Aufgabe 2. Sei $U \subset E$ offen in einem Banachraum. Zeig, dass jeder Punkt $x \in U$ positiven Abstand vom Rand ∂U hat. Sei $V \subset E$ eine beliebige Teilmenge mit nicht-leerem Rand und $x \in V$ ein Punkt, der zum Rand Abstand 0 hat. Folgt dann $x \in \partial V$?

Lösung: Suppose that there exists $x_0 \in \partial U$ such that $d(x, x_0) = \|x - x_0\| = 0$ for some $x \in U$. Since U is open and $x \in U$, there exists $r > 0$ such that an open ball $B(x, r)$ so that $B(x, r) \subset U$. Since $\|x - x_0\| = 0$, we have $x_0 \in B(x, r)$ and hence $x_0 \in U$. Then there exists a sequence $\{x_j\} \subset U$ such that $x_j \to x_0$ as $j \to \infty$. Then U is closed in E. □

Aufgabe 3. Sei $X : I \times U \to E$ ein stetiges Vektorfeld, lokal uniform Lipschitz-stetig im Raum und beschränkt auf Mengen der Form $[a, b] \times V$, wobei $V \subset U$ positiven Abstand zur Rand ∂U hat. Sei $\gamma : J \to U$ eine maximale Integralkurve mit $t_+ = \sup J < \sup I$. Zeige:

$$\lim_{t \to t^+} \min \left\{ d\left(\partial U, \gamma(t)\right), \frac{1}{\|\gamma(t)\|} \right\} = 0.$$

Lösung: We need to show that one has either

$$d(\partial U, \gamma(t)) = 0$$

or

$$\lim_{t \to t^+} \|\gamma(t)\| = +\infty.$$

We assume now that two statements cannot be held simultaneously, i.e., we assume that $\|\gamma(t)\|$ is bounded on an interval $[t_1 1, t^+]$, say $\|\gamma(t)\| \le c$ on the interval $[t_1, t^+]$. Simultaneously, there exists $\epsilon > 0$ such that $d(\partial U, \gamma(t)) = \epsilon$, on an interval $[t_2, t^+]$.
Now we set an ϵ-neighborhood of the boundary ∂U and denote it by $\widetilde{\partial U}$. Set $C = \{y \in U : \|y\| \le c\}$ and let $t_0 = \max\{t_1, t_2\}$. Then the set K given by

$$K = ([t_0, t^+] \times C) \setminus \widetilde{\partial U}$$

is a closed and bounded subset of $I \times U$. Hence K is compact and $K \subset V$ and $(t, \gamma(t)) \in K$ for $t \in [t_0, t^+]$, which contradicts to our assumptions. □