# BURNSIDE'S THEOREM: STATEMENT AND APPLICATIONS

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Let $k$ be a field, $G$ a finite group, and denote by $\operatorname{mod} G$ the category of finite dimensional $G$-modules. This category coincides $\operatorname{mod} kG$, the category of finite dimensional modules of the group algebra $kG$. Given $M \in \operatorname{mod} G$, we let $x_M : M \longrightarrow M$ denote the multiplication effected by the element $x \in kG$. The linear map

$$\chi_M : kG \longrightarrow k \;\; ; \;\; x \mapsto \operatorname{tr}(x_M)$$

is referred to as the *character* of the $G$-module $M$. The linear function $\chi_M$ is determined on the basis $G \subset kG$. We introduce a multiplication on $(kG)^*$: Given linear forms $\varphi$, $\psi \in (kG)^*$, we define their product $\varphi \cdot \psi$ to be the linear form satisfying

$$(\varphi \cdot \psi)(g) = \varphi(g)\psi(g) \qquad \forall\ g \in G.$$

In this fashion $(kG)^*$ obtains the structure of a commutative $k$-algebra. We let

$$\mathcal{A}_G := k[\{\chi_M \ ; \ M \in \operatorname{mod} G\}]$$

be the subalgebra of $(kG)^*$, generated by the characters of $G$.

*Problem.* For which characters $\chi_M : kG \longrightarrow k$ is

$$\mathcal{A}_G = k[\{\chi_S \ ; \ \chi_S \text{ is a summand of } \ \chi_M^\ell \text{ for some } \ell \geq 1\}]$$

the subalgebra of $(kG)^*$ generated by the summands of powers of $\chi_M$?

In his book [2] Burnside gave an affirmative answer in case $k = \mathbb{C}$ is the field of complex numbers. Subsequently, his proof was simplified and generalized in several directions [1, 6, 5, 4].

Since characters are given by modules, let us try to understand the above problem in terms of module theory. Given $G$-modules $M$, $N$ the tensor product $M \otimes_k N$ obtains the structure of a $G$-module via

$$g.(m \otimes n) := (g.m) \otimes (g.n) \qquad \forall\ g \in G,\ m \in M,\ n \in N.$$

We have the following properties:

(1) If $(0) \longrightarrow M' \longrightarrow M \longrightarrow M'' \longrightarrow (0)$ is an exact sequence of $G$-modules, then

$$\chi_M = \chi_{M'} + \chi_{M''}.$$

(2) If $M$ and $N$ are $G$-modules, then

$$\chi_{M \otimes_k N} = \chi_M \cdot \chi_N.$$

In fact, these two properties may be summarized by saying that $M \mapsto \chi_M$ induces a homomorphism from the *Grothendieck algebra* onto $\mathcal{A}_G$. Moreover, if $\{S_1, \ldots, S_n\}$ is a complete set of representatives of the simple $G$-modules, then (1) implies

$$\chi_M = \sum_{i=1}^{n} [M\!:\!S_i]\, \chi_{S_i},$$

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so that $\mathcal{A}_G$ is generated by the characters of the simple modules. Accordingly, our problem has an affirmative answer if we can produce a $G$-module $V$ such that each simple $G$-module $S_i$ is a composition factor of some tensor power $V^{\otimes \ell}$ of $V$.

If $V$ is a $G$-module, we let $\varrho_V : G \longrightarrow \mathrm{GL}(V)$ be the representation afforded by $V$. Since

$$\ker \varrho_V \subset \ker \varrho_{V^{\otimes \ell}} \quad \forall\ \ell \geq 1,$$

we obtain

$$\ker \varrho_V \subset \bigcap_{i=1}^{n} \ker \varrho_{S_i}$$

as a necessary condition. If $\mathrm{char}(k) = 0$, then Maschke's Theorem implies the semisimplicity of $kG$, so that the right-hand side is trivial. In that case $V$ has to be a *faithful* $G$-module, that is, $\ker \varrho_V = \{e\}$.

**Theorem** (Burnside). *Let $G$ be a finite group, $V$ a faithful, complex $G$-module. Then each simple $G$-module is a direct summand of some tensor power $V^{\otimes \ell}$.* □

Ideally, results of this type lead to concrete realizations of simple modules. In the context of complex Lie algebras the familiar $\mathfrak{sl}(2)$-theory provides an example: Every simple $\mathfrak{sl}(2)$-module is a composition factor of some tensor power of the 2-dimensional standard module $L(1)$. In fact, the simple modules are just the homogeneous parts of the symmetric algebra $S(L(1))$.

**Example.** Let $G$ be an abelian group. Then all simple $\mathbb{C}G$-modules are one dimensional, with each of them corresponding to a group homomorphism $\lambda : G \longrightarrow \mathbb{C}^\times$ (or, equivalently to an algebra homomorphism $\lambda : \mathbb{C}G \longrightarrow \mathbb{C}$). If one of these modules, $k_\lambda$ say, is faithful, then Burnside's Theorem in conjunction with $k_\mu \otimes_k k_\nu \cong k_{\mu\cdot\nu}$ implies that every homomorphism $\mu : G \longrightarrow \mathbb{C}^\times$ is of the form $\mu = \lambda^\ell$. This corresponds to the fact that the finite subgroups of $\mathbb{C}^\times$ are cyclic.

Burnside's Theorem also provides information on McKay quivers. Let $G$ be a finite group. We fix a complete set $\{S_1, \ldots, S_n\}$ of representatives of the complex, simple $G$-modules. Given a $G$-module $V$, we define an integral $(n \times n)$-matrix $A := (a_{ij})$ via

$$V \otimes_k S_j \cong \bigoplus_{i=1}^{n} a_{ij} S_i.$$

In other words, $A$ is the matrix representing multiplication by $V$ in the Grothendieck ring (relative to the standard basis).

**Definition.** The quiver $\Psi_V$ with underlying set of vertices $\{1, \ldots, n\}$ and $a_{ij}$ arrows $i \to j$ is called the *McKay quiver* of $G$ relative to $V$.

Given any quiver $Q$, we let $Q(i, j; m)$ be the set of paths of length $m$ starting at $i$ and terminating at $j$.

**Lemma 1.** *Let $V$ be a complex $G$-module. Then we have*

$$[V^{\otimes m} \otimes_k S_j : S_i] = |\Psi_V(i, j; m)|.$$

*Proof.* Using induction on $m$, we assume that $m \geq 2$. Note that

$$(*) \qquad \Psi_V(i,j;m) \cong \bigsqcup_{\ell=1}^{n} \Psi_V(\ell,j;m-1) \times \Psi_V(i,\ell;1).$$

The inductive hypothesis provides a decomposition

$$V^{\otimes(m-1)} \otimes_k S_j \cong \bigoplus_{t=1}^{n} b_t S_t,$$

where $b_t = |\Psi_V(t,j;m-1)|$. Consequently,

$$V^{\otimes m} \otimes_k S_j \cong \bigoplus_{t=1}^{n} b_t (V \otimes_k S_t) \cong \bigoplus_{r=1}^{n} (\sum_{t=1}^{n} a_{rt} b_t) S_r,$$

and $(*)$ implies

$$[V^{\otimes m} \otimes_k S_j : S_i] = \sum_{t=1}^{n} a_{it} b_t = |\Psi_V(i,j;m)|,$$

as desired. □

**Corollary 2.** *If $V$ is a faithful, complex $G$-module, then the McKay quiver $\Psi_V$ is connected.*

*Proof.* Let $S_1 = k$ be the trivial $G$-module. Then we have $V^{\otimes m} \otimes_k S_1 \cong V^{\otimes m} \ \ \forall\, m \geq 1$. Given a vertex $i \in \{1, \ldots, n\}$, Burnside's Theorem provides $m \in \mathbb{N}$ with

$$0 \neq [V^{\otimes m} : S_i] = [V^{\otimes m} \otimes_k S_1 : S_i] = |\Psi_V(i,1;m)|.$$

Hence there is a path from $i$ to 1. □

*Remarks.* (1) The McKay quiver also tells us that the first $m$ with $[V^{\otimes m} : S_i] \neq 0$ is the length of the shortest path from $i$ to the vertex corresponding to the trivial module.

(2) In many interesting cases, the structure of the McKay quiver is well-understood. If $V$ is a self-dual, two-dimensional, faithful representation, then the matrix defining $\Psi_V$ is symmetric, and the underlying graph is a Euclidean diagram [3].

## References


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