Selected topics in representation theory

– Families of representations of quivers –

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1 Kac's Theorem

Let Q be a quiver without oriented cycles and k an algebraically closed field. If the path algebra kQ is not representation finite, then it is well known there are dimension vectors of Q for which there exist families of representations.

Let $\mathbf{x} = (x_i)_{i\in Q_0} \in \mathbb{Z}^{Q_0}$. For a quiver $Q = (Q_0, Q_1, s, t)$ without loops we have reflections $r_i : \mathbb{Z}^{Q_0} \to \mathbb{Z}^{Q_0}$, $i \in Q_0$, which are defined by $r_i(\mathbf{x}) := (r_i(\mathbf{x})_j)_{j\in Q_0}$ with

$$r_i(\mathbf{x})_j = x_j \text{ for } j \neq i, \text{ and } r_i(\mathbf{x})_i = -x_i + \sum_{j\in \mathrm{adj}(i)} x_j,$$

where $\mathrm{adj}(i)$ is the set of vertices adjacent to i.

Let $W := W_Q := \langle r_i \mid i \in Q_0\rangle$ be the subgroup of $\mathrm{Aut}(\mathbb{Z}^{Q_0})$ generated by the reflections.

Let $(-,-) : \mathbb{Z}^{Q_0} \times \mathbb{Z}^{Q_0} \to \mathbb{Z}$ denote the symmetric bilinear form corresponding to the Tits form of Q. It is given by

$$(\mathbf{d_1}, \mathbf{d_2}) = 2 \cdot \sum_{i\in Q_0} d_{1i}d_{2i} - \sum_{\alpha\in Q_1} d_{1,s(\alpha)}d_{2,t(\alpha)} - \sum_{\alpha\in Q_1} d_{2,s(\alpha)}d_{1,t(\alpha)}$$

for $\mathbf{d_1} = (d_{1i})_{i\in Q_0}, \mathbf{d_2} = (d_{2i})_{i\in Q_0} \in \mathbb{Z}^{Q_0}$.

By $\Pi_Q := \{\mathbf{e_i} \mid i \in Q_0\}$ we denote the set of *simple roots* for Q. Here, $\mathbf{e_i} = (e_{ij})_{j\in Q_0} \in \mathbb{Z}^{Q_0}$ with $e_{ij} = \delta_{ij}$.

We have the *fundamental region* associated with Q:

$$F_Q := \{\mathbf{d} \in \mathbb{N}_0^{Q_0}\backslash\{0\} \mid (\mathbf{d}, \mathbf{e_i}) \leq 0 \ \forall i \in Q_0 \text{ and } \mathbf{d} \text{ has connected support}\}.$$

In [4] Kac gave a description of the *(positive) root system* $\Delta_+(Q)$ assigned to a quiver Q in purely combinatorial terms:

$$\Delta_+(Q) = \Delta_+^{\mathrm{re}}(Q) \,\dot\cup\, \Delta_+^{\mathrm{im}}(Q),$$

where $\Delta_+^{\mathrm{re}}(Q) = W\Pi_Q \cap \mathbb{N}_0^{Q_0}$ and $\Delta_+^{\mathrm{im}}(Q) = WF_Q$.

Let $\mu_{\mathbf{d}}(Q)$ denote the maximal number of parameters on which a family of indecomposable representations of Q over an algebraically closed field with dimension vector $\mathbf{d}$ depends.

In [5, Theorem C] Kac has shown the following (cf. also [6, Theorem § 1.10]), which is a generalisation and an extension of Gabriel's theorem in [2]:

Theorem 1 (Kac)**.** *Let $\mathbf{d} \in \mathbb{N}_0^{Q_0}$ be a dimension vector of representations of a quiver Q without loops and K be an algebraically closed field.*

a) There is an indecomposable representation over K with dimension vector $\mathbf{d}$ if and only if $\mathbf{d} \in \Delta_+(Q)$.

b) If $\mathbf{d} \in \Delta_+^{\mathrm{re}}(Q)$, there is a unique indecomposable representation over K with dimension vector $\mathbf{d}$.

c) If $\mathbf{d} \in \Delta_+^{\mathrm{im}}(Q)$, then $\mu_{\mathbf{d}}(Q) = 1 - q(\mathbf{d})$. Furthermore, there is a unique $\mu_{\mathbf{d}}(Q)$-parameter family of indecomposable representations with dimension vector $\mathbf{d}$.

2 A tame phenomenon?

By results of V. Dlab and C. M. Ringel ([1]) we know that for a tame quiver the indecomposable representations are either preprojective or preinjective or regular, and that the regular ones occur in tubes, at most three of which are non-homogeneous.

Consider, for example, the quiver $\tilde{\mathbb{D}}_4$ with subspace orientation. Given the critical dimension vector $\mathbf{d} = \begin{tikzcd} 1 \arrow[dr, dash] & & 1 \arrow[dl, dash] \\ & 2 & \\ 1 \arrow[ur, dash] & & 1 \arrow[ul, dash] \end{tikzcd}$, there is (almost) a one parameter family of indecomposable representations $M_\lambda := \begin{tikzcd} k \arrow[dr, "\binom{1}{0}"] & & k \arrow[dl, "\binom{0}{1}"] \\ & k^2 & \\ k \arrow[ur, "\binom{1}{\lambda}"] & & k \arrow[ul, "\binom{1}{1}"'] \end{tikzcd}$, $\lambda \in k \backslash \{0, 1\}$, all of which lie in homogeneous tubes.

But there are exactly six other isomorphism classes of indecomposable representations for $\mathbf{d}$, namely

$$M_a := \begin{tikzcd} k \arrow[dr, "\binom{1}{0}"] & & k \arrow[dl, "\binom{1}{0}"] \\ & k^2 & \\ k \arrow[ur, "\binom{1}{1}"] & & k \arrow[ul, "\binom{0}{1}"'] \end{tikzcd}, \quad M_b := \begin{tikzcd} k \arrow[dr, "\binom{0}{1}"] & & k \arrow[dl, "\binom{1}{1}"] \\ & k^2 & \\ k \arrow[ur, "\binom{1}{0}"] & & k \arrow[ul, "\binom{1}{0}"'] \end{tikzcd}, \quad M_c := \begin{tikzcd} k \arrow[dr, "\binom{1}{0}"] & & k \arrow[dl, "\binom{0}{1}"] \\ & k^2 & \\ k \arrow[ur, "\binom{1}{1}"] & & k \arrow[ul, "\binom{1}{0}"'] \end{tikzcd},$$

$$M_d := \begin{tikzcd} k \arrow[dr, "\binom{0}{1}"] & & k \arrow[dl, "\binom{1}{0}"] \\ & k^2 & \\ k \arrow[ur, "\binom{1}{0}"] & & k \arrow[ul, "\binom{1}{1}"'] \end{tikzcd}, \quad M_e := \begin{tikzcd} k \arrow[dr, "\binom{1}{0}"] & & k \arrow[dl, "\binom{0}{1}"] \\ & k^2 & \\ k \arrow[ur, "\binom{1}{0}"] & & k \arrow[ul, "\binom{1}{1}"'] \end{tikzcd} \text{ and } M_f := \begin{tikzcd} k \arrow[dr, "\binom{0}{1}"] & & k \arrow[dl, "\binom{1}{0}"] \\ & k^2 & \\ k \arrow[ur, "\binom{1}{1}"] & & k \arrow[ul, "\binom{1}{0}"'] \end{tikzcd}.$$

They do *not* lie in homogeneous tubes, but in tubes of rank 2, in the second "layers":

$$\begin{tikzcd}[column sep=small]
\cdots \arrow[dr] & & \cdots \arrow[dr] & & \cdots \\
& M_a \arrow[ur] \arrow[dr] & & M_b \arrow[ur] \arrow[dr] & \\
L_a \arrow[ur] & & L_b \arrow[ur] & & L_a
\end{tikzcd}
\quad
\begin{tikzcd}[column sep=small]
\cdots \arrow[dr] & & \cdots \arrow[dr] & & \cdots \\
& M_c \arrow[ur] \arrow[dr] & & M_d \arrow[ur] \arrow[dr] & \\
L_c \arrow[ur] & & L_d \arrow[ur] & & L_c
\end{tikzcd}
\quad
\begin{tikzcd}[column sep=small]
\cdots \arrow[dr] & & \cdots \arrow[dr] & & \cdots \\
& M_e \arrow[ur] \arrow[dr] & & M_f \arrow[ur] \arrow[dr] & \\
L_e \arrow[ur] & & L_f \arrow[ur] & & L_e
\end{tikzcd}$$

They do have proper regular subrepresentations, namely of the forms $L_a := \begin{tikzcd}[sep=tiny] k \arrow[dr] & & k \arrow[dl] \\ & k & \\ 0 \arrow[ur] & & 0 \arrow[ul] \end{tikzcd}$,

$L_b := \begin{tikzcd}[sep=tiny] 0 \arrow[dr] & & 0 \arrow[dl] \\ & k & \\ k \arrow[ur] & & k \arrow[ul] \end{tikzcd}$, $L_c := \begin{tikzcd}[sep=tiny] k \arrow[dr] & & 0 \arrow[dl] \\ & k & \\ 0 \arrow[ur] & & k \arrow[ul] \end{tikzcd}$, $L_d := \begin{tikzcd}[sep=tiny] 0 \arrow[dr] & & k \arrow[dl] \\ & k & \\ k \arrow[ur] & & 0 \arrow[ul] \end{tikzcd}$, $L_e := \begin{tikzcd}[sep=tiny] k \arrow[dr] & & 0 \arrow[dl] \\ & k & \\ k \arrow[ur] & & 0 \arrow[ul] \end{tikzcd}$ and $L_f := \begin{tikzcd}[sep=tiny] 0 \arrow[dr] & & k \arrow[dl] \\ & k & \\ 0 \arrow[ur] & & k \arrow[ul] \end{tikzcd}$,
resp., whereas all other representations of dimension vector $\mathbf{d}$ do *not* have any proper regular subrepresentations.

Question: Is this a typical tame phenomenon? Or can this also happen in the wild case?

3 Families of indecomposable representations for wild quivers

In [3], I constructed some families of indecomposable representations explicitly, namely those for the s-hypercritical and the s-tame dimension vectors.

One of the s-tame dimension vectors is

$$\begin{tikzcd}[sep=small]
& 2 \arrow[d, dash] & & \\
1 \arrow[r, dash] & 3 \arrow[r, dash] \arrow[d, dash] & 2 \arrow[r, dash] & 1\ , \\
& 2 & &
\end{tikzcd}$$

which has Tits form 0.

It is possible to construct a one parameter family of indecomposable representations for $\mathbf{d}$ as follows:

First we restrict $\mathbf{d}$ to a dimension vector of a smaller quiver by deleting one vertex and one arrow in Q.

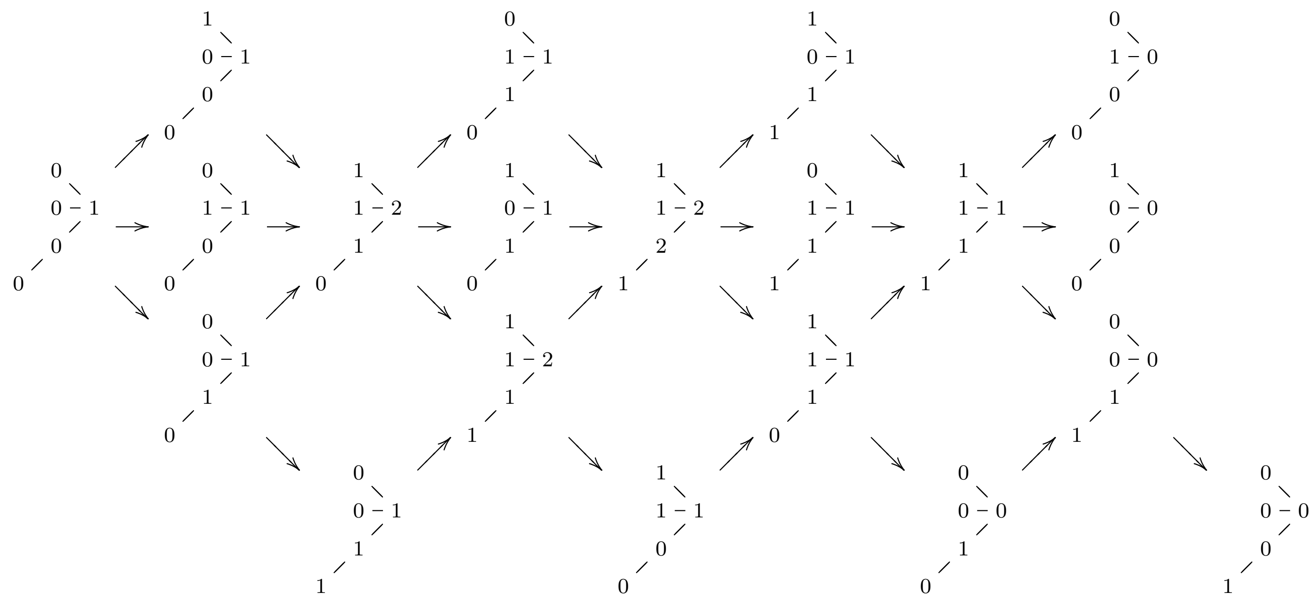
We delete the 1-entry in $\mathbf{d}$ and obtain

$$\mathbf{d}' = \begin{tikzcd}[sep=small]
& 2 \arrow[dr, dash] & \\
& 2 \arrow[r, dash] & 3 \\
& 2 \arrow[ur, dash] & \\
1 \arrow[ur, dash] & &
\end{tikzcd}$$

Now we calculate the canonical decomposition (see [7]) for $\mathbf{d}'$ and create a representation for the smaller quiver according to the canonical decomposition.

In order to find the canonical decomposition, it is useful to have the AR-quiver for a quiver of type $\mathbb{D}_5$ with subspace orientation.

It looks as follows:



The canonical decomposition of

$$\mathbf{d}' = \begin{array}{ccccc} & & 2 & & \\ & & & \diagdown & \\ & & 2 & - & 3 \\ & & & \diagup & \\ & & 2 & & \\ & \diagup & & & \\ 1 & & & & \end{array}$$

is

$$\begin{array}{ccccc} & & 1 & & \\ & & & \diagdown & \\ & & 1 & - & 2 \\ & & & \diagup & \\ & & 2 & & \\ & \diagup & & & \\ 1 & & & & \end{array} \quad + \quad \begin{array}{ccccc} & & 1 & & \\ & & & \diagdown & \\ & & 1 & - & 1 \\ & & & \diagup & \\ & & 0 & & \\ & \diagup & & & \\ 0 & & & & \end{array}$$

We take the (in this case) up to isomorphism uniquely determined indecomposable representations corresponding to the dimension vectors occurring in the canonical decomposition.

Finally we try to find suitable embeddings for the remaining one dimensional vector space providing us a family of indecomposable representations.

We can choose the embeddings as follows:

$$k \xrightarrow{\begin{pmatrix}1\\ \lambda\\ \hline 1\end{pmatrix}} k^2 \oplus k$$

with $\lambda \in k$.

All the representations are indecomposable, since their endomorphism rings are just k:

Since both representations with dimension vectors of the canonical decomposition are indecomposable and have endomorphism rings k and there are no homomorphisms from one representation to the other, we have to consider the following diagram (which has to commute):

$$\begin{tikzcd}
k \arrow[r, "\begin{pmatrix}1\\ \lambda\\ \hline 1\end{pmatrix}"] \arrow[d, "(\alpha)"'] & k^2 \oplus k \arrow[d, "\begin{pmatrix} a & 0 & 0\\ 0 & a & 0\\ 0 & 0 & b\end{pmatrix}"] \\
k \arrow[r, "\begin{pmatrix}1\\ \lambda'\\ \hline 1\end{pmatrix}"'] & k^2 \oplus k
\end{tikzcd}$$

which clearly implies that $\alpha = a = b$ and also $\lambda = \lambda'$ if $\alpha \neq 0$, i. e. the map is non zero.

But we could also choose the decomposition

$$\begin{tikzcd}[column sep=small, row sep=small]
 & 1 \arrow[dr, dash] & \\
 & 1 \arrow[r, dash] & 2 \\
 & 1 \arrow[ur, dash] & \\
1 \arrow[ur, dash] & &
\end{tikzcd}
\quad + \quad
\begin{tikzcd}[column sep=small, row sep=small]
 & 1 \arrow[dr, dash] & \\
 & 1 \arrow[r, dash] & 1 \\
 & 1 \arrow[ur, dash] & \\
0 \arrow[ur, dash] & &
\end{tikzcd}$$

(which is not the canonical decomposition for $\mathbf{d}'$).

Choosing

$$k \xrightarrow{\begin{pmatrix}0\\ 1\\ \hline 1\end{pmatrix}} k^2 \oplus k$$

as an embedding provides us with another indecomposable representation which is *not* isomorphic to any of the other ones.

We consider the following diagram (which has to commute):

$$\begin{tikzcd} k \arrow[r, "\left(\begin{array}{c}0\\1\\\hline 1\end{array}\right)"] \arrow[d, "(\alpha)"'] & k^2 \oplus k \arrow[d, "\begin{pmatrix} a & 0 & c\\ 0 & a & 0\\ 0 & 0 & b\end{pmatrix}"] \\ k \arrow[r, "\left(\begin{array}{c}0\\1\\\hline 1\end{array}\right)"'] & k^2 \oplus k \end{tikzcd}$$

This implies that $\alpha = a = b$ and that $c = 0$.

Furthermore, the subrepresentations are different which can already be seen from the restricted dimension vectors and the AR-quiver of $\mathbb{D}_5$.

So we see that the phenomenon from the second section is not just limited to tame quivers.

For example, the (up to isomorphism) unique indecomposable representation corresponding to

$$\begin{array}{ccccccc} & & & & 1 & & \\ & & & & & \diagdown & \\ & & & & 1 & - & 2 \\ & & & & & \diagup & \\ & & & & 2 & & \\ & & & \diagup & & & \\ & & 1 & & & & \\ & \diagup & & & & & \\ 0 & & & & & & \end{array}$$

is clearly a subrepresentation of each indecomposable representation in the family constructed above, but *not* of the indecomposable representation constructed afterwards.

References

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[7] A. Schofield, *General representations of quivers*, Proc. London Math. Soc. (3) **65** (1992), no. 1, 46–64.