Selected topics in representation theory 6
The volume of a tilting module

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The aim of this lecture is to define a volume for a tilting module. There are several natural applications of such a construction. First of all we want to realize the simplicial complex of tilting modules as a fan in the Grothendieck group. Secondly, each simplex should have a volume, so that we can measure the "density" of the tilting modules in the Grothendieck group. The construction works very well for hereditary algebras (section 4) and for tilting modules of projective dimension less or equal to one for all algebras (section 5). In case the complex of all tilting modules is finite (or at least tame), we can check for a given list of tilting modules whether it is already complete. This might be very useful, since one can iteratively construct tilting modules (e. g. via mutations) starting with the projective or injective tilting module. Moreover, we note that the volume is nothing natural, we can, depending on some choices, define many different notions of a volume. It turns out that at least two of them are easily to compute and very useful with respect to the applications mentioned above: the lattice invariant volume $\mathrm{vol}_{\mathbb{Z}}$ and the Δ–volume vol_{Δ}.

1 The cone of a module

Let A be a finite dimensional algebra k–algebra. We denote by K_0 the Grothendieck group of finite dimensional A–modules. Given a finite dimensional module X we associate to X a cone $C(X)$ in the $\mathbb{R}$–vector space $K_{\mathbb{R}} := K_0 \otimes \mathbb{R}$. Note that we only get useful results if we consider a module X with $\mathrm{Ext}^1(X, X) = 0$. Let $v_1, \ldots, v_r$ be elements in $K_{\mathbb{R}}$. The cone C generated by these elements (as a cone) is the set

$$C := \mathrm{cone}\{v_1, \ldots, v_r\} := \{\sum_{i=1}^{r} a_i v_i \mid a_i \in \mathbb{R}_{\geq 0} \text{ for all } i = 1, \ldots, r\}.$$

Similarly, we define the convex hull of the elements $v_1, \ldots, v_r$

$$\mathrm{conv}\{v_1, \ldots, v_r\} := \{\sum_{i=1}^{r} a_i v_i \mid a_i \in \mathbb{R}_{\geq 0} \text{ for all } i = 1, \ldots, r; \sum_{i=1}^{r} a_i = 1\}.$$

We consider $K_{\mathbb{R}}$ with the standard basis $e_i = [S(i)] = \underline{\dim}\, S(i)$, where $\{S(i)\}_{i=1}^{t}$ is a complete set of representatives of the indecomposable simple A–modules.

Definition. The cone $C(X)$ is defined as the cone generated (as a cone) by the elements $\underline{\dim}\, X_i = [X_i]$ in $K_{\mathbb{R}}$, where $X = \oplus X_i$ is a decomposition into indecomposable direct summands.

Lemma. *For each finite dimensional module X the cone $C(X)$ is a rational, convex, strictly convex, polyhedral cone in $K_{\mathbb{R}}$.*

The lemma itself is obvious, we only need to explain the terminology in detail (the reader should compare the result with the notion of a cone in toric geometry, see e. g. [2, 6]). A cone C is *rational* if it is generated by lattice elements that is by elements in K_0. It is *convex* if for any two elements u and v in C also the line segment between u and v is contained in C: $(1-t)u + tv \in C$ for all $t \in [0,1]$. A cone is *strictly convex* if it does not contain a line (affine or linear, both conditions are equivalent). This follows, since $C(X)$ is contained in the positive quadrant. Finally, a cone is *polyhedral* if it is the intersection of finitely many half spaces. If the cone is already rational, then this condition is equivalent to the following: the cone is generated by a finite number of lattice points.

DEFINITION. For the rest of the note a *cone* is always a rational, convex, strictly convex, polyhedral cone in $K_{\mathbb{R}}$. A cone C is called *smooth* if it is generated by a part of a $\mathbb{Z}$–basis of K_0 (the notion comes from toric geometry: a smooth cone defines a smooth affine toric variety, see [6, 2]). A cone is called *simplicial* if it is generated by a linearly independent set of elements. An $\mathbb{R}$–linear function f on $K_{\mathbb{R}}$ defines a half space $H_f^+ := \{v \in K_{\mathbb{R}} \mid f(v) \geq 0\} \subset K_{\mathbb{R}}$ (we allow $f = 0$ here!). The space H_f is called supporting if $f(v) \geq 0$ for all $v \in C$. For such a cone C we define a *face* of C to be the intersection of C with a supporting half space $C \cap H_f$. Note that C itself is a face of C (take $f = 0$) and also 0 is a face (take f generic). A set of cones $\Sigma = \{C_i \mid i \in I\}$ is called a *fan*, if the following conditions are satisfied:

1) each face of a cone in Σ is in Σ and

2) the intersection of two cones in Σ is a face of both.

Note that we do *not* assume Σ to be finite, as done in toric geometry. We will show later that we can associate to any finite dimensional algebra a fan associated to the set of tilting modules of projective dimension at most one. Moreover, if we intersect the fan with a sphere $S \subset K_{\mathbb{R}}$ (chose any euclidean metric here to define S, e. g. take the standard scalar product) then we obtain a simplicial complex. Under certain additional assumptions (e. g. A is hereditary) we obtain the simplicial complex of tilting modules in this way. Using this approach there are several additional structures on the simplicial complex: a lattice geometric structure, a natural volume form, and a convex geometric structure.

2 The volume of a module

To define a volume for a tilting module T we have to intersect the cone $C(T)$ with a set V, so that we can define a volume on $C(T) \cap V$ for each cone $C(T)$. The best definition would be to define the volume invariant under lattice automorphisms, in particular, we would like to have the definition invariant under tilting or even derived equivalences. Then we obtain the $\mathbb{Z}$–volume $\mathrm{vol}_{\mathbb{Z}}$. It turns out that we can measure only the number of tilting modules with this volume, it is useful only if there is only a finite number of tilting modules. An approach that works well also in the infinite case can be obtained as follows: consider the standard euclidean metric on $K_{\mathbb{R}}$ and consider any bounded measurable subset V in $K_{\mathbb{R}}$. Then define the volume as the intersection of the cone with this subset V. The simplest set V might be the standard simplex

$$\overline{\Delta} := \{\sum_{i=1}^{t} a_i e_i \mid \sum a_i = 1, a_i \geq 0\}.$$

To keep computations easier we normalize the euclidean metric (just multiply it by a constant) so that $\operatorname{vol}\overline{\Delta} = 1$.

Lemma. *Assume A is a finite dimensional algebra of finite global dimension and T is a partial tilting module. Then $C(T)$ is a smooth cone. If T has t indecomposable direct summands and $\det(\underline{\dim}\, T_i)_{i=1}^{t}$ is non-zero, then $C(T)$ is simplicial.*

Proof. If B is the endomorphism algebra of a tilting module, then the Cartan matrices of A, respectively B, are conjugate over $\mathbb{Z}$. Indead, if T is a tilting module, then $\{\underline{\dim}\, T_i\}_{i=1}^{t}$ is $\mathbb{Z}$–basis of $K_0 \subset K_{\mathbb{R}}$, where $T = \oplus T_i^{c_i}$ is a decomposition into indecomposable non-isomorphic direct summands. Consequently, for a partial tilting module the set $\dim T_i$ is a part of a $\mathbb{Z}$–basis. The second claim is obvious, since $\underline{\dim}\, T_i$ are an $\mathbb{R}$–basis of $K_{\mathbb{R}}$. □

Definition. Let T be an A–module and let $T = \oplus_{i=1}^{r} T_i^{c_i}$ be the decomposition into indecomposable non-isomorphic direct summands. We define

$$\operatorname{vol}_{\mathbb{Z}}(T) := \operatorname{vol}(\operatorname{conv}\{0, \underline{\dim}\, T_1, \ldots, \underline{\dim}\, T_r\}),$$

where vol is just the natural lattice invariant volume form normalized so that the volume of the t-dimensional standard simplex $\{\sum a_i e_i \mid \sum a_i \leq 1, a_i \geq 0\}$ is one. Moreover, we define

$$\operatorname{vol}_{\Delta}(T) := \operatorname{vol}(C(T) \cap \overline{\Delta}).$$

Lemma. *Assume A is of finite global dimension and T is a tilting module (with decomposition as above) then*
1) $\operatorname{vol}_{\mathbb{Z}}(T) = 1$ *and*
2) $\operatorname{vol}_{\Delta}(T) = 1/\prod_{i=1}^{t} \dim T_i$.

Proof. We first show 1). Since $\underline{\dim}\, T_i$ form a $\mathbb{Z}$–basis, we have $|\det(\underline{\dim}\, T_i)| = 1$. Consequently, the volume of the simplex $\operatorname{conv}\{0, \underline{\dim}\, T_1, \ldots, \underline{\dim}\, T_t\}$ coincides with the volume of $\operatorname{conv}\{0, e_1, \ldots, e_t\}$, that is one by our assumption. Both assertions are equivalent since

$$\operatorname{vol}_{\mathbb{Z}}(T) = \prod_{i=1}^{t} \dim(T_i) \operatorname{vol}_{\Delta}(T)$$

follows from $C(T) \cap \overline{\Delta} = \operatorname{conv}\{0, \underline{\dim}\, T_1/\dim T_1, \ldots, \underline{\dim}\, T_t/\dim T_t\}$. □

3 The fan of an algebra

Theorem. *Let A be a finite dimensional algebra. Assume for each dimension vector d there is at most one isomorphism class of an A–module T with $\operatorname{Ext}^1(T,T) = 0$. Then the set $\{C(T) \mid T \in A - \operatorname{mod}, \operatorname{Ext}^1(T,T) = 0\}$ is a fan. If A is even of finite global dimension then the fan is also smooth (that is each cone is smooth).*

The assumption in the theorem is rather strong, it is satisfied for hereditary algebras and only a few others. So it might be more useful to consider a smaller class of modules: in fact for modules of projective dimension at most one we get a similar result. We prove both results together.

Theorem. *Let A be any finite dimensional algebra A with invertible Cartan matrix. Then the set*

$$\Sigma(A) := \{C(T) \mid T \in A-\mathrm{mod}, \mathrm{pd}(T) \leq 1, \mathrm{Ext}^1(T,T) = 0\}$$

is a fan. If A is of finite global dimension then $\Sigma(A)$ is a smooth fan. In particular, if A is hereditary then the simplicial complex of tilting modules is the intersection of the fan $\Sigma(A)$ with the unit sphere in $K_{\mathbb{R}}$.

Proof. We first need to show that for each dimension vector d there is at most one module T satisfying $\underline{\dim}\, T = d$, $\mathrm{pd}(T) \leq 1$, and $\mathrm{Ext}^1(T,T) = 0$. For we show that the subset

$$\mathcal{R}(A;d)^1 := \{M \in \mathcal{R}(A;d) \mid \mathrm{pd}(M) \leq 1\}$$

of the representation space $\mathcal{R}(A;d)$ of representations of A with dimension vector d is an irreducible subvariety. Take a module M with $\underline{\dim}\, M = d$ and consider an exact sequence with P^1 projective and minimal

$$P^1 \longrightarrow P^0 := \oplus_i P(i) \otimes \dim M_i \longrightarrow M.$$

If $\mathrm{pd}(M) \leq 1$ then the map $P^1 \longrightarrow P^0$ is injective and $\underline{\dim}\, P^1$ is determined by $\underline{\dim}\, P^1 = \underline{\dim}\, P^0 - \underline{\dim}\, M$. By our assumption on the Cartan matrix the isomorphism class of P^1 is determined by $\underline{\dim}\, P^1$, so it is independent of M. The space of injective A–module homomorphisms $\mathrm{Hom}_A^{\mathrm{inj}}(P^1, P^0)$ is obviously irreducible (it is open in an affine space). Standard arguments show that then also $\mathcal{R}(A;d)^1$ is irreducible (see Section 5, proof of the theorem). Now we know that for each dimension vector d there is at most one module T with $\underline{\dim}\, T = d$, $\mathrm{pd}(T) \leq 1$, and $\mathrm{Ext}^1(T,T) = 0$ (since such a module has an open dense orbit in $\mathcal{R}(A;d)$, thus also in $\mathcal{R}(A;d)^1$). The following arguments also work if we use the assumption in the first theorem und omit the condition on the projective dimension. Now we prove the assertion about the fan: assume we have two modules T and R with $\mathrm{Ext}^1(T,T) = 0 = \mathrm{Ext}^1(R,R)$. Assume $C(T) \cap C(R) \neq 0$ and take $0 \neq u \in C(T) \cap C(R)$. Since both cones are rational we can assume $u \in K_0$. Since $u \in C(T)$ we obtain $au = \sum \underline{\dim}\, T_i^{a_i}$ for some positive integer a and non-negative integers a_i. Similarly, $bu = \sum \underline{\dim}\, R_i^{b_i}$. Thus, $abu = \sum \underline{\dim}\, T_i^{a_i b} = \sum \underline{\dim}\, R_i^{ab_i}$. Since for the dimension vector abu there exists at most one module X with this dimension vector, projective dimension less or equal one and no selfextensions (note that both modules $\oplus T_i^{a_i b}$ and $\oplus R_i^{ab_i}$ satisfy this conditions) we conclude $\oplus T_i^{a_i b} \simeq \oplus R_i^{ab_i}$. Comparing indecomposable direct summands and renumber, if necessary, we finally obtain $T_i \simeq R_i$ for some indices i. We conclude that $C(T)$ must be simplicial, each face is of the form $C(T')$ for some direct summand T' of T (in particular T' satisfies the condition), and the intersection of two cones $C(T)$ and $C(R)$ is also of the form $C(T')$, where T' is a direct summand of both T and R.

4 Hereditary algebras

Assume in this section A is hereditary and

$$\Sigma(A) = \{C(T) \mid T \in A-\mathrm{mod}, \mathrm{Ext}^1(T,T) = 0\}$$

is the smooth fan for A, consisting of the cones $C(T)$ for the various isomorphism classes of basic partial tilting modules T. We define

$$\mathrm{vol}_{\mathbb{Z}}(A) := \sum_T \mathrm{vol}_{\mathbb{Z}}(T) \qquad \text{and} \qquad \mathrm{vol}_{\Delta}(A) := \sum_T \mathrm{vol}_{\Delta}(T),$$

where the sums run over all isomorphism classes of basic partial tilting modules T.

Theorem. *1) The fan $\Sigma(A)$ is finite precisely when A is representation finite.*
2) Condition 1) is equivalent to $\mathrm{vol}_{\mathbb{Z}}(A) < \infty$.
3) For the Δ–volume we find $\mathrm{vol}_{\Delta}(A) \leq 1$ and $\mathrm{vol}_{\Delta}(A) = 1$ if and only if A is tame.

Example. Consider the n–arrow Kronecker quiver.
$n = 1$: There are two tilting modules with dimension vector $(2,1) = (1,0) + (1,1)$ and $(1,2) = (1,1) + (0,1)$, both have volume $1/2$.
$n = 2$: There are infinitely many tilting modules. Each is determined by its dimension vector $(2a-1, 2a+1) = (a-1, a)+(a, a+1)$ (the preprojective ones) with volume $1/((2a-1)(2a+1))$ and $(2a+1, 2a-1) = (a, a-1) + (a+1, a)$ (the preinjective ones) with volume $1/((2a-1)(2a+1))$. Consequently

$$2\sum_{a=1}^{\infty} 1/((2a-1)(2a+1)) = 1.$$

$n \geq 3$: There are also infinitely many tilting modules (each one is either preprojective or preinjective) and $\mathrm{vol}_{\Delta}(A) < 1$.

Proof. A tame hereditary algebra has already infinitely many isomorphism classes of preprojective tilting modules, in particular, infinitely many isomorphism classes of tilting modules. Thus $\mathrm{vol}_{\mathbb{Z}}(A) = \infty$ for A tame or wild (second lemma in section 2). For A representation finite we obviously get $\mathrm{vol}_{\mathbb{Z}}(A) < \infty$. Thus we have proven 1) and 2).
The inequality in 3) is also obvious, since $\mathrm{vol}_{\Delta}(A) \leq \mathrm{vol}(\overline{\Delta})$. If A is tame then for each dimension vector d that is not a multiple of the imaginary root there exists a module M with $\underline{\dim}\, M = d$ and $\mathrm{Ext}^1(M,M) = 0$. Consequently, the subset $\cup C(T)$ has the same volume as $\overline{\Delta}$. If A wild then the subset of all d where the quadratic from q satisfies $q(d) \leq 0$ has a positive volume, whereas $q(\underline{\dim}(T)) > 0$ for each tilting modules T. Thus $\cup C(T)$ has a volume smaller than 1. □

5 Modules of projective dimension at most 1

Assume A is finite dimensional with invertible Cartan matrix (we only need to assume $\dim P(i)$ for $i = 1, \ldots, t$ is an $\mathbb{R}$–basis). We consider the fan

$$\Sigma(A) := \{C(T) \mid T \in A-\mathrm{mod}, \mathrm{pd}\, T \leq 1, \mathrm{Ext}^1(T,T) = 0\}.$$

We consider the open subvariety in the representation space consisting of modules of projective dimension at most one

$$\mathcal{R}^1(A;d) \subset \mathcal{R}(A;d) = \{\phi \in \oplus_\alpha \operatorname{Hom}(k^{d(s(\alpha))}, k^{d(t(\alpha))}) \mid M(\phi) \text{ is an } A-\text{module}\}.$$

Theorem. *The variety $\mathcal{R}^1(A;d)$ is irreducible and reduced. In particular, there exists at most one dense orbit and if M is in a dense orbit then $\operatorname{Ext}^1(M,M)=0$.*

Proof. We only sketch the proof here. We define P^1 and P^0 as in the proof in Section 3. Then we need to define a morphism

$$\operatorname{Hom}_A^{\text{inj}}(P^1,P^0) \longrightarrow \mathcal{R}^1(A;d), \quad f \mapsto M(f).$$

Note that $\operatorname{Coker}(f)$ comes equipped with a basis, just take the image of a fixed basis in the top of P^0. Thus $M(f) := \operatorname{Coker}(f)$ is a module together with a basis, thus an element in the representation space. Consequently, we have a well–defined map. It is not hard to check that this map is a morphism, it is dominant and a morphism between affine schemes. Since $\operatorname{Hom}_A^{\text{inj}}(P^1,P^0)$ is a variety, its coordinate ring is integral and since the morphism is dominant, the coordinate ring of $\mathcal{R}^1(A;d)$ is a subring. Consequently, it is also integral and $\mathcal{R}^1(A;d)$ is irreducible and reduced. □

Now we can state a similar result, as the theorem in the previous section, for arbitrary algebras. In this case we need to define the cone C^1 as the cone of all dimension vectors d, so that there exists a module M with $\underline{\dim}\, M = d$ and $\operatorname{pd} M \leq 1$. Moreover, we define

$$\operatorname{vol}_{\mathbb{Z}}(A) := \sum \operatorname{vol}_{\mathbb{Z}}(T), \qquad \text{and} \qquad \operatorname{vol}_{\Delta}(A) := \sum \operatorname{vol}_{\Delta}(T),$$

where the sums run over all isomorphism classes of basic modules with $\operatorname{pd} T \leq 1$ and $\operatorname{Ext}^1(T,T)=0$. Note that this definition generalizes the definition in the previous section.

Theorem. *1) The fan $\Sigma(A)$ is finite precisely when A admits only a finite number of tilting modules with $\operatorname{pd} T \leq 1$.*
2) Condition 1) is equivalent to $\operatorname{vol}_{\mathbb{Z}}(A) < \infty$.
3) For the Δ–volume we find $\operatorname{vol}_\Delta(A) \leq \operatorname{vol}_\Delta(C^1)$. If in addition for each dimension vector d, except for some set of volume zero, there exists a module T with $\operatorname{pd} T \leq 1$ and the orbit of T is dense in $\mathcal{R}^1(A;d)$ (thus $\operatorname{Ext}^1(T,T)=0$) then $\operatorname{vol}_\Delta(A) = \operatorname{vol}_\Delta(C^1)$. If there are only finitely many isomorphism classes of indecomposable modules with $\operatorname{pd} \leq 1$ then the previous condition is satisfied.

Proof. The proof of the theorem in Section 4 also works here. □

Remark. One can not expect a characterisation of the equality $\operatorname{vol}_\Delta(A) = \operatorname{vol}_\Delta(C^1)$ in terms of the representation type of the set of modules with $\operatorname{pd} \leq 1$. There exist examples of algebras, where the category of modules with projective dimension at most 1 is wild, however the set of isomorphism classes of basic tilting modules with projective dimension at most one is finite (see e. g. [3]).

References

[1] I. Assem, *Tilting theory—an introduction.* Topics in algebra, Part 1 (Warsaw, 1988), 127–180, Banach Center Publ., 26, Part 1, PWN, Warsaw, 1990.

[2] W. Fulton, *Introduction to toric varieties.* Annals of Mathematics Studies, 131. The William H. Roever Lectures in Geometry. Princeton University Press, Princeton, NJ, 1993.

[3] T. Brüstle, L. Hille, C. M. Ringel, G. Röhrle, *The Δ-filtered modules without self-extensions for the Auslander algebra of $k[T]/\langle T^n \rangle$.* Algebr. Represent. Theory 2 (1999), no. 3, 295–312.

[4] D. Happel, C. M. Ringel, *Tilted algebras.* Trans. Amer. Math. Soc. 274 (1982), no. 2, 399–443.

[5] C. M. Ringel, *Tame algebras and integral quadratic forms.* Lecture Notes in Mathematics, 1099. Springer-Verlag, Berlin, 1984. xiii+376 pp.

[6] T. Oda, *Convex bodies and algebraic geometry.* An introduction to the theory of toric varieties. Translated from the Japanese. Ergebnisse der Mathematik und ihrer Grenzgebiete (3) [Results in Mathematics and Related Areas (3)], 15. Springer-Verlag, Berlin, 1988.