Aufgabe 1. Construction of integral curves for a given vector field using polygonal line.

Lösung: Firstly we want to review the argument used in the lecture. Let $X : I \times U \to \mathbb{R}^d$ be a continuous vector field. Suppose that X is bounded, i.e., $\|X\| \leq C$. The existence of an integral curve is as follows:
Let $p \in U$. Then for some $\epsilon > 0$, there exists an integral curve $\gamma : [0, \epsilon) \to U$ such that $\gamma(0) = p$.
Now we make the use of an appropriate translation, shrinking U, and a dilation so that we may assume that p is the origin, i.e., 0, and U is an open disc around the origin, and the image $X(U)$ lies in a disc D of radius 1, i.e., $\|X\| \leq 1$. Then the statement of the existence of integral curves is as follows:
There exists an integral curve $\gamma : [0, 1] \to D$ for X such that $\gamma(0) = 0$.
We shall construct the integral curve γ as a limit of polygonal line $\gamma_n : [0, 1] \to D$. To do this, we define γ_n by the piecewise-linear curve as follows:
Divide the interval $[0, 1]$ into n intervals $[\frac{k}{n}, \frac{k+1}{n}]$ for $0 \leq k \leq n$. The curve γ_n starts at 0, then on the interval $[0, \frac{1}{n}]$ follows the straight line in the direction of $X(0)$, then on $[\frac{1}{n}, \frac{2}{n}]$ follows the straight line in the direction of $X(\gamma_n(\frac{1}{n}))$, and so on. In summary, γ_n is formulated by

$$\gamma_n(t) = \gamma_n(\frac{k}{n}) + (t - \frac{k}{n})X(\gamma(\frac{k}{n})),\ 0 \leq k < n,\ t \in [\frac{k}{n}, \frac{k+1}{n}],$$

and $\gamma_n(0) = 0$.
Since $\|X\| \leq 1$ we have $\|\gamma_n(t)\| \leq 1$ for all $t \in [0, 1]$. Hence we can use the Arzela-Ascoli theorem to obtain a continuous function $\gamma : [0, 1] \to D$, an uniform limit of a subsequence of γ_n. To prove that γ is truly the integral curve for the vector field X, we shall prove the following inequality holds for all $\epsilon > 0$: there exists $\delta > 0$ and $N \in \mathbb{N}$ such that if $|t - s| < \delta$ then

$$\left\| \frac{\gamma(t) - \gamma(s)}{t - s} - X(\gamma(s)) \right\| < \epsilon.$$

To do this, we first prove the similar inequality for the polygonal line: there exists $\delta > 0$ and $N \in \mathbb{N}$ such that if $|t - s| < \delta$ and $n > N$, then

$$\left\| \frac{\gamma_n(t) - \gamma_n(s)}{t - s} - X(\gamma_n(s)) \right\| < \epsilon,$$

where $s < t$. To prove this, we first note that

$$\frac{\gamma_n(t) - \gamma_n(s)}{t - s} = \int_s^t \gamma_n'(u)\, du,$$

and hence it is the average of the quantity $\gamma_n'(u)$ as u range over the interval $[s, t]$. Thus our proof is reduced to the problem of showing that the average has the distance ϵ with $X(\gamma_n(s))$. By construction, for $u \in [s, t]$ we have $\gamma_n'(u) \in X(\gamma_n([s - \frac{1}{n}, t]))$. Since $X(\gamma_n(s)) \in X(\gamma_n([s - \frac{1}{n}, t]))$ and the average of a vector-valued function must lie in the convex hull of its image, it is sufficient to show that the diameter of the image $X(\gamma_n([s - \frac{1}{n}, t]))$ is at most ϵ.

Since $\|X\| \leq 1$, the diameter of the image $X(\gamma_n([s-\frac{1}{n}, t]))$ is at most $t - s + \frac{1}{n}$. Since X is bounded and continuous on the closed and bounded disc D, X is uniformly continuous and hence the required claim is true. Indeed, it suffices to show that for $n \geq N$, $x, y \in D$ with $\|x - y\| < \delta + \frac{1}{n}$, we have $\|X(x) - X(y)\| < \epsilon$, which directly follows from the uniform continuity of X. Finally, we arrive at the step of proving the inequality

$$\left\| \frac{\gamma(t) - \gamma(s)}{t - s} - X(\gamma(s)) \right\| < \epsilon,$$

provided that $|t - s| < \delta$, $n \geq N$. We see that

$$\begin{aligned} \left\| \frac{\gamma(t) - \gamma(s)}{t - s} - X(\gamma(s)) \right\| \leq \left\| \frac{\gamma_n(t) - \gamma_n(s)}{t - s} - X(\gamma_n(s)) \right\| + \left\| \frac{\gamma_n(s) - \gamma(s)}{t - s} \right\| \\ + \left\| \frac{\gamma_n(t) - \gamma(t)}{t - s} \right\| + \|X(\gamma_n(s)) - X(\gamma(s))\| < 4\epsilon. \end{aligned}$$

□

Aufgabe 2. Seien $a, b > 0$ Konstanten. Betrachte das zeitunabhängige Vektorfeld

$$X : \mathbb{R}^2 \to \mathbb{R}^2,\ X(x, y) = (a - bx + x^2y - x, bx - x^2y).$$

Zeige, dass durch einen Punkt (x_0, y_0) im ersten Quadranten (daher $x_0, y_0 > 0$) genau eine Integralkurve geht, die in positiver Zeitrichtung unendlich lang existiert und den ersten Quadranten nicht verlässt.

Lösung: We follow the hind given and consider firstly the vector field on the x-axis:

$$X(x, 0) = (a - (b + 1)x, bx),$$

and hence the integral curve $(x(t), y(t))$ of the vector field $X(x, 0)$ satisfies the following differential equations:

$$x'(t) = a - (b + 1)x(t),\ y'(t) = bx(t)$$

Then we have the solutions $x(t)$ and $y(t)$:

$$\begin{aligned} x(t) &= \frac{a}{b+1} + \left(x_0 - \frac{a}{b+1}\right) e^{-(b+1)t}, \\ y(t) &= \frac{ab}{b+1}t + y_0 + \frac{b}{b+1}\left(x_0 - \frac{a}{b+1}\right)(1 - e^{-(b+1)t}). \end{aligned}$$

It is easy to see that $x(t), y(t) > 0$ for all $t > 0$. For example, the inequality $x(t) \leq 0$ is equivalent to the inequality

$$\frac{a}{b+1}(e^{(b+1)t} - 1) + x_0 \leq 0,$$

which is impossible when $a, b, x_0 > 0$. On the other hand, the vector field $X(0, y)$ on the y-axis is given by

$$X(0, y) = (a, 0),$$

and hence the integral curve is simply given by

$$x(t) = at + x_0,\ y(t) = y_0,$$

which always lies in the first quadrant. □

Aufgabe 3. Seien $f, g : \mathbb{R} \to \mathbb{R}$ stetig. Die Funktion f sei überdies Lipschitz-stetig. Betrachte das zeitunabhängige Vektorfeld

$$X : \mathbb{R}^2 \to \mathbb{R}^2,\ X(x, y) = (f(x), yg(x)).$$

1. Zeige unter Angabe eines Beispiels für f und g, dass X nicht umbedingt lokal Lipschitz-stetig ist.

2. Zeige, dass X dennoch eindeutige Integralkurve hat, daher: zu jedem Punkt (x_0, y_0) und zu jeder Zeit t_0 gibt es genaue eine maximale Integralkurve mit: $\gamma(t_0) = (x_0, y_0)$.

Lösung: 1. It suffices to choose g a continuous function, but not even locally Lipschitz continuous function. For example, $g(x) = x^{\frac{3}{2}} \sin(\frac{1}{x})$, or even simpler, $g(x) = \sqrt{x}$, which is not Lipschitz-continuous near $x = 0$.
2. Let γ_1 and γ_2 be two integral curves for the vector field X with $\gamma_1(t_0) = \gamma_2(t_0) = (x_0, y_0)$ defined on an interval $I \subset \mathbb{R}$. Set

$$J = \{t \in I : \gamma_1(t) = \gamma_2(t)\}.$$

Assume that the complement of J in I is non-empty, i.e., there exists some points $t' \in I \setminus J$ such that $\gamma_1(t') \neq \gamma_2(t')$. We put $t_1 := \sup\{t \in J\}$. Then for $t > t_1$, we have $\gamma_1(t) \neq \gamma_2(t)$. We may assume that the length of the set $I \backslash J$ is at most ϵ. Now we write $\gamma_1(t) = (x_1(t), y_1(t))$ and $\gamma_2(t) = (x_2(t), y_2(t))$, respectively. We put

$$D = \sup_{t \in I \backslash J} (|x_1(t) - x_2(t)| + |y_1(t) - y_2(t)|) > 0,$$

and hence $D > 0$ is a fixed constant. Since g is not locally Lipschitz-continuous, we may assume that g is not Lipschitz-continuous around 0. To exploit this property, we also assume that the integral curves satisfy $|x_1(t)|, |x_2(t)| \leq \delta$, for some small $\delta > 0$. Then we see that

$$\begin{aligned}
\|\gamma_1(t) - \gamma_2(t)\| &\leq |x_1(t) - x_2(t)| + |y_1(t) - y_2(t)| \\
&\leq \int_0^t \|X(\gamma_1(t)) - X(\gamma_2(t))\|\, dt \\
&\leq \int_0^t |f(x_1(t)) - f(x_2(t))| + |y_1(t)g(x_1(t)) - y_2(t)g(x_2(t))|\, dt \\
&\leq \int_0^t |f(x_1(t)) - f(x_2(t))|\, dt + \int_0^t |y_1(t)||g(x_1(t)) - g(x_2(t))|\, dt \\
&\quad + \int_0^t |y_1(t) - y_2(t)||g(x_2(t))|\, dt.
\end{aligned}$$

For the first integral, we simply use the Lipschitz-continuity of f to obtain the bound

$$\begin{aligned}\int_0^t |f(x_1(t)) - f(x_2(t))|\,dt &\le \int_0^t L_f|x_1(t) - x_2(t)|\,dt \\ &\le L_f\epsilon D.\end{aligned}$$

On the other hand, we can bound the third integral easily since the function g is continuous and $|x_2(t)| \le \delta$, and the quantity $|g(D_\delta)|$ is bounded where D_δ is a disc of radius $\delta > 0$. Indeed, we have

$$\begin{aligned}\int_0^t |y_1(t) - y_2(t)||g(x_2(t))|\,dt &\le \int_0^t M_g|y_1(t) - y_2(t)|\,dt \\ &\le M_g\epsilon D.\end{aligned}$$

For the second integral, since g is at least continuous, and $|x_1(t)|, |x_2(t)| \le \delta$, we deduce that $|x_1(t) - x_2(t)| \le 2\delta$. Hence if necessary by shrinking $\delta > 0$, we obtain $|g(x_1(t)) - g(x_2(t))| \le D\epsilon$. Thus we have

$$\int_0^t |y_1(t)||g(x_1(t)) - g(x_2(t))|\,dt \le MD\epsilon,$$

where we used the continuity of $y_1(t)$ on a small interval $[0, t]$. In conclusion, we obtain

$$D \le \epsilon D(M_g + M + L_f).$$

By shrinking ϵ sufficiently we have $\epsilon(M_g + M + L_f) < \frac{1}{2}$ and hence we obtain $D < \frac{1}{2}D$, which contradicts to our assumption that $D > 0$. Thus we have $\gamma_1 = \gamma_2$ on I. □

Aufgabe 4. Betrachte das Vektorfeld

$$X : \mathbb{R} \times \mathbb{R} \to \mathbb{R},\ X(t, u) = 2t - \sqrt{u_+}.$$

Dabei ist $u_+ = \max(u, 0)$. Die Picard-Iteration zum Anfangswertproblem $\gamma(0)$ für dieses Vektorld lautet:

$$\gamma_{n+1}(t) = 0 + \int_0^t 2s - \sqrt{\gamma_n(s)_+}\,ds.$$

Zeige, dass die Folge (γ_n) der Picard-Iterationen zur Anfangskurve $\gamma_0(t) = 0$ abwechselnd zwischen zwei Funktionen hin- und herspringt. Insbesondere konvergiert die Picard-Iteration nicht. Welche Voraussetzung(en) des Satzes von Picard-Lindelöf ist (sind) nicht erfüllt?

Lösung: The map $(t, u) \mapsto 2t - 2\sqrt{u_+}$ is not locally Lipschitz in space, since the derivative of $\sqrt{u_+}$ is not bounded around 0. □