STABLE REPRESENTATION QUIVERS: SUBBADDITIVE FUNCTIONS AND WEBB'S THEOREM

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Given an artin algebra Λ, Zhang's Theorem (cf. [13, 5]) provides the structure of the regular, tree-infinite AR-components with small growth numbers. By definition, the growth numbers measure the exponential growth of the DTr-orbits within the given component. There are many instances, where Zhang's bound follows from homological properties, the classical case being the group algebra of a finite group.

From now on, Λ is assumed to be a finite dimensional algebra over a field k. As usual, $\operatorname{mod}\Lambda$ denotes the category of finite dimensional left Λ-modules.

Definition ([1])**.** Given $M \in \operatorname{mod}\Lambda$ and a minimal projective resolution $(P_n)_{n\geq 0}$ of M, we define the *complexity* of M via

$$\mathrm{cx}_\Lambda(M) := \min\{c \geq 0 \ ; \ \exists\ \lambda > 0 \text{ with } \dim_k P_n \leq \lambda\, n^{c-1} \ \ \forall\ n \geq 1\} \in \mathbb{N}_0 \cup \{\infty\}.$$

We shall focus henceforth only on self-injective algebras. General theory [2, Chap. IV] then provides the formula

$$\mathrm{DTr} = \nu \circ \Omega^2 = \Omega^2 \circ \nu,$$

where Ω and ν denote the Heller operator and the Nakayama functor of Λ, respectively. If Λ is symmetric, then DTr and Ω^2 coincide, implying that a module of finite complexity has growth number 1. If such a module lies in a regular, tree-infinite component of the AR-quiver, then Zhang's Theorem applies.

The foregoing observations apply for group algebras of finite groups. They also obtain whenever the Nakayama functor has finite order, which is the case for cocommutative Hopf algebras. In that context, the finiteness of the complexities follows from the Friedlander-Suslin Theorem [7], which asserts the finite generation of the cohomology ring of finite group schemes.

There are important examples, however, for which Zhang's Theorem provides no information. If

$$\Lambda = k[\Delta] \ltimes k[\Delta]^*$$

is the trivial extension of a tame hereditary algebra, then the regular components of the symmetric algebra Λ are infinite tubes, while the remaining two components are of type $\mathbb{Z}[\Delta]$ (cf. [8, 11]).

In this lecture, we will delineate an approach, which preceded Zhang's Theorem, and whose modern version employs subadditive functions defined via Ext-groups. The presence of subadditive functions was systematically exploited by Happel-Preiser-Ringel, cf. [9]. Shortly thereafter, Webb [12] implemented this method for group algebras of finite groups. Webb's fairly complicated arguments were simplified and refined by Okuyama [10], with subsequent improvements given by Erdmann-Skowroński, cf. [4].

Webb's Theorem concerns the components of the *stable Auslander-Reiten quiver* $\Gamma_s(\Lambda)$ of a self-injective algebra Λ. By definition, this quiver is obtained from the ordinary AR-quiver by removing

Date: February 1, 2006.

all projective (injective) vertices. In this fashion, $\Gamma_s(\Lambda)$ has the structure of a stable representation quiver, whose tree class we would like to determine via subadditive functions.

Theorem (Webb [12])**.** *Let G be a finite group with group algebra kG. Then the tree classes of the connected components of $\Gamma_s(kG)$ are finite or infinite Dynkin diagrams, or Euclidean diagrams.*

Given a quiver $Q := (V, A)$ any map $\nu : V \times V \longrightarrow \mathbb{N}_0 \times \mathbb{N}_0$ such that $\nu(a, b) \in \mathbb{N}\times\mathbb{N} \Leftrightarrow (a, b) \in A$ is called a *valuation* of Q. In that case (V, A, ν) is called a *valued quiver*. Let $\sigma : \mathbb{N}_0 \times \mathbb{N}_0 \longrightarrow \mathbb{N}_0 \times \mathbb{N}_0$ denote the flip.

Definition. A quadruple $Q := (V, A, \tau, \nu)$ is a *valued stable representation quiver* if
(a) (V, A, τ) is a stable representation quiver,
(b) (V, A, ν) is a valued quiver,
(c) $\nu(\tau(x), y) = \sigma(\nu(x, y)) \;\; \forall\, x, y \in V$.

The quiver $\Gamma_s(\Lambda)$ obtains the structure of a valued stable translation quiver by defining

$$\nu([M], [N]) := (\dim_{\Delta_M^{\mathrm{op}}} \mathrm{Irr}(M, N), \dim_{\Delta_N} \mathrm{Irr}(M, N)).$$

Here $\mathrm{Irr}(M, N)$ is the space of irreducible maps between M and N, and Δ_M is the divison algebra of the local algebra $\mathrm{End}_\Lambda(M)$. By general theory, the valuation (m, n) of an arrow $[M] \to [N]$ gives the multiplicities of M and N in the middle terms of the almost split sequences terminating in N and originating in M, respectively.

If the underlying field k is algebraically closed, then $\Delta_M = k = \Delta_N$, so that the valuation is of the form (n, n).

Subadditive functions are defined on *locally finite* quivers. By definition, every vertex of such a quiver has only finitely many successors and predecessors.

Definition. Let $Q = (V, A, \tau, \nu)$ be a locally finite valued stable representation quiver. A map $f : V \longrightarrow \mathbb{N}_0$ is called *subadditive* if

$$f(y) + f(\tau(y)) \geq \sum_{x\in y^-} \mathrm{pr}_1(\nu(x, y))\, f(x) \qquad \forall\, y \in V.$$

In case equality holds, f is referred to as *additive*.

The subadditive function f is *τ-periodic* if there exists $n \in \mathbb{N}$ such that $f \circ \tau^n = f$.

Consider the locally finite quiver $\Gamma_s(\Lambda)$. Directly from the definitions we obtain that

$$[M] \mapsto \dim_k M$$

defines a subadditive function of $\Gamma_s(\Lambda)$. However, this function does usually not provide any information on the tree class of a connected component $\Theta \subset \Gamma_s(\Lambda)$. The requisite additional property is that of τ-periodicity:

Theorem 1. *Let $Q = (V, A, \tau, \nu)$ be a connected, locally finite, valued stable representation quiver. Suppose that $f : V \longrightarrow \mathbb{N}$ is a τ-periodic, subadditive function.*
(1) *The tree class $\bar{T}_Q$ of Q is a Dynkin diagram, a Euclidean diagram, or $A_\infty, A_\infty^\infty, D_\infty, B_\infty, C_\infty$.*
(2) *If f is not additive, then $\bar{T}_Q$ is a Dynkin diagram or A_∞.*
(3) *If f is unbounded, then $\bar{T}_Q = A_\infty$.*

Proof. The tree class $\bar{T}_Q$ obtains the structure of a labelled graph with Cartan matrix $C : I \times I \longrightarrow \mathbb{Z}$. Since f is τ-periodic, there exists $n \in \mathbb{N}$ such that the function

$$\varphi : V \longrightarrow \mathbb{N} \ \ ; \ \ v \mapsto \sum_{i=0}^{n-1} f(\tau^i(v))$$

defines a subadditive function on V with $\varphi \circ \tau = \varphi$. Such a function gives rise to a subadditive function $d : I \longrightarrow \mathbb{N}$ of the Cartan matrix C. Our result now follows from the work of Happel-Preiser-Ringel [9]. □

Given $M \in \operatorname{mod} \Lambda$, we define a function

$$f_M : \Gamma_s(\Lambda) \longrightarrow \mathbb{N}_0 \ \ ; \ \ [N] \mapsto \dim_k \operatorname{Ext}^1_\Lambda(M, N).$$

Suppose there exists a projective module P such that $\operatorname{DTr}(M) \oplus P \cong M$. The formula $\operatorname{DTr} = \Omega^2 \circ \nu$ then implies that

$$f_M \circ \operatorname{DTr} = f_M.$$

Moreover, we have:

Proposition 2. *Let $\Theta \subset \Gamma_s(\Lambda)$ be a connected component of the stable AR-quiver of Λ. Assume there exists a Λ-module M such that*

(a) $\operatorname{DTr}(M) \oplus (\text{proj.}) \cong M$, *and*

(b) *there exists* $[X_0] \in \Theta$ *with* $\operatorname{Ext}^1_\Lambda(M, X_0) \neq (0)$.

Then $\bar{T}_\Theta$ is a finite or infinite Dynkin diagram, or a Euclidean diagram.

Proof. Let $[N]$ be a vertex in Θ. Then we have an AR-sequence

$$(0) \longrightarrow \operatorname{DTr}(N) \longrightarrow \sum_{[X] \in [N]^-} \operatorname{pr}_1(\nu([X],[N]))\, X \oplus (\text{proj.}) \longrightarrow N \longrightarrow (0).$$

Since projective modules are injective, application of $\operatorname{Ext}^1_\Lambda(M, -)$ implies that f_M is subadditive. Since Θ is connected, condition (b) ensures that

$$f_M([X]) \in \mathbb{N} \quad \forall\ [X] \in \Theta.$$

Our result now follows from Theorem 1. □

Of course, the utility of Proposition 2 hinges entirely on the validity of the seemingly innocuous conditions (a) and (b), which require the presence of enough periodic modules in $\operatorname{mod} \Lambda$. Let us begin with the case where Λ is a symmetric algebra, so that

$$\operatorname{DTr} = \Omega^2.$$

If $\Lambda_n = k[T]/(T^n)$ is a truncated polynomial ring, then we have

$$\Omega^2_{\Lambda_n}(k) \cong k.$$

Moreover, since Λ_n is local, the condition $\operatorname{Ext}^1_{\Lambda_n}(k, X) = (0)$ yields $\operatorname{Ext}^1_{\Lambda_n}(-, X) = (0)$, so that X is injective. Consequently, (a) and (b) hold for $M = k$ globally on $\Gamma_s(\Lambda_n)$.

Now suppose that $\Lambda_n \subset \Lambda$ such that Λ is a projective right Λ_n-module. If $\Theta \subset \Gamma_s(\Lambda)$ is a component such that $X_0|_{\Lambda_n}$ is not projective for some $X_0 \in \Theta$, then we define

$$M := \Lambda \otimes_{\Lambda_n} k.$$

The exactness of $\Lambda \otimes_{\Lambda_n} -$ then yields $\Omega^2_\Lambda(M) \oplus (\text{proj.}) \cong M$ and Frobenius reciprocity implies

$$\operatorname{Ext}^1_\Lambda(M, X_0) \cong \operatorname{Ext}^1_{\Lambda_n}(k, X_0) \neq (0).$$

The foregoing observations lead to the study of rank varieties [3] and flat homomorphisms $\alpha : \Lambda_n \longrightarrow \Lambda$ [6]. These considerations ensure the existence of enough periodic modules for cocommutative Hopf algebras of positive characteristic. (By Cartier's Theorem, such algebras are semisimple whenever $\mathrm{char}(k) = 0$.)

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