INDUCED MODULES: FIRST PROPERTIES OF DEFECT GROUPS

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Let k be a field of characteristic $p > 0$. If G is a finite group, then the group algebra kG has a block decomposition

$$kG = \mathcal{B}_1 \oplus \mathcal{B}_2 \oplus \cdots \oplus \mathcal{B}_r,$$

where each block $\mathcal{B}_i \trianglelefteq kG$ is an indecomposable two-sided ideal. Equivalently, each block $\mathcal{B} \subseteq kG$ is an indecomposable $kG \otimes_k kG^{\rm op}$-module. Since the map

$$(g, h) \mapsto g \otimes h^{-1}$$

induces an isomorphism $k(G \times G) \longrightarrow kG \otimes_k kG^{\rm op}$ of associative k-algebras, the latter condition amounts to $\mathcal{B}$ being an indecomposable submodule of the $(G\times G)$-module kG, relative to the action

$$(g, h).x := gxh^{-1} \qquad \forall\ g, h \in G,\ x \in kG.$$

One can thus speak of the vertex of the $(G \times G)$-module $\mathcal{B}$, see [4] for the definition.

Let $\Delta : G \longrightarrow G \times G\ ;\ g \mapsto (g, g)$ be the diagonal embedding, whose induced algebra homomorphism $kG \longrightarrow k(G \times G)$ will also be denoted Δ.

Definition. Let $\mathcal{B} \subseteq kG$ be a block. A p-subgroup $D \subseteq G$ is called a *defect group of* $\mathcal{B}$ if $\Delta(D)$ is a vertex of the $(G \times G)$-module $\mathcal{B}$. If $\mathrm{ord}(D) = p^d$, then d is called the *defect* of $\mathcal{B}$.

The name defect derives from an early result of the theory, which states that a block $\mathcal{B} \subseteq kG$ is semi-simple (and hence simple) if and only if $d = 0$. Thus, d may be viewed as a measure for the deviation of $\mathcal{B}$ from being semi-simple.

Defects were first defined by Brauer [1], with the definition of a defect group following shortly thereafter [2]. In his seminal articles [1, 2, 3] Brauer established important properties of defect groups that were later reformulated by Green [6, 7], whose approach is the basis of our exposition.

Recall that G acts on k via

$$g.\alpha = \alpha \qquad \forall\ g \in G,\ \alpha \in k.$$

Our first result establishes the existence of defect groups and shows that the defect of a block is well-defined.

Theorem 1. *Let $\mathcal{B} \subseteq kG$ be a block of kG.*

(1) *$\mathcal{B}$ possesses a defect group $D \subseteq G$.*

(2) *If $D, D' \subseteq G$ are defect groups of $\mathcal{B}$, then there exists $g \in G$ with $D' = gDg^{-1}$.*

Proof. (1) We consider $k(G \times G)$ as a left and right G-module via Δ. The bilinear map

$$\varphi : k(G \times G) \times k \longrightarrow kG\ \ ;\ \ ((g, h), \alpha) \mapsto \alpha gh^{-1}$$

is kG-balanced: Given $x \in G$, we have

$$\varphi((g, h).x, \alpha) = \varphi((gx, hx), \alpha) = \alpha gh^{-1} = \varphi((g, h), x.\alpha).$$

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Hence there exists a surjective, k-linear map

$$\psi : k(G \times G) \otimes_{kG} k \longrightarrow kG \;\; ; \;\; (g,h) \otimes \alpha \mapsto \alpha g h^{-1},$$

which is readily seen to be $k(G \times G)$-linear. Since both spaces involved have dimension $\operatorname{ord}(G)$, ψ is in fact an isomorphism, so that kG is a relatively $\Delta(G)$-projective $k(G \times G)$-module. Being a direct summand of kG, the block $\mathcal{B}$ enjoys the same property. According to [4, Prop.4] there exists a p-subgroup $D \subseteq G$ such that $\Delta(D)$ is a vertex of $\mathcal{B}$.

(2) Let D, D' be defect groups of $\mathcal{B}$. Owing to [4, Prop.4], there exists an element $(g,h) \in G \times G$ such that

$$\Delta(D') = (g,h)\Delta(D)(g,h)^{-1},$$

whence $D' = gDg^{-1}$. □

We would like to relate the defect group of a block to the vertices of its indecomposable modules. This necessitates the following subsidiary result, which shows that induction commutes with taking tensor products over k. Recall that the tensor product $M \otimes_k N$ of two G-modules obtains the structure of a G-module via

$$g\bullet(m \otimes n) := g.m \otimes g.n$$

for all $g \in G$, $m \in M$ and $n \in N$.

Lemma 2 (Tensor Identity)**.** *Let $H \subseteq G$ be a subgroup of the finite group G. If V is a finite-dimensional G-module and M is a finite-dimensional H-module, then we have an isomorphism*

$$kG \otimes_{kH} (M \otimes_k V|_H) \cong (kG \otimes_{kH} M) \otimes_k V$$

of G-modules.

Proof. Given $g \in G$, we consider the k-linear map

$$\lambda_g : M \otimes_k V \longrightarrow (kG \otimes_{kH} M) \otimes_k V \;\; ; \;\; m \otimes v \mapsto (g \otimes m) \otimes g.v$$

If $a = \sum_{g \in G} \alpha_g g$ is an element of kG, we define $\lambda_a := \sum_{g\in G} \alpha_g \lambda_g$. There results a bilinear map

$$\psi : kG \times (M \otimes_k V) \longrightarrow (kG \otimes_{kH} M) \otimes_k V \;\; ; \;\; (a,x) \mapsto \lambda_a(x).$$

Since $\lambda_{ah}(x) = \lambda_a(hx)$ for all $a \in kG$, $h \in H$ and $x \in M \otimes_k V$, the map ψ is kH-balanced and there exists a k-linear map

$$\omega : kG \otimes_{kH} (M \otimes_k V) \longrightarrow (kG \otimes_{kH} M) \otimes_k V \;\; ; \;\; a \otimes x \mapsto \lambda_a(x).$$

This map is actually kG-linear: Let $g, g' \in G$, $m \in M$ and $v \in V$. Then we have

$$\begin{aligned} \omega(g'\bullet(g \otimes (m \otimes v))) &= \omega(g'g \otimes (m \otimes v)) = (g'g \otimes m) \otimes g'g.v = g'\bullet((g \otimes m) \otimes g.v) \\ &= g'\bullet\omega(g \otimes (m \otimes v)). \end{aligned}$$

Directly from the definition, we obtain the surjectivity of ω. Since both G-modules involved have dimension $|G/H|(\dim_k M)(\dim_k V)$, the map ω is bijective. □

Recall that any block $\mathcal{B} \subseteq kG$ is of the form $\mathcal{B} = kGe$, where $e \in kG$ is a central, primitive idempotent of kG. Given an indecomposable kG-module M, we thus have $e.M = (0)$ or $e.M = M$. In the latter case, we say that M *belongs* to $\mathcal{B}$.

Theorem 3. *Let $\mathcal{B} \subseteq kG$ be a block with defect group D. Then every indecomposable kG-module M belonging to $\mathcal{B}$ has a vertex $D_M \subseteq D$.*

Proof. We let G act on kG via conjugation, i.e.,

$$g.a := gag^{-1} \qquad \forall\, a \in kG,\, g \in G.$$

Note that this amounts to pulling back the $(G \times G)$-action on kG along Δ. Since $\mathcal{B} \trianglelefteq kG$ is a two-sided ideal, $\mathcal{B} \subseteq kG$ is a G-submodule relative to this operation. The multiplication

$$\mu : \mathcal{B} \otimes_k M \longrightarrow M \;\;;\;\; b \otimes m \mapsto bm$$

is a homomorphism of G-modules: Given $g \in G, b \in \mathcal{B}$ and $m \in M$, we have

$$\mu(g(b \otimes m)) = \mu(g.b \otimes gm) = \mu(gbg^{-1} \otimes gm) = gbg^{-1}gm = g(bm) = g\mu(b \otimes m).$$

Let $e \in kG$ be the central primitive idempotent of $\mathcal{B}$, so that $\mathcal{B} = kGe$. Then

$$\iota : M \longrightarrow \mathcal{B} \otimes_k M \;\;;\;\; m \mapsto e \otimes m$$

is a homomorphism of G-modules. Since M belongs to $\mathcal{B}$, we obtain $\mu \circ \iota = \mathrm{id}_M$, so that M is a direct summand of $\mathcal{B} \otimes_k M$.

As $D \subseteq G$ is a defect group of $\mathcal{B}$, the G-module $\mathcal{B}$ is relatively D-projective. Consequently, $\mathcal{B}$ is a direct summand of $kG \otimes_{kD} \mathcal{B}|_D$. In view of Lemma 2, the tensor product $\mathcal{B} \otimes_k M$ is a direct summand of $(kG \otimes_{kD} \mathcal{B}|_D) \otimes_k M \cong kG \otimes_{kD} (\mathcal{B}|_D \otimes_k M|_D)$. By the above, this implies that M is relatively D-projective, so that D contains a vertex of M, cf. [4, Prop.4]. □

There exists exactly one block $\mathcal{B}_0(G) \subseteq kG$ to which the trivial G-module k belongs. The block $\mathcal{B}_0(G)$ is customarily referred to as the *principal block*. The following result shows why $\mathcal{B}_0(G)$ is thought of as being the "most complicated" block of kG:

Corollary 4. *Every defect group $D \subseteq G$ of the principal block $\mathcal{B}_0(G)$ is a Sylow-p-subgroup of G.*

Proof. Owing to Theorem 3, D contains a vertex D' of the trivial module k. Being a p-group, D' is contained in a Sylow-p-subgroup $P \subseteq G$. As k is relatively D'-projective, k is a summand of $kG \otimes_{kD'} k$. By Mackey's Theorem [4], the trivial P-module $k|_P$ is a summand of

$$\bigoplus_{PgD'} kP \otimes_{k(P \cap D'^g)} k^g = \bigoplus_{PgD'} kP \otimes_{k(P \cap D'^g)} k,$$

where $D'^g := gD'g^{-1}$. Repeated application of Green's Indecomposability Theorem [5] (to a chain of normalizers in P starting with $\mathrm{Nor}_P(P \cap D'^g)$) implies that each summand is an indecomposable kP-module.[1] The Theorem of Krull-Remak-Schmidt now ensures that $k|_P$ is isomorphic to one of these summands. Hence there exists an element g with $P = D'^g$, so that $P = D'$. □

Corollary 5. *Let $\mathcal{B} \subseteq kG$ be a block with defect group D.*
(1) *If D is cyclic, then $\mathcal{B}$ has finite representation type.*
(2) *If $D = \{1\}$, then $\mathcal{B}$ is simple.*

Proof. Suppose that $\mathrm{ord}(D) = p^r$. As D is cyclic, the group algebra $kD \cong k[X]/(X^{p^r})$ has finite representation type, with indecomposable modules $N_1, \ldots, N_{p^r}$. In view of Theorem 3, every indecomposable $\mathcal{B}$-module is relatively D-projective, and hence a direct summand of some $kG \otimes_{kD} N_i$. Consequently, there are only finitely many isomorphism classes of such modules. If $D = \{1\}$, then each indecomposable $\mathcal{B}$-module M is a direct summand of $kG \otimes_k k \cong kG$ and is thus projective. This implies that $\mathcal{B}$ is simple. □

[1] This argument actually shows that induction functors of p-groups preserve indecomposables. In our situation, Frobenius reciprocity gives $\mathrm{Hom}_{kP}(kP \otimes_{k(P \cap D'^g)} k, k) \cong \mathrm{Hom}_{k(P \cap D'^g)}(k, k)$, which, in view of kP being local, implies that the top of the induced module is simple.

Remark. The converse statements of (1) and (2) of Corollary 5 also hold, but their proofs necessitate the so-called Brauer correspondence of blocks.

References

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