	1	$q(q+1)$	$q(q-1)$	q^2-1
$(\vartheta \circ \det).1_{\mathbf{G}^F}$	$\vartheta(a^2)$	$\vartheta(ab)$	$\vartheta(aF(a))$	$\vartheta(a^2)$
$(\vartheta \circ \det).St_{\mathbf{G}}$	$q\vartheta(a^2)$	$\vartheta(ab)$	$-\vartheta(aF(a))$	0
$R_{\mathbf{T}_1}^{\mathbf{G}}(\vartheta_1, \vartheta_2)$ $\vartheta_1 \neq \vartheta_2 \in \mathrm{Irr}(\mathbb{F}_q^*)$	$(q+1)\vartheta_1(a)\vartheta_2(a)$	$\vartheta_1(a)\vartheta_2(b)$ $+\vartheta_1(b)\vartheta_2(a)$	0	$\vartheta_1(a)\vartheta_2(a)$
$-R_{\mathbf{T}_s}^{\mathbf{G}}\vartheta$ $\vartheta^q \neq \vartheta \in \mathrm{Irr}(\mathbb{F}_{q^2}^*)$	$(q-1)\vartheta(a)$	0	$-\vartheta(a) - \vartheta(F(a))$	$-\vartheta(a)$

References

[DM91] F. Digne and J. Michel. *Representations of Finite Groups of Lie Type*, volume 21 of *LMS Student Texts*. Cambridge University Press, 1991.