Aufgabe 1. Bestimme die allegemeine Lösung des homogenen linearen System

$$\begin{bmatrix} x'(t) \\ y'(t) \\ z'(t) \end{bmatrix} = \begin{bmatrix} t & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & \frac{1}{t} \end{bmatrix} \begin{bmatrix} x(t) \\ y(t) \\ z(t) \end{bmatrix}.$$

Inwiefern hilft die Diagonalgestalt der Koeffizientenmatrix?

Lösung: Note that thanks to the diagonal matrix, we can easily rewrite the homogeneous linear system as follows:

$$\begin{aligned} x'(t) &= tx(t), \\ y'(t) &= y(t), \\ z'(t) &= \frac{1}{t} z(t). \end{aligned}$$

Then a simple integration gives

$$x(t) = Ce^{\frac{t^2}{2}},\ y(t) = Ce^t,\ z(t) = Ct,$$

where C is integration constant. □

Aufgabe 2. Bestimme die allegemeine Lösung des zeitunabhängigen homogen linearen System

$$\begin{bmatrix} x'(t) \\ y'(t) \end{bmatrix} = \begin{bmatrix} -11 & -6 \\ 18 & 10 \end{bmatrix} \begin{bmatrix} x(t) \\ y(t) \end{bmatrix}.$$

Lösung: We use the given hint to take advantage of diagonal matrix. We have

$$\begin{bmatrix} -11 & -6 \\ 18 & 10 \end{bmatrix} = \begin{bmatrix} 2 & -1 \\ -3 & 2 \end{bmatrix} \begin{bmatrix} -2 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 2 & -1 \\ -3 & 2 \end{bmatrix}^{-1}$$

and hence we substitute the above multiplication of matrix to rewrite

$$\begin{bmatrix} x'(t) \\ y'(t) \end{bmatrix} = \begin{bmatrix} 2 & -1 \\ -3 & 2 \end{bmatrix} \begin{bmatrix} -2 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 2 & -1 \\ -3 & 2 \end{bmatrix}^{-1} \begin{bmatrix} x(t) \\ y(t) \end{bmatrix},$$

and we also have

$$\begin{bmatrix} 2 & -1 \\ -3 & 2 \end{bmatrix} \begin{bmatrix} x'(t) \\ y'(t) \end{bmatrix} = \begin{bmatrix} 2 & -1 \\ -3 & 2 \end{bmatrix} \begin{bmatrix} -2 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} x(t) \\ y(t) \end{bmatrix}.$$

We put

$$\begin{bmatrix} u(t) \\ v(t) \end{bmatrix} = \begin{bmatrix} 2 & -1 \\ -3 & 2 \end{bmatrix}^{-1} \begin{bmatrix} x(t) \\ y(t) \end{bmatrix},$$

or equivalently,

$$\begin{aligned} x(t) &= 2u(t) - v(t) \\ y(t) &= -3u(t) + 2v(t). \end{aligned}$$

Then we can rewrite the homogeneous linear system in terms of (u, v) instead of (x, y) as follows:

$$\begin{bmatrix} u'(t) \\ v'(t) \end{bmatrix} = \begin{bmatrix} -2 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} u(t) \\ v(t) \end{bmatrix},$$

where the solutions are given by

$$u(t) = C_1 e^{-2t},\ v(t) = C_2 e^t.$$

Consequently, the solutions are given by

$$x(t) = 2C_1 e^{-2t} - C_2 e^t,\ y(t) = -3C_1 e^{-2t} + 2C_2 e^t.$$

□

Aufgabe 3.

1. Betrachte $\Gamma(t) = \begin{bmatrix} \cos t & \sin t \\ -\sin t & \cos t \end{bmatrix}$. Gibt es eine zeitunabhängige Matrix A, sodass Γ ein Fundamentalsystem der DGL

$$\Gamma'(t) = A\Gamma(t)$$

ist?

2. Betrachte $\Gamma(t) = \begin{bmatrix} \sin t & \cos t \\ e^t & e^t \end{bmatrix}$. Gibt es eine stetige Funktion $A : \mathbb{R} \to \mathbb{M}_{2\times 2}(\mathbb{R})$, sodass Γ ein Fundamentalsystem der DGL

$$\Gamma'(t) = A(t)\Gamma(t)$$

ist?

Lösung:

1. We write $\Gamma(t) = \begin{bmatrix} \gamma_1(t) & \gamma_2(t) \end{bmatrix}$, where $\gamma_1(t) = \begin{bmatrix} \cos t \\ -\sin t \end{bmatrix}$ and $\gamma_2(t) = \begin{bmatrix} \sin t \\ \cos t \end{bmatrix}$. Then the system $\Gamma'(t) = A\Gamma(t)$ is rewritten as

$$\gamma_1'(t) = A\gamma_1(t),\ \gamma_2'(t) = A\gamma_2(t).$$

Thus the problem is reformulated as a problem of finding a vector field such that γ_1 and γ_2 become integral curves. The required matrix A is given by

$$A = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}.$$

2. We write $\Gamma(t) = \begin{bmatrix}\gamma_1(t) & \gamma_2(t)\end{bmatrix}$, where $\gamma_1(t) = \begin{bmatrix}\sin t \\ e^t\end{bmatrix}$ and $\gamma_2(t) = \begin{bmatrix}\cos t \\ e^t\end{bmatrix}$. Then the system $\Gamma'(t) = A(t)\Gamma(t)$ is rewritten as

$$\gamma_1'(t) = A(t)\gamma_1(t),\ \gamma_2'(t) = A(t)\gamma_2(t).$$

As a trial, we put

$$A(t) = \begin{bmatrix} u(t) & v(t) \\ 0 & 1 \end{bmatrix}.$$

Then the equations $\gamma_1'(t) = A(t)\gamma_1(t),\ \gamma_2'(t) = A(t)\gamma_2(t)$ are given by

$$\begin{aligned} u(t)\sin t + v(t)e^t &= \cos t, \\ u(t)\cos t + v(t)e^t &= -\sin t. \end{aligned}$$

Multiplication by $\cos t$ on both sides for the first equation and multiplication by $\sin t$ on both sides for the second equation and then subtraction in order give

$$v(t)e^t(\cos t - \sin t) = 1,$$

which yields

$$v(t) = \frac{e^{-t}}{\cos t - \sin t}.$$

On the other hand, multiplication by $\sin t$ on both sides for the first equation and multiplication by $\cos t$ on both sides for the second equation and then addition in order give

$$u(t) + v(t)e^t(\sin t + \cos t) = 0,$$

which yields

$$u(t) = -\frac{\sin t + \cos t}{\cos t - \sin t},\ t \neq \frac{\pi}{4}.$$

□

Aufgabe 4. Bestimme ein Fundamentalsystem für die homogene lineare DGL

$$\begin{bmatrix} x'(t) \\ y'(t) \end{bmatrix} = \begin{bmatrix} e^t & 2e^t \\ t & t - e^t \end{bmatrix} \begin{bmatrix} x(t) \\ y(t) \end{bmatrix}.$$

Lösung: We follow the method by d'Alembert. We first consider the solution $(x(t), y(t)) = (\psi(t), -\psi(t))$. One can easily obtain that $\psi(t) = ce^{-e^t}$, where we put $c = 1$. We adopt the notation from the lecture, and we write

$$A(t) = \begin{bmatrix} e^t & 2e^t \\ t & t - e^t \end{bmatrix} = \begin{bmatrix} a_{11}(t) & a_{12}(t) \\ a_{21}(t) & a_{22}(t). \end{bmatrix}$$

We put $\zeta(t) = \begin{bmatrix} 0 \\ \zeta_2(t) \end{bmatrix}$, where

$$\begin{aligned} \zeta_2'(t) &= \left(a_{22}(t) - \frac{\gamma_2(t)}{\gamma_1(t)} a_{12}(t) \right) \zeta_2(t) \\ &= (t + e^t)\zeta_2(t). \end{aligned}$$

Hence we have $\zeta_2(t) = e^{t^2/2+e^t}$. Then

$$\begin{aligned} \varphi'(t) &= \frac{1}{\gamma_1(t)} a_{12}(t)\zeta_2(t) \\ &= e^{e^t} \cdot (2e^t) \cdot e^{t^2/2+e^t} \\ &= e^{t^2/2+2e^t}. \end{aligned}$$

Since $\eta(t) = \varphi(t)\gamma(t) + \zeta(t)$, we see that

$$\eta(t) = \Phi(t) \begin{bmatrix} e^{-e^t} \\ -e^{-e^t} \end{bmatrix} + \begin{bmatrix} 0 \\ e^{t^2/2+e^t}, \end{bmatrix}$$

where $\Phi(t) = \int e^{t^2/2+2e^t} dt$. Then the solutions are given by $\alpha\gamma(t) + \beta\eta(t)$, $\alpha, \beta \in \mathbb{R}$. $\square$