# The $\Delta$-filtered modules without self-extensions for the Auslander algebra of $k[T]/\langle T^n\rangle$

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ABSTRACT. It is well-known that the Auslander algebra of any representation finite algebra is quasi-hereditary. We consider the Auslander algebra $A_n$ of $k[T]/\langle T^n\rangle$ (here, $k$ is a field, $T$ a variable and $n$ a natural number). We determine all $\Delta$-filtered $A_n$-modules without self-extensions; they can be described purely combinatorially. We show, given any $\Delta$-filtered module $N$, there is (up to isomorphism) a unique $\Delta$-filtered module $M$ without self-extensions which has the same dimension vector. In case $k$ is an infinite field, $N$ is a degeneration of this module $M$. In particular, we see that in this case, the set of $\Delta$-filtered modules with a fixed dimension vector is the closure of an open orbit (thus irreducible). As observed in [HiRö], the problem of describing all $\Delta$-filtered $A_n$-modules is the same as that of describing the conjugacy classes of elements in the unipotent radical of a parabolic subgroup $P$ of $\mathrm{GL}(m,k)$ under the action of $P$, thus we recover Richardson's dense orbit theorem in this instance.

## 1. Preliminaries.

Let $k$ be a field. We consider $k$-algebras, their modules, and maps between these modules. All modules are left modules, and all maps are written on the left of the argument. Thus, if $X, Y$ and $Z$ are modules, the composition of homomorphisms $f\colon X \to Y$ and $g\colon Y \to Z$ is denoted by $gf$.

Let $R_n = k[T]/\langle T^n\rangle$, this is a representation finite algebra with precisely $n$ isomorphism classes of indecomposable representations $M(i)$, $1 \le i \le n$, and it is convenient to choose the indexing[1] so that $M(i)$ is of length $n-i+1$.

Let $A_n = \mathrm{End}(\bigoplus_i M(i))$ be the Auslander-algebra of $R_n$; a description of $A_n$ in terms of quivers and relations is given in Section 5 below. Our indexing of the simple $A_n$-modules $E(i)$ corresponds to that of the indecomposables $M(i)$. Thus the indecomposable projective $A_n$-modules $P(i)$ embed into each other as follows

$$P(1) \supset P(2) \supset \cdots \supset P(n-1) \supset P(n).$$

Note that $A_n$ is quasi-hereditary in a unique way, its weight set is the canonically ordered set $\{1, 2, \ldots, n\}$. We denote by $E(i)$ the simple factor module of $P(i)$ and we stress that the module $P(1)$ is just the injective envelope of $E(1)$.

We recall some well-known properties (see Lemma 7.1 of [DlRi]): For $1 \le i \le n$, the standard $A_n$-module $\Delta(i)$ is the serial module of length $i$ with socle $E(1)$. The class $\mathcal{F}(\Delta)$ of all $\Delta$-filtered (or $\Delta$-good) modules is just the set of all modules with socle generated by $E(1)$. In particular, $\mathcal{F}(\Delta)$ is closed under submodules and all projective $A_n$-modules belong to $\mathcal{F}(\Delta)$.

[1] With respect to the modules themselves, it would be more natural to use the actual length as index. However, this would lead to a conflict with respect to the traditional way of using indices for the weight set of a quasi-hereditary algebra: we would have to endow a sequence of natural numbers with the ordering opposite to the natural one.

For any $\Delta$-filtered module $M$, we denote by $(\mathbf{dim}_\Delta M)_i$ the multiplicity of $\Delta(i)$ in a $\Delta$-filtration of $M$ (these multiplicities are uniquely determined, since we deal with the standard modules of a quasi-hereditary algebra). The (ordered) $n$-tuple $\mathbf{dim}_\Delta M$ with coefficients $(\mathbf{dim}_\Delta M)_i$ is said to be the $\Delta$-*dimension vector* of $M$. The $\Delta$-*support* of $M$ is the set of indices $i$ such that $(\mathbf{dim}_\Delta M)_i \neq 0$.

The dual statements are also of interest: For $1 \leq i \leq n$, the costandard $A_n$-module $\nabla(i)$ is the serial module of length $i$ with top $E(1)$. The class $\mathcal{F}(\nabla)$ of all $\nabla$-filtered modules is just the set of all modules with top generated by $E(1)$. Similar to the $\Delta$-dimension vector and the $\Delta$-support of modules in $\mathcal{F}(\Delta)$, there is a notion of $\nabla$-dimension vector and $\nabla$-support for any module in $\mathcal{F}(\nabla)$, using filtrations by costandard modules.

We denote by $T(i)$ the unique indecomposable module[2] which has both a $\Delta$-filtration and a $\nabla$-filtration, one composition factor of the form $E(i)$, and all other composition factors of the form $E(j)$ with $j < i$. Note that $T(n) = P(1)$, this module is both projective and injective.

**Lemma 1.** *The modules in $\mathcal{F}(\Delta)$ are just the $A_n$-modules with projective dimension at most 1.*

Proof: In order to bound the projective dimension of the modules in $\mathcal{F}(\Delta)$, it is sufficient to consider the standard modules. But $\Delta(n) = P(n)$, and for $1 \leq i < n$, a projective resolution of $\Delta(i)$ is given by

$$0 \to P(i+1) \to P(i) \to \Delta(i) \to 0.$$

This shows that the projective dimension of any module in $\mathcal{F}(\Delta)$ is at most 1.

Conversely, assume that $M$ is an $A_n$-module with a projective resolution $0 \to P_1 \to P_0 \to M \to 0$. We want to show that the socle of $M$ is the direct sum of copies of $E(1)$. Let $S$ be a simple submodule of $M$. The inclusion map $S \subseteq M$ yields an induced exact sequence $0 \to P_1 \to X \to S \to 0$, where $X$ is a submodule of $P_0$. If this induced sequence splits, then $S$ is isomorphic to a submodule of $X$, thus of $P_0$, and therefore isomorphic to $E(1)$. If it does not split, then $\operatorname{Ext}^1(S, P_1) \neq 0$, thus there is an indecomposable projective module $P(i)$ with $\operatorname{Ext}^1(S, P(i)) \neq 0$ (and $i > 1$). Note that $P(i)$ embeds into $P(1)$ and that the quotient is just $T(i-1)$. If we apply $\operatorname{Hom}(S, -)$ to the exact sequence $0 \to P(i) \to P(1) \to T(i-1) \to 0$, we obtain an exact sequence

$$\operatorname{Hom}(S, T(i-1)) \to \operatorname{Ext}^1(S, P(i)) \to 0,$$

where we use again that $P(1)$ is injective. But the only simple module $S$ with $\operatorname{Hom}(S, T(i-1)) \neq 0$ is $S = E(1)$. This completes the proof.

## 2. The modules $\Delta(I)$ and $\Delta(\mathbf{d})$.

In this section we define some $\Delta$-filtered modules which fail to admit any self-extension. The indecomposable ones are labelled $\Delta(I)$, where $I$ is a non-empty

[2] Occasionally, these modules are referred to as "tilting modules". However, this is in conflict with the original terminology. The tilting modules we are going to consider in Section 7 are the traditional ones.

subset of $\{1,\ldots,n\}$. We also consider suitable direct sums of such modules $\Delta(I)$: they are denoted by $\Delta(\mathbf{d})$, where $\mathbf{d}=(d_1,\ldots,d_n)$ is an $n$-tuple of non-negative integers.

**Lemma 2.** (1) *A module $M$ is isomorphic to a non-zero submodule of $P(1)$ if and only if $\operatorname{soc} M = E(1)$. Such a module $M$ is indecomposable and belongs to $\mathcal{F}(\Delta)$.*

(2) *The map sending a submodule $M$ of $P(1)$ to its $\Delta$-support affords a bijection between the set of submodules of $P(1)$ and the subsets of $\{1,2,\ldots,n\}$. In particular, $P(1)$ has precisely $2^n$ submodules.*

Proof: The equivalence stated in (1) is due to the fact that $P(1)$ is the injective envelope of $E(1)$. This also implies that any non-zero submodule $M$ of $P(1)$ is indecomposable. Such a module $M$ belongs to $\mathcal{F}(\Delta)$, since all projective $A_n$-modules belong to $\mathcal{F}(\Delta)$ and $\mathcal{F}(\Delta)$ is closed under submodules.

Let $M$ be a submodule of $P(1)$. First consider the case when $(\mathbf{dim}_\Delta M)_n = 0$. Note that $A_{n-1}$ is the factor algebra of $A_n$ modulo the trace ideal of $P(n)$, thus the category of all $A_n$-modules $N$ with $(\mathbf{dim}_\Delta N)_n = 0$ may be identified with the category of all $A_{n-1}$-modules. Of course, in this way $T(n-1)$ is just the unique indecomposable $A_{n-1}$-module which is both projective and injective. By induction, the $\Delta$-support yields a bijection between the submodules of $T(n-1)$ and the subsets of $\{1,2,\ldots,n-1\}$.

Next, let $(\mathbf{dim}_\Delta M)_n \neq 0$, thus $\Delta(n)$ is a submodule of $M$ and we may consider the submodule $M/\Delta(n)$ of $T(n)/\Delta(n)\simeq T(n-1)$. Again, using induction, we see that the $\Delta$-support defines a bijection between the submodules of $T(n)$ which contain $\Delta(n)$ and the subsets $I$ of $\{1,2,\ldots,n\}$ with $n\in I$. This completes the proof.

Dual considerations show: (1) *A module $M$ is isomorphic to a non-zero factor module of $P(1)$ if and only if $\operatorname{top} M = E(1)$. Such a module $M$ is indecomposable and belongs to $\mathcal{F}(\nabla)$.*

(2) *The map sending a submodule $M$ of $P(1)$ to its $\nabla$-support affords a bijection between the set of factor modules of $P(1)$ and the subsets of $\{1,2,\ldots,n\}$.*

**Definition:** If $I$ is a subset of $\{1,2,\ldots,n\}$, let $\Delta(I)$ be the (unique) submodule of $P(1)$ with $\Delta$-support $I$ and $\nabla(I)$ the (unique) factor module of $P(1)$ with $\nabla$-support $I$.

**Remark 1.** *For any subset $I$ of $\{1,2,\ldots,n\}$ with complement $I'$, there is an exact sequence*

$$0\to\Delta(I)\to P(1)\to\nabla(I')\to 0.$$

In order to see this, we note the following: The embedding $\Delta(I)\to P(1)$ has as cokernel a module with top $E(1)$, thus this cokernel is of the form $\nabla(J)$, for some set $J$. Consider dimension vectors:

$$\sum_{i=1}^n \mathbf{dim}\,\Delta(i) = \mathbf{dim}\,P(1) = \mathbf{dim}\,\Delta(I) + \mathbf{dim}\,\nabla(J)$$
$$= \sum_{i\in I}\mathbf{dim}\,\Delta(i) + \sum_{i\in J}\mathbf{dim}\,\nabla(i).$$

Now $\mathbf{dim}\,\nabla(i) = \mathbf{dim}\,\Delta(i)$ for all $i$. Since the dimension vectors $\mathbf{dim}\,\Delta(i)$ are linearly independent, it follows that $J = I'$.

**Lemma 3.** *Let $I$ be a non-empty subset and $j$ an element of $\{1, 2, \ldots, n\}$ such that $i < j$ for each $i$ in $I$. Then $\operatorname{Ext}^1(\Delta(I), \Delta(j))$ is one-dimensional.*

Proof: First, we note that the cokernel $C$ of the embedding $\Delta(j) \to P(1)$ has a submodule $C'$ isomorphic to $T(j-1)$ such that $C/C'$ has a filtration with factors $\nabla(j+1), \ldots, \nabla(n)$. This can be seen by looking at the submodule chain $\Delta(j) \subseteq T(j) \subseteq T(n) = P(1)$; the factor module $T(j)/\Delta(j)$ is isomorphic to $T(j-1)$, the factor module $T(n)/T(j)$ has a filtration with factors $\nabla(j+1), \ldots, \nabla(n)$.

Second, observe that $\operatorname{Hom}(\Delta(I), \nabla(r)) = 0$ for all $r \notin I$, in particular for all $r \geq j$, thus $\operatorname{Hom}(\Delta(I), C) = \operatorname{Hom}(\Delta(I), C')$. Similarly, any homomorphism $\Delta(I) \to P(1)$ factors through the submodule $T(j-1)$, since $P(1)/T(j-1)$ has a filtration by the modules $\nabla(r)$ with $r \geq j$. This shows that the spaces $\operatorname{Hom}(\Delta(I), C)$ and $\operatorname{Hom}(\Delta(I), P(1))$ have the same dimension.

If we apply $\operatorname{Hom}(\Delta(I), -)$ to the exact sequence $0 \to \Delta(i) \to P(1) \to C \to 0$, we obtain an exact sequence

$$\begin{aligned} 0 &\to \operatorname{Hom}(\Delta(I), \Delta(i)) \to \operatorname{Hom}(\Delta(I), P(1)) \to \operatorname{Hom}(\Delta(I), C) \\ &\to \operatorname{Ext}^1(\Delta(I), \Delta(i)) \to 0. \end{aligned}$$

Thus we see that $\operatorname{Ext}^1(\Delta(I), \Delta(i))$ has the same dimension as $\operatorname{Hom}(\Delta(I), \Delta(i))$, but the latter space is one-dimensional.

**Proposition 1.** *Let $M$ be a module in $\mathcal{F}(\Delta)$ with $\Delta$-support in $I$. Then*

$$\operatorname{Ext}^1(M, \Delta(I)) = 0 = \operatorname{Ext}^1(\Delta(I), M).$$

Proof: It is sufficient to consider the case where $M = \Delta(i)$ with $i \in I$. If $i = n$, then $\Delta(n)$ is projective, thus $\operatorname{Ext}^1(\Delta(n), -) = 0$. For $1 \leq i < n$, take the projective resolution of $\Delta(i)$ as exhibited above and apply the functor $\operatorname{Hom}(-, \Delta(I))$. We obtain an exact sequence

$$\begin{aligned} 0 &\to \operatorname{Hom}(\Delta(i), \Delta(I)) \to \operatorname{Hom}(P(i), \Delta(I)) \to \operatorname{Hom}(P(i+1), \Delta(I)) \\ &\to \operatorname{Ext}^1(\Delta(i), \Delta(I)) \to 0. \end{aligned}$$

Using the $\Delta$-filtration of $\Delta(I)$, we see that $\dim \operatorname{Hom}(P(i), \Delta(I))$ equals the number of elements $j \geq i$ in $I$. Similarly, $\dim \operatorname{Hom}(P(i+1), \Delta(I))$ is equal to the number of elements $j > i$ in $I$, thus one less. On the other hand, $\operatorname{Hom}(\Delta(i), P(1))$ is one-dimensional, and all the homomorphisms $\Delta(i) \to P(1)$ map into $\Delta(I)$, thus $\operatorname{Hom}(\Delta(i), \Delta(I))$ is also one-dimensional. This shows that $\operatorname{Ext}^1(\Delta(i), \Delta(I)) = 0$.

An alternative proof of $\operatorname{Ext}^1(\Delta(i), \Delta(I)) = 0$ uses Remark 1: We apply the functor $\operatorname{Hom}(\Delta(i), -)$ to the exact sequence presented there and we obtain an exact sequence

$$\operatorname{Hom}(\Delta(i), \nabla(I')) \to \operatorname{Ext}^1(\Delta(i), \Delta(I)) \to 0.$$

Since $\operatorname{Hom}(\Delta(i), \nabla(j)) = 0$ for all $i \neq j$, we have $\operatorname{Hom}(\Delta(i), \nabla(I')) = 0$ and therefore $\operatorname{Ext}^1(\Delta(i), \Delta(I)) = 0$.

In order to see that $\mathrm{Ext}^1(\Delta(I), \Delta(i)) = 0$, we write $I$ as the disjoint union of the subsets $I' = \{j \in I \mid j \leq i\}$ and $I'' = \{j \in I \mid j > i\}$. There is an exact sequence

$$0 \to \Delta(I'') \to \Delta(I) \to \Delta(I') \to 0,$$

thus we have an exact sequence

$$\mathrm{Ext}^1(\Delta(I'), \Delta(i)) \to \mathrm{Ext}^1(\Delta(I), \Delta(i)) \to \mathrm{Ext}^1(\Delta(I''), \Delta(i)).$$

Clearly, $\mathrm{Ext}^1(\Delta(I''), \Delta(i)) = 0$, since $\mathrm{Ext}^1(\Delta(j), \Delta(i)) = 0$ for $i \leq j$. It remains to be seen that $\mathrm{Ext}^1(\Delta(I'), \Delta(i)) = 0$. Let $J = I' \setminus \{i\}$, and consider the exact sequence

$$0 \to \Delta(i) \to \Delta(I') \to \Delta(J) \to 0;$$

we denote its equivalence class in $\mathrm{Ext}^1(\Delta(J), \Delta(i))$ by $\epsilon$. If we apply $\mathrm{Hom}(-, \Delta(i))$ to this sequence, we obtain an exact sequence

$$\mathrm{Hom}(\Delta(i), \Delta(i)) \xrightarrow{\delta} \mathrm{Ext}^1(\Delta(J), \Delta(i)) \to \mathrm{Ext}^1(\Delta(I'), \Delta(i)) \to 0,$$

since $\mathrm{Ext}^1(\Delta(i), \Delta(i)) = 0$. According to Lemma 3, we know that the vector space $\mathrm{Ext}^1(\Delta(J), \Delta(i))$ is one-dimensional. The connecting homomorphism $\delta$ sends the identity morphism of $\Delta(i)$ to $\epsilon$ and this is a non-zero element; it follows that $\delta$ is surjective. This shows that $\mathrm{Ext}^1(\Delta(I'), \Delta(i)) = 0$.

**Corollary.** *Let $I, J$ be subsets of $\{1, 2, \ldots, n\}$ such that $I \subseteq J$ or $I \supseteq J$. Then*

$$\mathrm{Ext}^1(\Delta(I), \Delta(J)) = 0.$$

**Definition:** For any $\mathbf{d} \in \mathbb{N}_0^n$, we define $\Delta(\mathbf{d})$ inductively as follows: Let $I$ be the support of $\mathbf{d}$ and let $\mathbf{d}_I$ be the characteristic function on $I$. Then $\Delta(I)$ is defined and by induction on the $\Delta$-dimension also $\Delta(\mathbf{d} - \mathbf{d}_I)$. Let $\Delta(\mathbf{d}) = \Delta(I) \oplus \Delta(\mathbf{d} - \mathbf{d}_I)$. We obtain in this way a $\Delta$-filtered module $\Delta(\mathbf{d})$ with $\Delta$-dimension vector $\mathbf{d}$ such that

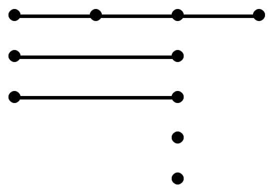
$$\mathrm{Ext}^1(\Delta(\mathbf{d}), \Delta(\mathbf{d})) = 0.$$

In a similar way, we may define the modules $\nabla(\mathbf{d})$.

It is convenient to use the following visualization for $\Delta(\mathbf{d})$. Let $\mathbf{d} = (d_1, \ldots, d_n)$ and $m = \sum_i d_i$. We start with an arrangement of $m$ vertices in $n$ columns, such that the $j$-th column contains precisely $d_j$ vertices. More precisely, using the convention for labelling matrix entries, there is a vertex with row index i and column index j precisely when $1 \leq i \leq d_j$. For example, for $n = 4$ and $\mathbf{d} = (3, 1, 5, 1)$, we work with the following arrangement:

$$\begin{array}{cccc}
\bullet & \bullet & \bullet & \bullet \\
\bullet & & \bullet & \\
\bullet & & \bullet & \\
 & & \bullet & \\
 & & \bullet &
\end{array}$$

Next, we connect these vertices by horizontal edges, whenever possible; in this manner, the rows give rise to the connected components of a graph:



Note that the rows represent the various summands $\Delta(I)$ of $\Delta(\mathbf{d})$; a row with vertices in the columns $i_1, \dots, i_s$ corresponds to a direct summand of the form $\Delta(\{i_1, \dots, i_s\})$. The graph exhibits nicely the direct decomposition of $\Delta(\mathbf{d})$ into indecomposables: In our example $\mathbf{d} = (3, 1, 5, 1)$, we see that

$$\Delta((3,1,5,1)) = \Delta(\{1,2,3,4\}) \oplus \Delta(\{1,3\}) \oplus \Delta(\{1,3\}) \oplus \Delta(\{3\}) \oplus \Delta(\{3\}).$$

(In case some of the $d_i$'s are zero, it may be necessary to add the labelling of the columns, in order to be able to recover $\mathbf{d}$ from the diagram.)

### 3. The $\Delta$-filtered modules without self-extensions

The aim of this section is the following characterization of $\Delta$-filtered modules without self-extensions.

**Theorem 1.** *For a $\Delta$-filtered module $M$ with $\Delta$-dimension vector $\mathbf{d}$, the following conditions are equivalent:*
(i) $\mathrm{Ext}^1(M, M) = 0$.
(ii) *$M$ is isomorphic to $\Delta(\mathbf{d})$.*

The implication (ii) $\Longrightarrow$ (i) is a direct consequence of the last Corollary.

The proof of the implication (i) $\Longrightarrow$ (ii) will be done in four steps. We will work with classes $\mathcal{M}(J, s)$ of $\Delta$-filtered modules, where $J$ is a subset of $\{1, 2, \dots, n\}$ of cardinality $t$, and $s$ is an integer with $1 \le s \le t$. The class $\mathcal{M}(J, s)$ is defined as follows: If $J = \{j_1 < j_2 < \dots < j_t\}$, then $\mathcal{M}(J, s)$ is the class of all $\Delta$-filtered $A_n$-modules $M$ with $\Delta$-support $J$ which have a factor module of the form $\Delta(\{j_1, j_2, \dots, j_s\})$.

(a) *If $M$ belongs to $\mathcal{M}(J, s)$, then either $s = t$ or else there is an extension $0 \to M \to N \to M \to 0$ such that $N$ belongs to $\mathcal{M}(J, s+1)$.*

Proof: Let us consider a module $M$ in $\mathcal{M}(J, s)$, where $s < t$. Denote by $\varphi$ an epimorphism $M \to \Delta(I)$, say with kernel $M'$.

Since $s + 1$ belongs to the $\Delta$-support of $M$, there are submodules $M'' \subset M'$ of $M$ such that $M'/M'' = \Delta(s+1)$ and such that $M/M'$ is $\Delta$-filtered with $\Delta$-support in $I$. Let $\varepsilon\colon M' \to \Delta(s+1)$ be an epimorphism with kernel $M''$. Finally, let $\iota : M' \to M$ be the inclusion map.

Let $I' = I \cup \{j_{s+1}\}$. Since $j_{s+1} > i$ for all $i \in I$, there is an exact sequence $0 \to \Delta(s+1) \to \Delta(I') \to \Delta(I) \to 0$. This sequence is used as the upper row of the following commutative diagram with exact rows:

$$\begin{tikzcd}
0 \arrow[r] & \Delta(s+1) \arrow[r] & \Delta(I') \arrow[r] & \Delta(I) \arrow[r] & 0 \\
0 \arrow[r] & \Delta(s+1) \arrow[r] \arrow[u, equal] & X \arrow[r] \arrow[u, "\varphi'"'] & M \arrow[r] \arrow[u, "\varphi"'] & 0 \\
0 \arrow[r] & M' \arrow[r] \arrow[u, "\varepsilon"] \arrow[d, "\iota"'] & Y \arrow[r] \arrow[u, "\varepsilon'"'] \arrow[d, "\iota'"] & M \arrow[r] \arrow[u, equal] \arrow[d, equal] & 0 \\
0 \arrow[r] & M \arrow[r] & N \arrow[r] & M \arrow[r] & 0
\end{tikzcd}$$

The second row is induced from the first one by $\varphi$. In order to obtain the third row, we note that with $\varepsilon$ also $\operatorname{Ext}^1(M,\varepsilon)\colon \operatorname{Ext}^1(M,M') \to \operatorname{Ext}^1(M,\Delta(s+1)$ is surjective, since $M$ has projective dimension at most 1. Finally, the last row is induced from the third one, using $\iota$.

Since $N$ is an extension of $M$ by itself, the $\Delta$-support of $N$ is the same as the $\Delta$-support of $M$, thus equal to $J$. We may assume that $\iota'$ is again an inclusion map, thus we have obtained a module $N$ with a submodule $Y$ and a surjection $\varphi'\varepsilon'\colon Y \to \Delta(I')$, let $N'$ be its kernel. We see that $N/N'$ has the submodule $Y/N'$ isomorphic to $\Delta(I')$ and the corresponding factor module is $(N/N')/(Y/N') \simeq N/Y \simeq M/M'$. But $M/M'$ is $\Delta$-filtered with $\Delta$-support in $I$, thus $N/N'$ is just the direct sum of $\Delta(I')$ and $M/M'$, according to Proposition 1. This shows that $\Delta(I')$ is a factor module of $N$. It follows that $N$ belongs to $\mathcal{M}(J,s+1)$.

(b) *If $M$ is any $\Delta$-filtered module with $\Delta$-support $J$, and if* $\operatorname{Ext}^1(M,M)=0$, *then $M^{2^{s-1}}$ belongs to $\mathcal{M}(J,s)$, for each $1\le s\le t$.*

Proof by induction on $s$. For $s=1$, we only have to note that any $\Delta$-filtered module $M$ with $\Delta$-support $J$ has a factor module $\Delta(j_1)$, where $j_1$ is the minimal element of $J$. Now assume we know that $M^{2^{s-1}}$ belongs to $\mathcal{M}(J,s)$ and that $s<t$. By (a), there is an exact sequence $0\to M^{2^{s-1}} \to N \to M^{2^{s-1}} \to 0$ such that $N$ belongs to $\mathcal{M}(J,s+1)$. Since $\operatorname{Ext}^1(M,M)=0$, this sequence splits, thus $N\simeq M^{2^s}$.

(c) *If $M$ is any $\Delta$-filtered module with $\Delta$-support $J$, and if* $\operatorname{Ext}^1(M,M)=0$, *then $\Delta(J)$ is a direct summand of $M$.*

Proof: From (b) we know that $M^{2^{t-1}}$ belongs to $\mathcal{M}(J,t)$, thus $M^{2^{t-1}}$ has a factor module of the form $\Delta(J)$. Since $J$ is the $\Delta$-support of $M^{2^{t-1}}$, this factor module is actually a direct summand, thanks to Proposition 1. But $\Delta(J)$ is indecomposable, thus it is already a direct summand of $M$.

(d) Let $M$ be a $\Delta$-filtered module with $\Delta$-support $J$ and with $\operatorname{Ext}^1(M,M)=0$. From (c) we know that $M=\Delta(J)\oplus\overline{M}$ for some module $\overline{M}$. Of course, $\overline{M}$ is again $\Delta$-filtered and satisfies $\operatorname{Ext}^1(\overline{M},\overline{M})=0$. By induction on the $\Delta$-dimension, we see that $\overline{M}$ is isomorphic to $\Delta(\mathbf{dim}_\Delta\overline{M})$, and therefore, $M$ is isomorphic to $\Delta(J)\oplus\Delta(\mathbf{dim}_\Delta\overline{M}) = \Delta(\mathbf{dim}_\Delta M)$. This proves the remaining implication (i) $\Longrightarrow$ (ii) of Theorem 1.

**Proposition 2.** *Let $I$ be a subset and $j$ an element of $\{1,2,\dots,n\}$ such that $i<j$ for each $i$ in $I$. Let $M$ be $\Delta$-filtered with $\Delta$-support in $J=I\cup\{j\}$. Then either $\Delta(J)$ is a direct summand of $M$ or else any surjective map $M\to\Delta(I)$ splits.*

Proof: Let $I = \{j_1 < j_2 < \cdots < j_{t-1}\}$ and $j = j_t$, thus $J = \{j_1 < j_2 < \cdots < j_t\}$. Let $M$ be $\Delta$-filtered with $\Delta$-support in $J$. Consider a surjective map $f\colon M \to \Delta(I)$. Let $M'$ be the largest submodule of $M$ which is a direct sum of copies of $\Delta(j)$. Of course, the kernel $U$ of $f$ contains $M'$ and the induced map $M/M' \to \Delta(I)$ splits, according to Proposition 1. Thus, there exists a submodule $U'$ of $M$ such that $U + U' = M$ and $U \cap U' = M'$. Note that $U'/M'$ is isomorphic to $M/U = \Delta(I)$. If the sequence $0 \to M' \to U' \to \Delta(I) \to 0$ splits, then we obtain a direct complement $U''$ to $M'$ in $U'$, and therefore $U \oplus U'' = M$. In this case, $f$ splits. But if the sequence does not split, then $U'$ is isomorphic to the direct sum of $\Delta(J)$ and copies of $\Delta(j)$, since $\mathrm{Ext}^1(\Delta(I), \Delta(j))$ is one-dimensional. In particular, $M$ contains a submodule $U''$ isomorphic to $\Delta(J)$. But such a submodule is a direct summand of $M$ by Proposition 1.

## 4. Homomorphisms between modules of the form $\Delta(I)$

**Lemma 4.** *Let $I = \{i_1 > i_2 > \cdots > i_s\}$ and $J = \{j_1 > j_2 > \cdots > j_t\}$ be subsets of $\{1, 2, \ldots, n\}$. The following conditions are equivalent:*

(i) *$\Delta(I) \subseteq \Delta(J)$ (as submodules of $P(1)$).*
(ii) *There is a monomorphism $\Delta(I) \to \Delta(J)$.*
(iii) *We have: $s \le t$ and $i_r \le j_r$ for $1 \le r \le s$.*

Proof: The implications (iii) $\implies$ (i) and (i) $\implies$ (ii) are obvious. Let us consider the implication (ii) $\implies$ (iii). Note that $\Delta(I)$ has a submodule of the form $\Delta(i_s)$. Assume that there exists a monomorphism $\Delta(I) \to \Delta(J)$. Then $\Delta(i_s)$ must embed into $\Delta(J)$, but this implies $i_1 \le j_1$. The map induces a monomorphism $\Delta(I)/\Delta(i_1) \to \Delta(J)/\Delta(j_1)$, this allows us to use induction.

**Corollary.** *If $I \subseteq J$, then $\Delta(I) \subseteq \Delta(J)$.*

For our next lemma, we recall the definition of the *rational completion* of a module $M$. Let $Q(M)$ be an injective envelope of $M$. The rational completion $R(M)$ of $M$ is the intersection of the kernels of all endomorphisms $\phi$ of $Q(M)$ which vanish on $M$; in particular, we always have $M \subseteq R(M) \subseteq Q(M)$. In case $Q(M)$ is of finite length, this submodule $R(M)$ of $Q(M)$ is characterized as follows: it contains $M$, no composition factor of $R(M)/M$ embeds into $M$, whereas every composition factor of the socle of $Q(M)/R(M)$ does embed into $M$.

**Lemma 5.** *Let $I$ be a subset of $\{1, 2, \ldots, n\}$ of cardinality $t \ge 1$. Then $\Delta(I) \subseteq P(n+1-t)$ and this is the rational completion of $\Delta(I)$.*

Proof: Note that $P(i) = \Delta(\{i, i+1, \ldots, n\})$, thus $P(n+1-t) = \Delta(\{n-t+1, n-t+2, \ldots, n\})$. Now $I$ is a subset of $\{1, 2, \ldots, n\}$ of cardinality $t$, whereas $\{n-t+1, n-t+2, \ldots, n\}$ is the subset of the $t$ largest elements of $\{1, 2, \ldots, n\}$. According to Lemma 4 we see that $\Delta(I) \subseteq P(n+1-t)$.

In order to see that the inclusion $\Delta(I) \subseteq P(n+1-t)$ is a rational completion, we first note that the socle of $\Delta(I)$ is equal to $E(1)$ and that $P(n+1-t)$ is a submodule of the injective envelope $P(1)$ of $\Delta(I)$. The Jordan-Hölder multiplicity of the simple module $E(1)$ in both modules $\Delta(I)$ and $P(n+1-t)$ is $t$, thus $E(1)$ is not a subfactor of the factor module $P(n+1-t)/\Delta(I)$. On the other hand, for $t < n$, we know that $P(1)/P(n+1-t) = T(n-t)$, thus its socle is equal to $E(1)$, whereas $P(1)/P(n+1-t) = 0$ for $t = n$.

**Lemma 6.** *Let $I = \{i_1 < i_2 < \cdots < i_s\}$ and $J = \{j_1 < j_2 < \cdots < j_t\}$ be subsets of $\{1, 2, \dots, n\}$ such that $I \subseteq J$ or $J \subseteq I$. Then* $\dim \operatorname{Hom}(\Delta(I), \Delta(J)) = \min(s, t)$.

Proof: Let $r = \min(s, t)$. Let $I^q = \{i_1 < i_2 < \cdots < i_q\}$ for $1 \le q \le r$. Clearly, $\Delta(I^q)$ is a factor module of $\Delta(I)$. We are going to show that $\Delta(I^q)$ is a submodule of $\Delta(J)$. First, assume that $I \subseteq J$. Then $I^q \subseteq I \subseteq J$ shows that $\Delta(I^q) \subseteq \Delta(J)$. Second, assume that $J \subseteq I$. Then $J_q = \{j_{t-q+1} < j_{t-q+2} < \cdots < j_t\}$ is a subset of $J$, thus $\Delta(J_q) \subseteq \Delta(J)$. We may consider both $I^q$ and $J_q$ as subsets of $I$ of the same cardinality $q$. Since $I^q$ are the $q$ smallest elements of $I$, it follows that $i_p \le j_{t-q+p}$, for all $1 \le p \le q$, thus $\Delta(I^q) \subseteq \Delta(J_q)$, according to Lemma 3. Thus, we see that $\Delta(I^q) \subseteq \Delta(J_q) \subseteq \Delta(J)$. Altogether, the projection $\Delta(I) \to \Delta(I^q)$ composed with the inclusions $\Delta(I^q) \subseteq \Delta(J)$ yields a homomorphism with image $\Delta(I^q)$. Clearly, these homomorphisms are linearly independent. This shows that $\dim \operatorname{Hom}(\Delta(I), \Delta(J)) \ge \min(s, t)$.

Any homomorphism $\Delta(I) \to \Delta(J)$ can be extended uniquely to a homomorphism $P(n+1-s) \to P(n+1-t)$, since these modules are just the rational completions of $\Delta(I)$ and $\Delta(J)$, respectively. But $\dim \operatorname{Hom}(P(n+1-s), P(n+1-t)) = \min(s, t)$. This completes the proof.

**Corollary.** *Let $\mathbf{d} = (d_1, \dots, d_n) \in \mathbb{N}^n$. If the support of $\mathbf{d}$ is contained in $I$, then*

$$\dim \operatorname{Hom}(\Delta(I), \Delta(\mathbf{d})) = \sum_{i\in I} d_i = \dim \operatorname{Hom}(\Delta(\mathbf{d}), \Delta(I)).$$

**Proposition 3.** *Let $\mathbf{d} = (d_1, \dots, d_n) \in \mathbb{N}^n$. Then* $\dim \operatorname{End}(\Delta(\mathbf{d})) = \sum d_i^2$.

Proof: Let $I$ be the support of $\mathbf{d}$, and $\mathbf{d}' = \mathbf{d} - \mathbf{d}_I$ (where again $d_I$ is the characteristic function on $I$). Then $\Delta(\mathbf{d}) = \Delta(I) \oplus \Delta(\mathbf{d}')$. By induction, we can assume that $\dim \operatorname{End}(\Delta(\mathbf{d}')) = \sum_{i\in I}(d_i - 1)^2$. It follows from the Corollary that $\dim \operatorname{Hom}(\Delta(I), \Delta(\mathbf{d}')) = \sum_{i\in I}(d_i - 1) = \dim \operatorname{Hom}(\Delta(\mathbf{d}'), \Delta(I))$ and that $\dim \operatorname{End}(\Delta(I)) = |I|$. It remains to see that

$$\sum_{i\in I} d_i^2 = \sum_{i\in I}(d_i - 1)^2 + 2\sum_{i\in I}(d_i - 1) + |I|.$$

As a special case, we recover the well-known fact that the dimension of $A_n$ is equal to $\sum_{i=1}^n i^2$, since $\Delta((1, 2, \dots, n))$ is just the regular representation of $A_n$.

## 5. Orbit closures

Assume now that $k$ is an infinite field. Let $Q$ be the quiver of the algebra $A_n$; its set of vertices is $Q_0 = \{1, 2, \dots, n\}$, the set $Q_1$ consists of the arrows $\alpha_a\colon a \to a-1$, for $1 < a \le n$ and $\beta_a\colon a \to a+1$, for $1 \le a < n$. In general, given an arrow $\gamma \in Q_1$, let $s(\gamma)$ be the vertex where it starts, and $t(\gamma)$ the vertex where it ends. Also, there is the set $\rho$ of relations $\beta_{a-1}\alpha_a = \alpha_{a+1}\beta_a$ for $1 < a < n$ and $\beta_{n-1}\alpha_n = 0$.

Given an $n$-tuple $\mathbf{d} = (d_1, \dots, d_n)$ of natural numbers, let $\mathbf{v} = (v_1, \dots, v_n)$, where $v_a = \sum_{r\ge a} d_r$. Choose a $k$-space $V_a$ of dimension $v_a$. We consider the vector space

$$\mathcal{R}(Q; V) = \prod_{\gamma\in Q_1} \operatorname{Hom}_k(V_{s(\gamma)}, V_{t(\gamma)}),$$

its elements are written in the form $m = (m_\gamma)_{\gamma\in Q_1}$, where $m_\gamma\colon V_{s(\gamma)} \to V_{t(\gamma)}$ is a $k$-linear map; these elements $m$, or, more precisely, the pairs $(V, m)$, may be considered as representations of the path algebra of the quiver $Q$. Let $\mathcal{R}(Q, \rho, \Delta; V)$ be the subset of $\mathcal{R}(Q; V)$ of all tuples $m = (m_\gamma)_{\gamma\in Q_1}$ subject to the requirements, first, that the relations $\rho$ are satisfied, and second, that all the maps $m_{\alpha_a}$ are injective. Clearly, we obtain in this way representations of the algebra $A_n$ (since the relations $\rho$ are satisfied) which are $\Delta$-filtered (since we require that the maps $m_{\alpha_a}$ are injective), and the $\Delta$-dimension vector of any such representation is $\mathbf{d}$. Also, conversely, for any $\Delta$-filtered $A_n$-module with $\Delta$-dimension vector $\mathbf{d}$, there is an isomorphic one which belongs to $\mathcal{R}(Q, \rho, \Delta; V)$. The group $G(V) = \prod_{a\in Q_0} \mathrm{GL}(V_a)$ operates on $\mathcal{R}(Q, \rho, \Delta; V)$ via conjugation: given an element $g = (g_a)_a$ with $g_a \in \mathrm{GL}(V_a)$, then $g * m = (g_{t(\gamma)} m_\gamma g_{s(\gamma)}^{-1})_\gamma$. Two elements in $\mathcal{R}(Q, \rho, \Delta; V)$ belong to the same $G(V)$-orbit if and only if the corresponding representations are isomorphic.

**Theorem 2.** *Let $M$ be a $\Delta$-filtered module with $\Delta$-dimension vector $\mathbf{d}$. Then $M$ is in the Zariski closure of the orbit of $\Delta(\mathbf{d})$.*

Proof: Let $\mathbf{d} = (d_1, \dots, d_n)$. Denote by $\mathcal{O}$ the set of elements of $\mathcal{R}(Q, \rho, \Delta; V)$ which are isomorphic to $\Delta(\mathbf{d})$. We want to show that $M$ belongs to the Zariski closure $\overline{\mathcal{O}}$ of $\mathcal{O}$.

For $1 \le a \le n$, let $W_a$ be a vector space of dimension $d_a$ and let $V_a = \bigoplus_{r\ge a} W_r$. The canonical inclusion map $V_a \to V_{a-1}$ will be denoted by $\iota_a$, and we consider the subset $\mathcal{R}_\iota = \mathcal{R}_\iota(Q, \rho, \Delta; V)$ of all elements $(m_\gamma)_{\gamma\in Q_1}$ of $\mathcal{R}(Q, \rho, \Delta; V)$ such that $m_{\alpha_a} = \iota_a$. Also, let $\mathcal{O}_\iota = \mathcal{O} \cap \mathcal{R}_\iota$. Any $\Delta$-filtered module with $\Delta$-dimension vector $\mathbf{d}$ is isomorphic to a module belonging to $\mathcal{R}_\iota$. Thus, up to isomorphism (or, equivalently, up to the operation of the group $G(V)$) we may assume that both $M$ and $\Delta(\mathbf{d})$ are elements of $\mathcal{R}_\iota$.

We observe the following properties: First of all, $\mathcal{R}_\iota$ is a linear subspace of $\mathcal{R}(Q; V)$ (this follows from the fact that the relations in $\rho$ are homogeneous and linear in the $\beta$-maps). In fact, the elements $m$ of $\mathcal{R}_\iota$ can be described as follows (here, we write $\beta_a$ instead of $m_{\beta_a}$) : choose arbitrary linear maps $\beta_{ij}\colon W_j \to W_i$, for $1 \le j < i \le n$, and let $\beta_{ij} = 0$ otherwise; if we set $\beta_a = (\beta_{ij})_{a+1\le i\le n, a\le j\le n}$, then this gives an element of $\mathcal{R}_\iota$, and all the elements are obtained in this way. It follows that the $k$-dimension of the vector space $\mathcal{R}_\iota$ is just $\sum_{i>j} d_i d_j$.

Let $P$ be the subgroup of $G(V)$ of all elements $g = (g_a)_a$ with $g_a \in \mathrm{GL}(V_a)$ such that $g_{a-1}\iota_a g_a^{-1} = \iota_a$ for $2 \le a \le n$. As in the case of $\mathcal{R}_\iota$, we can describe the elements $g$ of $P$ as sequences of matrices in block form, where again $g = (g_a)_a$ is uniquely determined by $g_1$, and it follows that $P$ is an affine group of dimension $\sum_{i\ge j} d_i d_j$.

Obviously, the restriction to $P$ of the operation of $G(V)$ on $\mathcal{R}$ maps $\mathcal{R}_\iota$ into itself and $\mathcal{O}_\iota$ is a $P$-orbit. The stabilizer $P_0$ of $\Delta(\mathbf{d})$ in $P$ is just the automorphism group of $\Delta(\mathbf{d})$, thus the group of units of the endomorphism ring $\mathrm{End}(\Delta(\mathbf{d}))$. The dimension of this group $P_0$ is the same as the that of $\mathrm{End}(\Delta(\mathbf{d}))$, thus, according to Proposition 3, equal to $\sum d_a^2$. The dimension of the orbit $\mathcal{O}_\iota$ is the difference of $\dim P$ and $\dim P_0$, thus equal to $\sum_{i>j} d_i d_j$. As a consequence, $\mathcal{O}_\iota$ is dense in $\mathcal{R}_\iota$. This shows that $M$ belongs to the closure of $\mathcal{O}_\iota$.

**Remark 2.** The subgroup $P$ of $\mathrm{GL}(V_1)$ used in the proof is a parabolic subgroup, and $\mathcal{R}_\iota$ is the Lie algebra of the unipotent radical of $P$. To be more precise: As we have mentioned, any element $m$ of $\mathcal{R}_\iota$ is uniquely determined by the map

$\beta_1$, the additional maps $\beta_a$ being just restrictions of $\beta_1$; similarly, any element $g$ of $P$ is uniquely determined by $g_1$, again the additional components $g_a$ are just restrictions of $g_1$. The projection of $P$ given by $g \mapsto g_1$ yields an embedding of $P$ into $\mathrm{GL}(V_1)$, its image is the parabolic subgroup defined by the flag of subspaces $V_n \subseteq V_{n-1} \subseteq \cdots \subseteq V_2 \subseteq V_1$, and we may identify $\mathcal{R}_\iota$ with the Lie algebra of the unipotent radical of $P$. These arguments just reproduce some of the considerations in [HiRö]. We present an explicit example at the end of Section 8.

**Corollary.** *The algebraic varieties $\mathcal{R}(Q,\rho,\Delta;V)$ are irreducible.*

**Remark 3.** For any $Q_0$-graded vector space $V$, we denote by $\mathcal{R}(Q,\rho;V)$ the set of all elements in $\mathcal{R}(Q;V)$ which satisfy the relations $\rho$. Of course, the elements of $\mathcal{R}(Q,\rho;V)$ can be considered as representations of $A_n$ with dimension vector $\mathbf{v} = (v_1,\dots,v_n)$, where $v_a = \dim V_a$, for any $a \in Q_0$. As before, the orbits of the group action of $G(V)$ on $\mathcal{R}(Q,\rho;V)$ via conjugation just yield the isomorphism classes. Note that $\mathbf{v}$ is the dimension vector of some $\Delta$-filtered module $M$ if and only if $v_1 \geq v_2 \geq \cdots \geq v_n$, and then the $\Delta$-dimension vector is $\mathbf{d}$, where $d_a = v_a - v_{a+1}$ for $1 \leq a < n$ and $v_n = d_n$.

By Theorem 2, the orbit of $\Delta(\mathbf{d})$ is dense in $\mathcal{R}(Q,\rho,\Delta;V)$; in particular, $\mathcal{R}(Q,\rho,\Delta;V)$ is irreducible. Since $\mathrm{Ext}^1(\Delta(\mathbf{d}),\Delta(\mathbf{d})) = 0$, the orbit of $\Delta(\mathbf{d})$ is also open in $\mathcal{R}(Q,\rho;V)$, and thus its closure in $\mathcal{R}(Q,\rho;V)$ yields an irreducible component of $\mathcal{R}(Q,\rho;V)$. However, in general there will be several irreducible components in $\mathcal{R}(Q,\rho;V)$. Namely, the dimension vectors of $\Delta$-filtered modules are at the same time the dimension vectors of $\nabla$-filtered modules, thus we may similarly consider the nabla-analogues $\nabla(\mathbf{d})$ of the modules $\Delta(\mathbf{d})$. Clearly, the closure of the orbit of $\nabla(\mathbf{d})$ will also be an irreducible component of $\mathcal{R}(Q,\rho;V)$, and it will be different from the closure of the orbit of $\Delta(\mathbf{d})$, except in case $\Delta(\mathbf{d}) = \nabla(\mathbf{d})$, thus in case the module $\Delta(\mathbf{d})$ is a direct sum of modules of the form $T(a_i)$.

## 6. More closures

Again, we assume that the field $k$ is infinite. We consider the sets $\mathcal{M}(J,s)$ defined at the beginning of Section 3.

**Proposition 4.** *Let $J$ be a subset of $\{1,2,\dots,n\}$ of cardinality $t$ and let $1 \leq s \leq t$. The set $\mathcal{M}(J,s-1)$ is contained in the closure of the set $\mathcal{M}(J,s)$.*

Proof: Consider a module $M$ with $\Delta$-dimension vector $\mathbf{d}$ which belongs to $\mathcal{M}(J,s-1)$. We want to show that $M$ belongs to the closure of $\mathcal{M}(J,s)$. Thus, assume that $M$ does not belong to $\mathcal{M}(J,s)$. Let $J = \{j_1 < \cdots < j_t\}$. We consider the set $I = \{j_1 < \cdots < j_{s-1}\}$ and have to deal with the two vertices $i = j_{s-1}$ and $j = j_s$. Let $M'$ be the maximal submodule of $M$ with a $\Delta$-filtration with factors of the form $\Delta(j')$, where $j' > j$. According to the splitting lemma, we know that we can find submodules $X, Y$ of $M$ such that $X + Y = M$, $X \cap Y = M'$, and $X/M'$ is isomorphic to $\Delta(I)$. Choose submodules $M' \subset X' \subseteq X$ and $M' \subset Y' \subseteq Y$ such that $X'/M'$ is isomorphic to $\Delta(i)$ and $Y'/M'$ is isomorphic to $\Delta(j)$.

Let $\mathbf{v} = (v_1,\dots,v_n)$ be the dimension vector of $M$ and fix vector spaces $V_a$ of dimension $v_a$. Take a realization $(m_\gamma)_{\gamma\in Q_1}$ of $M$ in $\mathcal{R}(Q,\rho,\Delta;V)$. We are going to work with bases $\mathcal{B}_a$ of the vector spaces $V_a$ which are compatible with the action of the monomorphisms $m_{\alpha_a}$. In order to do so, we start with a $\Delta$-filtration of $M$, say

$$0 = M^{(0)} \subset M^{(1)} \subset \cdots \subset M^{(u)} = M$$

and choose for $M^{(i)}/M^{(i-1)} \simeq \Delta(a_i)$ an element of $M^{(i)}_{a_i} \setminus M^{(i-1)}_{a_i}$ and take these elements as well as their images under the compositions of the various maps $m_{\alpha_a}$. In this way, we obtain all the bases $\mathcal{B}_a$ simultaneously.

The $\Delta$-filtration which we want to use should be a refinement of the following chain of submodules of $M$

$$(*) \qquad 0 \subseteq M' \subset Y' \subset X'+Y' \subseteq X+Y' \subseteq M.$$

Note that the factors $Y'/M' \simeq \Delta(j)$ and $(X'+Y')/Y' \simeq X'/M' \simeq \Delta(i)$ are already standard modules, thus do not have to be refined. Let $x = x_i$ be the chosen base elements for the factor $Y'/M'$, thus $x_i$ is an element of $Y'_i \setminus M'_i$. Let $y = y_j$ be the chosen base elements for the factor $(X'+Y')/Y'$, we may assume that $y_i$ is an element of $X'_i \setminus M'_i$. Inductively, we define $x_{a-1} = m_{\alpha_a}(x_a)$, for $1 < a \leq i$, and $y_{a-1} = m_{\alpha_a}(y_a)$, for $1 < a \leq j$.

Let us consider the subset $\mathcal{R}_m(Q,\rho,\Delta;V)$ of all elements $m'$ in $\mathcal{R}(Q,\rho,\Delta;V)$ with $m'_\gamma = m_\gamma$ for all $\gamma = \alpha_i$; note that $\mathcal{R}_m(Q,\rho,\Delta;V)$ is a linear subspace, since the relations in $\rho$ are linear, once we fix the maps $\alpha_i$. To be more precise, we introduce an addition $\underset{\alpha}{+}$ on $\mathcal{R}_m(Q,\rho,\Delta;V)$ by

$$(m' \underset{\alpha}{+} m'')_\gamma = \begin{cases} m_\gamma & \text{if } \gamma = \alpha_i, \\ m'_\gamma + m''_\gamma & \text{if } \gamma = \beta_i. \end{cases}$$

Similarly, there is a scalar multiplication $\underset{\alpha}{\cdot}$ on $\mathcal{R}_m(Q,\rho,\Delta;V)$ defined by

$$(\lambda \underset{\alpha}{\cdot} m')_\gamma = \begin{cases} m_\gamma & \text{if } \gamma = \alpha_i, \\ \lambda m'_\gamma & \text{if } \gamma = \beta_i, \end{cases}$$

where $m'$ belongs to $\mathcal{R}_m(Q,\rho,\Delta;V)$ and $\lambda \in k$.

For example, we may consider the element $m^0$ of $\mathcal{R}_m(Q,\rho,\Delta;V)$ defined by $m^0_\gamma = m_\gamma$ for $\gamma = \alpha_i$ and $m^0_\gamma = 0$ for $\gamma = \beta_i$; this is just the zero element, and it corresponds to a representation of $A_n$ which has the same dimension vector as $M$, but is the direct sum of standard modules.

A special element of $\mathcal{R}_m(Q,\rho,\Delta;V)$ which turns out to be of interest, denoted by $m^{xy}$, is defined as follows: First of all, $m^{xy}_\gamma = m_\gamma$ for $\gamma = \alpha_i$. Second, we require that $m^{xy}_{\beta_i}(x_i) = y_{i+1}$, for $1 \leq a \leq i$. Finally, $m^{xy}_\gamma(z) = 0$ for all other elements $z$ of the bases $\mathcal{B}_a$. This element $m^{xy}$ yields a representation of $A_n$ which is the direct sum of one copy of $\Delta(\{i,j\})$ and additional standard modules.

Now we can define the desired one-parameter family: For $\lambda \in k$, let

$$m(\lambda) = m \underset{\alpha}{+} (\lambda \underset{\alpha}{\cdot} m^{xy}).$$

The $A_n$-module corresponding to $m(\lambda)$ is denoted by $M(\lambda)$.

Of course, $M(0) = M$. It remains to be shown that for $\lambda \neq 0$, the module $M(\lambda)$ belongs to $\mathcal{M}(J,s)$. Note that by construction, the chain of subspaces $(*)$ is still a chain of submodules, for any parameter $\lambda$, but we should add the parameter $\lambda$ in brackets $(-)$ in order to be specific:

$$0 \subseteq M'(\lambda) \subset Y'(\lambda) \subset (X'+Y')(\lambda) \subseteq (X+Y')(\lambda) \subseteq M(\lambda).$$

Actually, $M'(\lambda) = M'$, and also $M(\lambda)/Y'(\lambda) = M/Y'$, these modules remain unchanged. In particular, $M(\lambda)/Y'(\lambda) = M/Y'$ contains $(X+Y')(\lambda)/Y'(\lambda) = (X+Y')/Y'$ as a submodule, and this submodule is isomorphic to $\Delta(I)$. Let $\lambda \neq 0$. In this case, the module $(X+Y')(\lambda)/M'(\lambda)$ is no longer the direct sum of $\Delta(I)$ and $\Delta(j)$, it is a non-split extension of $\Delta(j)$ by $\Delta(I)$. Since $M(\lambda)/(X+Y')(\lambda) = M/(X+Y')$ has $\Delta$-support in $I \cup \{j\}$, it follows that the subfactor $(X+Y')(\lambda)/M'(\lambda)$ in $M(\lambda)$ is isomorphic to $\Delta(I \cup \{j\})$, according to Lemma 3, and it is a direct summand of $M(\lambda)(\lambda)/M'(\lambda)$ according to Proposition 1. In particular, we see that $M(\lambda)$ maps onto $\Delta(I \cup \{j\})$, thus $M(\lambda)$ belongs to $\mathcal{M}(J,s)$, as claimed.

We have shown that all the modules $M(\lambda)$ with $\lambda \neq 0$ belong to $\mathcal{M}(J,s)$, thus $M = M(0)$ belongs to the closure of $\mathcal{M}(J,s)$.

Observe that the proof of Proposition 4 is based on explicit degenerations along affine lines. We are going to use Proposition 4 in order to present a second proof of Theorem 2. In this way, we describe an explicit method of degenerations.

This alternative proof proceeds by induction on the dimension of $M$, and we use a second induction on $s$, where $M$ belongs to $\mathcal{M}(J,s)$, this time in reversed order. Thus, let $J$ be the $\Delta$-support of $M$, and assume that $J$ has cardinality $t$.

Proposition 4 asserts that for $1 \leq s \leq t$, the set $\mathcal{M}(J,s-1)$ is contained in the closure of the set $\mathcal{M}(J,s)$. Thus, inductively, we see that $\mathcal{M}(J,s)$ is contained in the closure of $\mathcal{M}(J,t)$.

It remains to be seen that $\mathcal{M}(J,t)$ is contained in the closure of $\Delta(\mathbf{d})$. If $M \neq 0$ belongs to $\mathcal{M}(J,t)$, then $M = \Delta(J) \oplus M'$ for some $\Delta$-filtered module $M'$ of smaller dimension. By induction, $M'$ belongs to the closure of the orbit of $\Delta(\mathbf{d}')$, where $\mathbf{d}'$ is the $\Delta$-dimension vector of $M'$. Since $M'$ belongs to the closure of $\Delta(\mathbf{d}')$, it follows that $M = \Delta(J) \oplus M'$ belongs to the closure of the orbit of $\Delta(\mathbf{d}) = \Delta(J) \oplus \Delta(\mathbf{d}')$.

## 7. The tilting modules $\Delta(\pi)$

Let $S_n$ be the set of all permutations of $\{1,2,\ldots,n\}$. For a fixed $\pi \in S_n$ let $I_\pi(s) = \{\pi^{-1}s,\ldots,\pi^{-1}n\}$, for $1 \leq s \leq n$ and define

$$\Delta(\pi) := \bigoplus_{s=1}^{n} \Delta(I_\pi(s)).$$

Observe that $(\mathbf{dim}_\Delta\, \Delta(\pi))_i = \pi i$ for each $1 \leq i \leq n$.

We use the following definition of a *tilting* module which is equivalent to the original one given in [HaRi]; this is a module of projective dimension at most 1, without self-extensions, and such that the number of isomorphism classes of indecomposable summands is equal to the number of simple $A_n$-modules. A module $M$ is said to be *basic* provided it can be written as a direct sum of pairwise non-isomorphic indecomposable modules.

**Theorem 3.** *The modules $\Delta(\pi)$ for $\pi \in S_n$ furnish a complete list of all the isomorphism classes of basic tilting modules.*

Proof: It is clear that the modules $\Delta(\pi)$ satisfy all the defining properties of a basic tilting module. Conversely, let $M$ be a basic tilting module. Since $M$ has projective dimension at most 1, we see that $M$ belongs to $\mathcal{F}(\Delta)$. Thus $M$ is isomorphic to $\Delta(\mathbf{d})$ for some $\mathbf{d}$. This means that $M$ is isomorphic to a direct sum of modules $\Delta(I(s))$, where $I(s)$ are subsets of $\{1, 2, \ldots, n\}$, and that these subsets are pairwise comparable. Since $M$ is basic, these subsets have to be also pairwise different. Since the number of summands $\Delta(I(s))$ has to be $n$, we see that the subsets $I(s)$ have to form a complete chain of subsets of the set $\{1, 2, \ldots, n\}$, thus $M$ is isomorphic to a module of the form $\Delta(\pi)$ for some permutation $\pi$ in $S_n$.

We are going to calculate the endomorphism rings of the basic tilting modules:

**Theorem 4.** *The endomorphism ring of any basic tilting module is isomorphic to $A_n$.*

**Proof.** Consider the module $\Delta(\pi)$ for some $\pi \in S_n$. Since $I_\pi(s)$ has cardinality $n - s + 1$, the rational completion of $\Delta(I_\pi(s))$ is the projective module $P(s)$, according to Lemma 5. Thus, the canonical embedding $\Delta(\pi) \subseteq \bigoplus_{s=1}^n P(s) = P$ is the rational completion. This shows that the endomorphism ring $E$ of $\Delta(\pi)$ embeds into the endomorphism ring the regular representation $P$ of $A_n$, thus into the opposite of $A_n$. Since $A_n$ is isomorphic to its opposite algebra, we see that $E$ can be considered as a subalgebra of $A_n$. On the other hand, Proposition 3 applied to $\mathbf{d} = (\pi^{-1}1, \ldots, \pi^{-1}n)$ shows that $\dim E = \sum_{i=1}^n i^2 = \dim A_n$. Thus $E = A_n$.

**Corollary.** *The endomorphism ring of a module of the form $\Delta(\mathbf{d})$ is equal to the endomorphism ring of a projective $A_n$-module.*

## 8. The description of the Richardson classes in $\mathrm{GL}(m, k)$

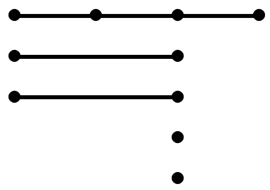
In this section let $k$ be an algebraically closed field and $m \in \mathbb{N}$. Let $P$ be a parabolic subgroup of $\mathrm{GL}(m, k)$ with unipotent radical $P_u$. Then $P$ has a Levi decomposition $P = LP_u$. A Levi subgroup $L$ of $P$ is (isomorphic to) a product of general linear groups, say $\mathrm{GL}(d_i, k)$, for $1 \le i \le n$, with $m = \sum d_i$. The ordered tuple $\mathbf{d} = (d_1, \ldots, d_n)$ uniquely determines the conjugacy class of $P$ in $\mathrm{GL}(m, k)$. To indicate this, we write $P = P(\mathbf{d})$.

By a fundamental result of R.W. Richardson [R] a parabolic subgroup $P$ of any reductive group $G$ always admits an open dense orbit when operating on $P_u$ as well as on its Lie algebra $\mathbf{p}_u = \mathrm{Lie}\, P_u$. The associated unipotent class in $G$ is known as the *Richardson class $C_P$* of $P$ in $G$, likewise for the adjoint action of $P$ on $\mathbf{p}_u$.

Following Spaltenstein [S], we give an explicit description of the Richardson class $C_P$ associated to $P$ in terms of partitions for $\mathrm{GL}(m, k)$. A partition $\mu = (1^{c_1}, 2^{c_2}, \ldots)$ of $m$ canonically defines a unipotent class $C_\mu$ whose elements have $c_i$ Jordan blocks of size $i$ for $i = 1, 2, \ldots$. $\lambda(\mathbf{d}) = \lambda = (\lambda_1 \ge \lambda_2 \ge \ldots)$ be the partition of $m$ associated to the tuple $\mathbf{d}$ (i.e., $\lambda_i = d_{\tau i}$ for some permutation $\tau$ of $\{1, 2, \ldots, n\}$). Let $\lambda^*$ be the partition dual to $\lambda$ (i.e., the partition whose parts are the lengths of the columns of the Young tableau of $\lambda$). Then $C_P = C_{\lambda^*}$, where $P = P(\mathbf{d})$ and $\lambda = \lambda(\mathbf{d})$, see [S], II §5. In particular, the Richardson class of $P$ only depends on the partition derived from the tuple $\mathbf{d}$, but not on $\mathbf{d}$ itself; this means that $C_P$ only depends on the conjugacy class of a Levi subgroup $L$ of $P$ in $\mathrm{GL}(m,k)$ as opposed to that of $P$, see [S], II 3.7. Moreover, in $\mathrm{GL}(m,k)$ every unipotent class is the Richardson class of some parabolic subgroup [S], III §1.1.

It has been observed in [HiRö] that the problem of describing the conjugacy classes of elements of $P_u$ or $\mathbf{p}_u$ under the action of $P = P(\mathbf{d})$ is the same as that of describing the isomorphism classes of all $\Delta$-filtered $A_n$-modules with $\Delta$-dimension $\mathbf{d}$. Consequently, Theorem 2 above recovers Richardson's theorem for $\mathrm{GL}(m,k)$. Moreover, with our detailed description of the modules $\Delta(\mathbf{d})$ in the dense $G(V)$-orbit on the variety $\mathcal{R}(Q,\rho,\Delta;V)$, we may obtain an explicit representative of the open dense $P$-orbit on $\mathbf{p}_u$ where $P = P(\mathbf{d})$. We illustrate this with an example.

**Example.** Let $n = 4$ and consider the 4-tuple $\mathbf{d} = (3,1,5,1)$. At the end of Section 2 we have attached to $\mathbf{d}$ a graph which now turns out to be very useful:



As mentioned above, the rows represent the various summands $\Delta(I)$ of $\Delta(\mathbf{d})$; a row with vertices in the columns $i_1,\dots,i_s$ corresponds to a direct summand of the form $\Delta(\{i_1,\dots,i_s\})$.

Note that $\Delta(\mathbf{d})$ is an $A_4$-module, thus a representation $(V_1,\dots,V_4,\alpha_i,\beta_i)$ of the following quiver:

$$\begin{tikzcd} 1 \arrow[r, shift left, "\beta_1"] & 2 \arrow[l, shift left, "\alpha_2"] \arrow[r, shift left, "\beta_2"] & 3 \arrow[l, shift left, "\alpha_3"] \arrow[r, shift left, "\beta_3"] & 4 \arrow[l, shift left, "\alpha_4"] \end{tikzcd}$$

Recall that $V_1$ is the direct sum $W_1 \oplus \cdots \oplus W_4$ (with $W_i$ the cokernel of $\alpha_{i+1}$). We may consider the vertices of the diagram as basis vectors of $V_1$ as follows: the vertices in the $i$-th column represent basis vectors of $W_i$ for $1 \le i \le 4$. It is of importance to observe the relevance of the edges: an edge, say $v$ —— $w$, indicates that $\alpha_2\beta_1$ maps the basis vector $v$ to the basis vector $w$. Thus, this diagram illustrates the operation of $\alpha_2\beta_1$ on each of the 5 summands. As a consequence, we can use the diagram in order to write down a matrix for $\alpha_2\beta_1$ explicitly. We label the basis vectors $v_1,\dots,v_{10}$ column-wise, as follows:

$$\begin{array}{ccccccc} 1 & \text{---} & 4 & \text{---} & 5 & \text{---} & 10 \\ 2 & \text{---} & \text{---} & \text{---} & 6 & & \\ 3 & \text{---} & \text{---} & \text{---} & 7 & & \\ & & & & 8 & & \\ & & & & 9 & & \end{array}$$

The resulting matrix of the nilpotent endomorphism $\alpha_2\beta_1$ in $\mathrm{Lie}\,\mathrm{GL}(V_1) = \mathrm{End}(V_1)$ is

$$
\left(\begin{array}{c|c|ccccc|c}
 & 1 & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\
0 & \cdot & \cdot & 1 & \cdot & \cdot & \cdot & \cdot \\
 & \cdot & \cdot & \cdot & 1 & \cdot & \cdot & \cdot \\
\hline
 & 0 & 1 & \cdot & \cdot & \cdot & \cdot & \cdot \\
\cline{2-8}
 & & & & & & & 1 \\
 & & & & & & & \cdot \\
0 & & & & 0 & & & \cdot \\
 & & & & & & & \cdot \\
 & & & & & & & \cdot \\
\cline{3-8}
 & & & & & & & 0
\end{array}\right)
$$

The only non-zero entries of this matrix are at the positions $(i, j)$ with $i$ —— $j$. Thus, it is straightforward to write down this representing matrix for any given $\mathbf{d}$. (We have exhibited the matrix together with its block structure: the diagonal blocks as well as the blocks below the diagonal are zero matrices; the remaining blocks form the "$\mathbf{p}_u$-part", the 0's inside the "$\mathbf{p}_u$-part" are represented by dots, where, as before, $P = P(\mathbf{d})$ and $\operatorname{Lie} P_u = \mathbf{p}_u$.)

Thanks to Theorem 2 this is a matrix representative of the Richardson class associated to the adjoint action of $P$ on $\mathbf{p}_u$, namely of the element $\alpha_2\beta_1$ in $\mathbf{p}_u$. According to Remark 2 above, $\Delta(\mathbf{d})$ is uniquely determined by the linear map $\beta_1$, or also by the map $\alpha_2\beta_1 \colon V_1 \to V_1$.

As we have mentioned in the last paragraph, the partition $\lambda = (5, 3, 1, 1)$ associated to $\mathbf{d}$ uniquely determines the Richardson class of $P$. However, the nilpotent element given by the Jordan blocks of the dual partition $\lambda^*$ of $\lambda$ usually does not lie in $\mathbf{p}_u$. This happens only when $\mathbf{d} = \lambda$.

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