Selected topics in representation theory

– Examples –

WS 2005/06

Here are two examples connected with the two lectures on "Modules with standard filtration I/II" in the lecture series "Selected topics in representation theory".

Example 1: A covariantly finite, but not contravariantly finite full subcategory of a module category

In the last lecture we saw an example of a full subcategory of a module category which was neither covariantly nor contravariantly finite. Here is an example showing that both properties are independent of each other.

Take again the Kronecker algebra. Let P be the simple projective module, and let M be an indecomposable module of length 2. Let $\Theta = \{P, M\}$. Then $\mathrm{Ext}^1(M, P) \neq 0$.

The full subcategory $\mathcal{F}(\Theta)$ consisting of all modules filtered by P and M is covariantly finite, but not contravariantly finite. (The argument that it is not contravariantly finite is the same as in the example from last time.) To see that it is covariantly finite, we remark that there are no maps from indecomposable preinjective modules or indecomposable modules from tubes other than the one containing M to $\mathcal{F}(\Theta)$. For the other indecomposables we can take the identity map as a right $\mathcal{F}(\Theta)$-approximation, as every indecomposable module M_n of length $2n$, $n \in \mathbb{N}$, lying in the same tube as M is an iterated extension of M by itself, and every indecomposable preprojective module P_n of length $2n+1$, $n \in \mathbb{N}$, is an extension of M_{2n} by P.

Example 2: An explicit construction of a right $\mathcal{F}(\Delta)$-approximation

Take the path algebra $A = kQ$ of the quiver $Q = \mathbb{D}_4$ with subspace orientation.

Here is the Auslander-Reiten quiver for the algebra:

$$\begin{tikzcd}
 & \begin{smallmatrix}1&\\0&1\\0&\end{smallmatrix} \arrow[dr] & & \begin{smallmatrix}0&\\1&1\\1&\end{smallmatrix} \arrow[dr] & & \begin{smallmatrix}1&\\0&0\\0&\end{smallmatrix} \\
\begin{smallmatrix}0&\\0&1\\0&\end{smallmatrix} \arrow[ur] \arrow[r] \arrow[dr] & \begin{smallmatrix}0&\\1&1\\0&\end{smallmatrix} \arrow[r] & \begin{smallmatrix}1&\\1&2\\1&\end{smallmatrix} \arrow[ur] \arrow[r] \arrow[dr] & \begin{smallmatrix}1&\\0&1\\1&\end{smallmatrix} \arrow[r] & \begin{smallmatrix}1&\\1&1\\1&\end{smallmatrix} \arrow[ur] \arrow[r] \arrow[dr] & \begin{smallmatrix}0&\\1&0\\0&\end{smallmatrix} \\
 & \begin{smallmatrix}0&\\0&1\\1&\end{smallmatrix} \arrow[ur] & & \begin{smallmatrix}1&\\1&1\\0&\end{smallmatrix} \arrow[ur] & & \begin{smallmatrix}0&\\0&0\\1&\end{smallmatrix}
\end{tikzcd}$$

Let $P_1, \ldots, P_4$ denote the indecomposable projective modules, and $S_1, \ldots, S_4$ the simple modules.

Write $\Delta_i = P_i / \sum_{j>i} \mathrm{im}(P_j \to P_i)$. Then $\mathrm{Ext}^1_A(\Delta_j, \Delta_i) = 0$ for all $j \geq i$.

Note that the Δ_i, $i = 1, \ldots, 4$, depend on the labelling chosen for the quiver.

A not so interesting labelling is the following:

$$\begin{tikzcd}[row sep=tiny, column sep=small]
2 \arrow[dr] & \\
3 \arrow[r] & 1 \\
4 \arrow[ur] &
\end{tikzcd}$$

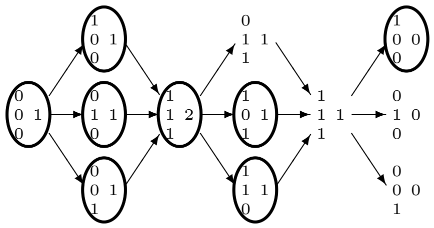
Then $\Delta_i = S_i$ for all $i = 1, \ldots, 4$, and $\mathcal{F}(\Delta) = \operatorname{mod} A$. Clearly, every module has a right $\mathcal{F}(\Delta)$-approximation, if we just take the identity map on it.

A more interesting case is the following:

$$\begin{tikzcd}[row sep=tiny, column sep=small] 1 \arrow[dr] & \\ 3 \arrow[r] & 2 \\ 4 \arrow[ur] & \end{tikzcd}$$

Then $\Delta_1 = S_1$ and $\Delta_i = P_i$ for $i = 2, 3, 4$.

The following diagram shows the indecomposable modules in $\mathcal{F}(\Delta)$:



Clearly, $\begin{smallmatrix}0&\\0&1\\0&\end{smallmatrix}$, $\begin{smallmatrix}0&\\1&1\\0&\end{smallmatrix}$, $\begin{smallmatrix}0&\\0&1\\1&\end{smallmatrix}$, and $\begin{smallmatrix}1&\\0&0\\0&\end{smallmatrix}$ lie in $\mathcal{F}(\Delta)$ (as modules in Δ).

Also $\begin{smallmatrix}1&\\0&1\\0&\end{smallmatrix}$ (as an extension of $\begin{smallmatrix}1&\\0&0\\0&\end{smallmatrix}$ by $\begin{smallmatrix}0&\\0&1\\0&\end{smallmatrix}$), $\begin{smallmatrix}1&\\1&2\\1&\end{smallmatrix}$ (as an extension of $\begin{smallmatrix}1&\\0&0\\0&\end{smallmatrix}$ by $\begin{smallmatrix}0&\\1&1\\0&\end{smallmatrix}$ and $\begin{smallmatrix}0&\\0&1\\1&\end{smallmatrix}$), $\begin{smallmatrix}1&\\0&1\\1&\end{smallmatrix}$ (as an extension of $\begin{smallmatrix}1&\\0&0\\0&\end{smallmatrix}$ by $\begin{smallmatrix}0&\\0&1\\1&\end{smallmatrix}$), and $\begin{smallmatrix}1&\\1&1\\0&\end{smallmatrix}$ (as an extension of $\begin{smallmatrix}1&\\0&0\\0&\end{smallmatrix}$ by $\begin{smallmatrix}0&\\1&1\\0&\end{smallmatrix}$) lie in $\mathcal{F}(\Delta)$.

The other indecomposables do *not* lie in $\mathcal{F}(\Delta)$.

We are going to construct a right $\mathcal{F}(\Delta)$-approximation for $\begin{smallmatrix}0&\\1&0\\0&\end{smallmatrix}$.

According to the second Lemma (from the last lecture), we should try to find a surjective map from a module $M \in \mathcal{F}(\Delta)$ onto $\begin{smallmatrix}0&\\1&0\\0&\end{smallmatrix}$.

We could e.g. choose $M = \begin{smallmatrix}1&\\1&2\\1&\end{smallmatrix}$. There is a non-zero map, say π from $\begin{smallmatrix}1&\\1&2\\1&\end{smallmatrix}$ to $\begin{smallmatrix}0&\\1&0\\0&\end{smallmatrix}$, which is also surjective.

The Lemma shows us how to construct a right $\mathcal{F}(\Delta)$-approximation of $\begin{smallmatrix}0&\\1&0\\0&\end{smallmatrix}$, namely in the following way:

We take the kernel K of π, and we should try to find a short exact sequence $0 \to K \to Y_K \to X_K \to 0$ with $X_K \in \mathcal{F}(\Delta)$ and $Y_K \in \operatorname{mod} A$ such that $\operatorname{Ext}^1(\Delta_i, Y_K) = 0$ for all $i = 1, \ldots, 4$. (Then a push out and its induced exact sequence give us the desired right $\mathcal{F}(\Delta)$-approximation of $\begin{smallmatrix}0&\\1&0\\0&\end{smallmatrix}$.)

Let us construct the sequence $0 \to K \to Y_K \to X_K \to 0$ by the method from the third Lemma (from the last lecture).

We have that $K = \begin{smallmatrix}1&\\0&1\\0&\end{smallmatrix} \oplus \begin{smallmatrix}0&\\0&1\\1&\end{smallmatrix}$.

We find that $\operatorname{Ext}^1(\Delta_i, K) = 0$ for all $i = 4, 3, 2$. Also, there is no extension of $\Delta_1 = \begin{smallmatrix}1&\\0&0\\0&\end{smallmatrix}$ by $\begin{smallmatrix}1&\\0&1\\0&\end{smallmatrix}$. But there is an extension $0 \to \begin{smallmatrix}0&\\0&1\\1&\end{smallmatrix} \xrightarrow{h} \begin{smallmatrix}1&\\0&1\\1&\end{smallmatrix} \to \begin{smallmatrix}1&\\0&0\\0&\end{smallmatrix} \to 0$

We take the induced exact sequence $0 \to \begin{smallmatrix}1&\\0&1\\0&\end{smallmatrix} \oplus \begin{smallmatrix}0&\\0&1\\1&\end{smallmatrix} \xrightarrow{\begin{pmatrix} \mathrm{id} & 0 \\ 0 & h \end{pmatrix}} \begin{smallmatrix}1&\\0&1\\0&\end{smallmatrix} \oplus \begin{smallmatrix}1&\\0&1\\1&\end{smallmatrix} \to \begin{smallmatrix}1&\\0&0\\0&\end{smallmatrix} \to 0.$

The right hand side $\Delta_1 = \begin{smallmatrix}1&\\0&0\\0&\end{smallmatrix}$ is clearly filtered by Δ-modules, and there are no extensions of Δ-modules by $\begin{smallmatrix}1&\\0&1\\0&\end{smallmatrix} \oplus \begin{smallmatrix}1&\\0&1\\1&\end{smallmatrix}$. So this is the required sequence for an application of the second Lemma.

Taking now the push out of the diagram

$$\begin{tikzcd}
\begin{smallmatrix}1&\\0&1\\0&\end{smallmatrix} \oplus \begin{smallmatrix}0&\\0&1\\1&\end{smallmatrix} \arrow[r] \arrow[d] & \begin{smallmatrix}1&\\0&1\\0&\end{smallmatrix} \oplus \begin{smallmatrix}1&\\0&1\\1&\end{smallmatrix} \\
\begin{smallmatrix}1&\\1&2\\1&\end{smallmatrix} &
\end{tikzcd},$$

we obtain the induced exact sequence $0 \to \begin{smallmatrix}1&\\0&1\\0&\end{smallmatrix} \oplus \begin{smallmatrix}1&\\0&1\\1&\end{smallmatrix} \to \begin{smallmatrix}0&\\1&1\\0&\end{smallmatrix} \oplus \begin{smallmatrix}1&\\0&1\\1&\end{smallmatrix} \xrightarrow{f} \begin{smallmatrix}0&\\1&0\\0&\end{smallmatrix} \to 0$, where the right $\mathcal{F}(\Delta)$-approximation of $\begin{smallmatrix}0&\\1&0\\0&\end{smallmatrix}$ is given by the map f.