Selected topics in representation theory – Tilting modules –

WS 2005/06

1 Definition

Let A be a finite dimensional k-algebra (k a field) of finite global dimension.[1] Denote by $\operatorname{mod} A$ the category of finite dimensional left A-modules. Every ${}_AM \in \operatorname{mod} A$ is also a right $\operatorname{End}({}_AM)$-module.

Definition. A module ${}_AT \in \operatorname{mod} A$ is called a *partial tilting module* if

1. $\operatorname{pd}_A T \leq 1$ and
2. $\operatorname{Ext}^1({}_AT, {}_AT) = 0$.

It is called a *tilting module* if, in addition, the following condition holds:

3. There exists a short exact sequence $0 \to_A A \to_A T' \to_A T'' \to 0$ with ${}_AT', {}_AT'' \in \operatorname{add}({}_AT)$.

(Here, $\operatorname{add}({}_AT)$ denotes the modules which are direct summands of sums of ${}_AT$.)

Remark. It can be shown that, if ${}_AT$ is a partial tilting module and A is basic, then 3. is equivalent to the following condition:

3.′ ${}_AT$ has exactly n indecomposable direct summands, where $n = \operatorname{rk} K_0(A)$, the rank of the Grothendieck group of A (=number of isomorphism classes of simple A-modules).

There are also *generalisations* of tilting modules, cf. the next chapter.

One of the aims of tilting theory is to obtain new algebras from given algebras keeping (some of) the structure of their module categories. We can construct a new algebra B from an algebra A by taking a tilting module ${}_AT \in \operatorname{mod} A$ and letting $B = \operatorname{End}({}_AT)$.

2 Brenner-Butler Theorem

Let A be a finite dimensional algebra of finite global dimension, and ${}_AT$ be an r*-tilting module*, i.e.

1. $\operatorname{pd}_A T \leq r < \infty$,

[1] The finite global dimension is used in order to have a triangle equivalence of the bounded derived category $\mathcal{D}^b(A)$ with the homotopy category $\mathcal{K}^b({}_A\mathcal{P})$ of bounded complexes of projective A-modules (see [4, Chapter I, Section 3.3].

2. $\mathrm{Ext}^i({}_AT, {}_AT) = 0$ for all $i > 0$, and

3. there exists an exact sequence $0 \to_A A \to_A T^{(0)} \to \cdots \to_A T^{(s)} \to 0$ with ${}_AT^{(j)} \in \mathrm{add}({}_AT)$ for all $j = 0, \ldots, s$.

One can show that we can always find a coresolution with $s \leq r$. (This has been only stated in the lecture, but here I will give a proof.)

Lemma. *Let ${}_AT$ be an r-tilting module. Then there is an exact sequence $0 \to_A A \to_A T^{(0)} \to \cdots \to_A T^{(s)} \to 0$ with ${}_AT^{(j)} \in \mathrm{add}({}_AT)$ for all $j = 0, \ldots, s$ and $s \leq r$.*

Proof. By assumption, there is an exact sequence $0 \to_A A \to_A T^{(0)} \xrightarrow{d^0} \cdots \xrightarrow{d^{s-1}}{}_A T^{(s)} \to 0$. Choose s minimal, and set $K^i = \ker d^i$ for $0 \leq i \leq s-1$. We get that $\mathrm{Ext}^j(T, K^{i+1}) = \mathrm{Ext}^{j+1}(T, K^i)$ for $j \geq 1$. If $s > r$, then $\mathrm{Ext}^1(T, K^{s-1}) = 0$, therefore d^{s-1} is a retraction, contradicting the minimality of s. □

Let ${}_AM \in \mathrm{mod}\, A$, $B = \mathrm{End}({}_AM)$. We can consider ${}_AM$ as an A-B-bimodule. The dual $D({}_AM_B)$ is in fact a B-A-bimodule.

We have a canonical ring homomorphism $A \to \mathrm{End}(D(M_B))$. (For tilting modules, this is even an isomorphism as the next lemma will show.)

Theorem. *There is a pair of adjoint functors $F = \mathrm{Hom}_A({}_AT, -)$ and $G = T \otimes_B -$ between $\mathrm{mod}\, A$ and $\mathrm{mod}\, B$ inducing inverse triangle equivalences between the bounded derived categories $\mathcal{D}^b(A)$ and $\mathcal{D}^b(B)$, see e.g. [4, Chapter III, Section 2].*

The following lemma shows that we obtain a tilting module for the algebra B by considering T over its endomorphism ring.

Lemma. *If ${}_AT$ is an r-tilting module, then $D(T_B)$ satisfies the following conditions:*

1. $\mathrm{id}\, D(T_B) \leq r$,

2. $\mathrm{Ext}^i_B(D(T_B), D(T_B)) = 0$ *for all* $i > 0$, *and*

3. *there is an exact sequence* $0 \to_B U^{(s)} \to \cdots \to_B U^{(0)} \to D(B_B) \to 0$ *with* ${}_BU^{(j)} \in \mathrm{add}(D(T_B))$ *for all* $j = 0, \ldots, s$.

Moreover, $A \xrightarrow{\sim} \mathrm{End}(D(T_B))$ *canonically.*

Proof. 1. Let us show that $\mathrm{pd}\, T_B \leq r$:

Using the third condition for ${}_AT$ being an r-tilting module, the previous lemma shows that there is a finite coresolution of ${}_AA$ by modules in $\mathrm{add}({}_AT)$ of length $s \leq r$. Apply now $\mathrm{Hom}_A(-, {}_AT)$ to this resolution. Note that all $\mathrm{Hom}_A({}_AT^{(j)}, {}_AT)$ are projective as B-modules.

2. Use

$$D(T_B) = \mathrm{Hom}_k(T_B, k) \cong \mathrm{Hom}_k(A \otimes_A T_B, k) \cong \mathrm{Hom}_A({}_AT_B, \mathrm{Hom}_k(A_A, k))$$

$$= \mathrm{Hom}_A({}_AT_B, D(A_A)) = F(D(A_A)).$$

Using the triangular equivalence of $\mathcal{D}^b(A)$ and $\mathcal{D}^b(B)$ via F, we obtain:

$$\mathrm{Ext}^i_B(D(T_B), D(T_B)) \cong \mathrm{Hom}_{\mathcal{D}^b(B)}(D(T_B), \Sigma^i(D(T_B)))) = \mathrm{Hom}_{\mathcal{D}^b(A)}(D(A_A), \Sigma^i(D(A_A)))$$

$$= \mathrm{Ext}^i_A(D(A_A), D(A_A)) = \begin{cases} 0 \text{ for } i > 0, \text{ since } D(A_A) \text{ is injective, and} \\ \mathrm{Hom}_A(D(A_A), \mathrm{Hom}_k(A_A, k)) \cong \mathrm{Hom}_k(D(A_A) \otimes_A A, k) \\ \cong \mathrm{Hom}_k(D(A_A), k) \cong A_A \text{ for } i = 0. \end{cases}$$

3. Take a projective resolution of ${}_AT$, say

$$0 \to_A P^{(r)} \to \cdots \to_A P^{(0)} \to_A T \to 0,$$

and apply $\mathrm{Hom}_A(-,_A T)$. This leads to a finite coresolution

$$0 \to \mathrm{End}_A({}_AT) \to \mathrm{Hom}_A({}_AP^{(0)},_A T) \to \cdots \to \mathrm{Hom}_A({}_AP^{(r)},_A T) \to 0$$

of $\mathrm{End}({}_AT) = B$. The resolution stops because $\mathrm{Ext}^i_A({}_AT,_A T) = 0$ for all $i > 0$, and $\mathrm{Hom}_A({}_AP^{(0)},_A T) \in \mathrm{add}(T_B)$ by the isomorphism $\mathrm{Hom}_A({}_AA,_A T) \cong T_B$. Dualise the resolution.

□

Let now i be a non-negative integer. We set

$$\mathcal{E}_i := \{{}_AX \in \mathrm{mod}\, A \mid \mathrm{Ext}^j_A({}_AT,_A X) = 0 \;\forall j \neq i\},$$

and

$$\mathcal{T}_i := \{{}_BY \in \mathrm{mod}\, B \mid \mathrm{Tor}^j_B(T_B,_B Y) = 0 \;\forall j \neq i\}.$$

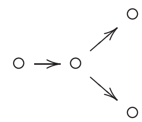
These are full subcategories.

The functors $\mathrm{Ext}^i_A({}_AT, -) : \mathrm{mod}\, A \to \mathrm{mod}\, B$ and $\mathrm{Tor}^i_B(T_B, -) : \mathrm{mod}\, B \to \mathrm{mod}\, A$ are denoted by F^i, G^i resp.

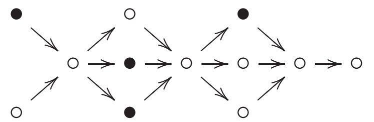
Theorem (Brenner-Butler). *The categories $\mathcal{E}_i$ and $\mathcal{T}_i$ are equivalent under the restrictions of the functors F^i and G^i.*

Proof. We have the triangle equivalence of $\mathcal{D}^b(A)$ and $\mathcal{D}^b(B)$ given by the corresponding functors F and G in the derived categories, and $F|_{\mathcal{E}_i} = \Sigma^{-i} F^i|_{\mathcal{E}_i}$, and $G|_{\mathcal{T}_i} = \Sigma^i G^i|_{\mathcal{T}_i}$. Consider $\mathcal{E}_i$ and $\mathcal{T}_i$ as full subcategories of the derived categories. $F^i|_{\mathcal{E}_i}$ has values in $\mathcal{T}_i$, and $G^i|_{\mathcal{T}_i}$ has values in $\mathcal{E}_i$. □

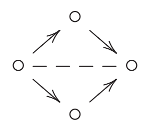
3 Examples

Take the path algebra of the quiver:  .

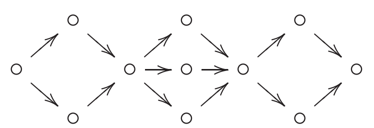
The Auslander-Reiten quiver has the following shape:



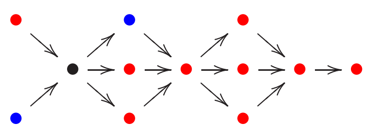
Take the sum of the modules corresponding to the dark dots as a tilting module.

The tilted algebra has the following quiver:  .

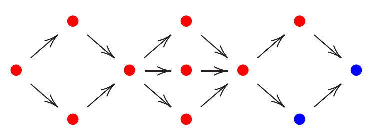
As AR-quiver we get the following:



We identify now the subcategories $\mathcal{E}_0$, $\mathcal{E}_1$ in mod A



and $\mathcal{T}_0$, $\mathcal{T}_1$ in mod B



References

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[2] K. Bongartz, *Tilted algebras.* Representations of algebras (Puebla, 1980), pp. 26–38, Lecture Notes in Math., 903, Springer, Berlin-New York, 1981.

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