Selected topics in representation theory 3
Quadratic algebras and Koszul algebras

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1 Quadratic algebras

We consider a finite dimensional vector space V over a field k. Let $I \subseteq V \otimes V$ be a subvector space and denote by (I) the ideal in the tensor algebra TV generated by I. In this section we restrict ourself to quotients of the tensor algebra, however similar definitions can be made for path algebras.

Definition. The algebra $A := TV/(I)$ is called a *quadratic algebra*. Its *quadratic dual* is defined as $A^! := TV^*/I^\perp$, where V^* is the dual vector space and

$$I^\perp := \{\sum a_i\phi \otimes \psi \mid \sum_{i,j} a_i b_j \phi_i(y_j)\psi_j(y_i) = 0 \text{ for all } \sum b_j x_j \otimes y_j \in I\}.$$

Note that A and $A^!$ are $\mathbb{Z}$–graded algebras.

Example. 1) If $I = \{0\}$ then $A = TV$ and $A^! = TV^*/(V^* \otimes V^*)$.
2) If $A = SV$ the symmetric algebra over V, then $A^! = \Lambda V^*$ is the exterior algebra over the dual vector space.
3) If $A = k[x,y]/x^2$ then $A^! = k\langle \xi, \eta\rangle/(\xi\eta + \eta\xi, \eta^2)$.

Definition. We define the Poincaré polynomial of a quadratic algebra A (it can be defined for any graded algebra) as $P_A(t) := \sum_{i=0}^\infty \dim A_i t^i$.

Example. Note that in all examples above we have $P_A(t)P_{A^!}(-t) = 1$:
1) $P_A(t) = \sum_{i=0}^\infty n^i t^i$ and $P_{A^!}(t) = 1 + nt$, where $n = dimV$.
2) $P_A(t) = \sum_{i=0}^\infty \binom{n+1+i}{i} t^i$ and $P_{A^!}(t) = \sum_{i=0}^\infty \binom{n}{i} t^i$.
3) $P_A(t) = 1 + 2\sum_{i=1}^\infty t^i$.

2 The Koszul complex

We keep the notation from the previous section: $A = TV/I$ is a quadratic algebra and $A^! = TV^*/I^\perp$ is the quadratic dual algebra. Moreover, let $B_i := (A^!_i)^*, B = \oplus_{i=0}^\infty B_i$ the graded dual of $A^!$, it is a graded coalgebra and an $A^!$–bimodule. Let M be a bimodule: a right $A^!$–module and a left A–module. Then we can define a k–linear map $d : M \longrightarrow M$ defined by $dm := \sum x_i m \xi_i$, where $\{x_i\}$ is a basis of V and $\{\xi_i\}$ is the dual basis of V^*. If M is a graded module, then $dM_i \subseteq M_{i+1}$.

Lemma. *1) Consider the composition of maps*

$$k \longrightarrow V \otimes V \longrightarrow (V \otimes V)/I \otimes (V^* \otimes V^*)/I^\perp, \quad 1 \mapsto \sum_i \xi_i x_i, \quad \phi \otimes v \mapsto \sum_i \phi x_i \otimes \xi_i v.$$

It is the restriction of the differential d to

$$A_0 \otimes A^!_0 \longrightarrow A_1 \otimes A^!_1 \longrightarrow A_2 \otimes A^!_2.$$

Then the composition is zero.
2) For any $A^! - A$ bimodule M, the map $d : m \mapsto \sum_i \xi_i m x_i$ is a differential, that is $d^2 = 0$.

Proof. Assertion 2) is a consequence of assertion 1). We consider the composition together with the adjoint map

$$k \longrightarrow (V \otimes V)/I \otimes (V^* \otimes V^*)/I^\perp, \qquad ((V \otimes V)/I)^* \simeq I \longrightarrow (V^* \otimes V^*)/I^\perp.$$

The second map is obviously the zero map, consequently $d^2 = 0$. The natural isomorphism $((V^* \otimes V^*)/I^\perp)^* \simeq I$ is an easy exercise in linear algebra. □

Definition. Using the differential d defined above we define a bigraded differential $A - A^!$–bimodule $K_\bullet, d$, with $K = A \otimes B$ and grading $K_i := A \otimes B_i$. Obviously $dK_i \subseteq K_{i-1}$. This differential bimodule becomes, with this grading, a complex of free left A–modules, called *Koszul complex*. We define A to be a *Koszul algebra*, if $H_i(K_\bullet, d) = 0$ for all $i \neq 0$.

Remark. 1) It is obvious that $H_0(K_\bullet, d) = S = A_0 \simeq A/A_{>0}$ is the natural simple A–module.
2) If A is Koszul, then $K_\bullet, d$ is a minimal projective resolution of S as a graded A–module and $\mathrm{Ext}^{\,l}(S, S) = \mathrm{Ext}^{\,l}_{\mathbb{Z}}(S, S\langle l\rangle) \simeq B^*_l = A^!_l$, where the first group denotes the usual extension group, $\mathrm{Ext}^{\,l}_{\mathbb{Z}}$ is the goups of extensions in the categry of $\mathbb{Z}$–graded A–modules and $\langle l\rangle$ is the lth shift by the grading.
3) For any quadratic algebra A we obtain $H_1(K_\bullet, d) = 0$ and $H_2(K_\bullet, d) = 0$.
4) If A is Koszul then we have for the Poincaré polynomials the following equation

$$P_A(t)P_{A^!}(-t) = 1.$$

All claims follow easily from the concrete description of the Koszul complex: we consider the graded pieces $K_{i,j} = A_i \otimes B_j \subseteq K_j$. Then $dK_{i,j} \subseteq K_{i+1,j-1}$, so $K(n) := \oplus_{i+j=n} K_{i,j}$ is a complex of vector spaces and coincides with the graded part of degree n of the Koszul complex (consider the Koszul complex as a complex of graded modules). In degree zero there is just one non-zero vector space $A_0 \otimes B_0$. In higher degrees the last map is always surjective, this shows 1) and also 2). To see 3) we write down the beginning of the Kosul complex and obtain for the graded parts:

$$\begin{array}{ccccccc}
 & & & & & & A_0 \otimes B_0 \\
 & & & & A_0 \otimes B_1 & \longrightarrow & A_1 \otimes B_0 \\
 & & A_0 \otimes B_2 & \longrightarrow & A_1 \otimes B_1 & \longrightarrow & A_2 \otimes B_0 \\
A_0 \otimes B_3 & \longrightarrow & A_1 \otimes B_2 & \longrightarrow & A_2 \otimes B_1 & \longrightarrow & A_3 \otimes B_0.
\end{array}$$

The A–module structure is the natural one coming from the action $V \otimes A_i \longrightarrow A_{i+1}$ from the left and shifts the row one step down (note that we used the A–right module structure to define the differential). A proof of the claim follows in the next lecture. To see 4) we just notice that $P_A(t)P_{A^!}(-t) = 1$ precisely when $\sum_{i=0}^n (-1)^i \dim A_i \dim B_{n-i} = 0$ for all $n > 0$.

3 Koszul algebras

Let A be a graded algebra of the form $A = TV/I$. We denote by $E(A)$ the algebra $\oplus_l \mathrm{Ext}^l(S,S)$ (the product is the Yoneda product) where $S = A/A_{>0}$ is the natural simple module. The main property of a Koszul algebra is that we can compute the algebra $E(A)$ very easily as the quadratic dual and obtain $E(E(A)) \simeq A$ is a natural isomorphism of $\mathbb{Z}$–graded algebras. However, there are several further characterizations of a Koszul algebra, based on the fact, that $E(E(A))$ is usually more complicated then A itself (it does always contain A). In particular, it is usually not generated in degree 1. Surprisingly, an algebra A is already Koszul, if one of the conditions above is satisfied. This shows, that Koszul is a very restrictive condition on an algebra and can also be defined without the assumption A to be quadratic (however this approach is less technical). So the following theorem can also be considered as a definition of Koszul for algebras not necessarily quadratic, then it is a result, that from Koszul follows quadratic.

Theorem. *Let A be a graded algebra of the form $A = TV/J$ for some homogeneous ideal J contained in $A_{\geq 2}$. Then the following conditions are equivalent:*
1) $E(A)$ is generated by $Ext^1(S,S)$,
2) $E(E(A)) \simeq A$ (as an algebra),
3) $E(E(A))_1 \simeq A_1$ (as a k–vector space),
4) A is quadratic and Koszul,
5) A is quadratic and $A^!$ is Koszul, and
6) A is qudratic and $P_A(t)P_{A^!}(-t) = 1$,
7) A is quadratic and $E(A) \simeq A^!$.

Proof (sketch). Let A be a graded algebra and consider the minimal projective resolution of S, then $\mathrm{Ext}^2(S,S)$ is the vector space of relations, that are homogeneous elements in I, that are not a product of an homogenous element in I with an homogeneous element of A. Since the Yoneda product is compatible with the grading, $E(A)$ is generated by $\mathrm{Ext}^1(S,S)$ only if A is qudratic. So conditions 1), 2) and 3) are necessary for an algebra to be qudratic and Koszul. If A is quadratic and Koszul, then 1) and 3) follow, and 2) follows from 5). To see 4) is equivalent to 6) we refer to the next lecture, one has to check, that there is precisely one non-vanishing homology group of minimal degree. If A is not Koszul, then the equation $P_A(t)P_{A^!} = 1$ can not be satisfied. Since 4) is equivalent to 6), we also obtain 4) is equivalent to 5). Using this equivalence, we can also show 2). From 7) follows 1) and conversely 7) follows from 5) (one has to use the coalgebra structure of B) or from 2) using the argument above on the structure of $E(A)$ and $E(E(A))$. So it remains to prove 5) is equivalent to 6), shown in the next lecture. □

Remark. The present note only contains a very brief introduction to the subject. We did not include any result on bounded (quadratic) path algebras, Koszul duality, Kszul algebras over semisimple rings and the theory of linear resolutions. Also the references select only a very small list of articles on the subject.

References

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