Aufgabe 1. Berechne die Matrixexponentialfunktion für Vielfache von Jordanblöcken $J_m(\lambda)$, d.h., berechne:

$$\exp(tJ_m(\lambda)) = \exp\left(t\begin{bmatrix}\lambda & 1 & 0 & \cdots & 0\\ 0 & \lambda & 1 & \ddots & 0\\ \vdots & 0 & \ddots & \ddots & \vdots\\ & & & & 1\\ 0 & \cdots & \cdots & \cdots & \lambda\end{bmatrix}\right).$$

Im Skript findet sich eine Darstellung als Matrixprodukt. In dieser Aufgabe sollen die Matrixeinträge von $\exp(tJ_m(\lambda))$ explizit bestimmt werden.

Lösung: For an expository purpose, I would like to give a somewhat brutal computation. (You may find a more fancy way of finding the explicit form!) We first consider the matrix $J_2(\lambda)$, with $m = 2$. By an inductive argument, one can show that

$$\begin{bmatrix}\lambda & 1\\ 0 & \lambda\end{bmatrix}^k = \begin{bmatrix}\lambda^k & k\lambda^{k-1}\\ 0 & \lambda^k\end{bmatrix},\ k \geq 2.$$

Indeed, the case $k = 2$ is obvious, and assume that the above identity holds for some $k' \geq 2$. Then

$$\begin{bmatrix}\lambda & 1\\ 0 & \lambda\end{bmatrix}^{k'+1} = \begin{bmatrix}\lambda^{k'} & k'\lambda^{k'-1}\\ 0 & \lambda^{k'}\end{bmatrix}\begin{bmatrix}\lambda & 1\\ 0 & \lambda\end{bmatrix} = \begin{bmatrix}\lambda^{k'+1} & (k'+1)\lambda^{k'}\\ 0 & \lambda^{k'+1}\end{bmatrix}.$$

Now we consider the matrix $J_3(\lambda)$ with $m = 3$. As the case $m = 2$, we obtain the similar result as follows:

$$\begin{bmatrix}\lambda & 1 & 0\\ 0 & \lambda & 1\\ 0 & 0 & \lambda\end{bmatrix}^k = \begin{bmatrix}\lambda^k & k\lambda^{k-1} & T_{k-1}\\ 0 & \lambda^k & k\lambda^{k-1}\\ 0 & 0 & \lambda^k\end{bmatrix}.$$

Here, for the term T_k, one can obtain the recursion formula given by

$$T_k = k\lambda^{k-1} + \lambda T_{k-1},\ k \geq 2,$$

and $T_0 = 0,\ T_1 = 1$. Then we have

$$T_k = \frac{k(k+1)}{2}\lambda^{k-1},\ k \geq 1.$$

Now we consider the matrix $J_4(\lambda)$ with $m = 4$. Then we have

$$\begin{bmatrix}\lambda & 1 & 0 & 0\\ 0 & \lambda & 1 & 0\\ 0 & 0 & \lambda & 1\\ 0 & 0 & 0 & \lambda\end{bmatrix}^k = \begin{bmatrix}\lambda^k & k\lambda^{k-1} & T_{k-1} & S_{k-1}\\ 0 & \lambda^k & k\lambda^{k-1} & T_{k-1}\\ 0 & 0 & \lambda^k & k\lambda^{k-1}\\ 0 & 0 & 0 & \lambda^k\end{bmatrix},$$

where S_k satisfies the following recursion formula:

$$S_k = T_{k-1} + \lambda S_{k-1},\ k \geq 4,$$

with $S_1 = S_2 = 0$ and $S_3 = 1$. Now we write the matrix for a general $m \geq 3$:

$$\begin{bmatrix} \lambda & 1 & 0 & \cdots & 0 \\ 0 & \lambda & 1 & \ddots & 0 \\ \vdots & 0 & \ddots & \ddots & \vdots \\ & & & & 1 \\ 0 & \cdots & \cdots & \cdots & \lambda \end{bmatrix}^k = \left[J_m^k(\lambda)\right]_{i,j=1,\cdots,m}.$$

We get

$$\left[J_m^k(\lambda)\right]_{11} = \lambda^k,\ \left[J_m^k(\lambda)\right]_{12} = k\lambda^{k-1},\ \left[J_m^k(\lambda)\right]_{13} = T_{k-1} = \frac{k(k-1)}{2}\lambda^{k-2}$$

and $\left[J_m^k(\lambda)\right]_{1j}$, $j = 4, \cdots, m$ satisfiy the recursion formula

$$[J_m^{k+1}(\lambda)]_{1j} = [J_m^k(\lambda)]_{1(j-1)} + \lambda[J_m^k(\lambda)]_{1j},$$

with $[J_m^k(\lambda)]_{1j} = 0$, for $k = 0, 1, 2, \cdots, j-3$, and $[J_m^k(\lambda)]_{1j} = 1$, for $k = j-2$. In order to obtain the explicit formula, we put $\lambda = 1$ and we simply write $[J_m^k(\lambda = 1)]_{1j} = [J_m^k]_{1j}$. Then we see that

$$[J_m^{k+1}]_{14} - [J_m^k]_{14} = [J_m^k]_{13} = \frac{k(k-1)}{2},$$

and hence

$$\begin{aligned} [J_m^1]_{14} - [J_m^0]_{14} &= 0, \\ [J_m^2]_{14} - [J_m^1]_{14} &= 0, \\ &\vdots \\ [J_m^{k+1}]_{14} - [J_m^k]_{14} &= \frac{k(k-1)}{2}, \end{aligned}$$

which gives

$$[J_m^{k+1}]_{14} = \sum_{j=1}^{k} \frac{j(j-1)}{2} = \frac{k(k-1)(k+1)}{6}.$$

Similarly, we obtain

$$[J_m^{k+1}]_{15} = \sum_{j=1}^{k} \frac{j(j-1)(j+1)}{6} = \frac{k(k-1)(k+1)(k+2)}{24}.$$

In general, we have

$$[J_m^{k+1}]_{1j} = \frac{(k-1)k(k+1)\cdots(k+j-3)}{(j-1)!},\ j \geq 3,$$

or equivalently,

$$[J_m^{k+1}]_{1(j+3)} = \frac{(k-1)k(k+1)\cdots(k+j)}{(j+2)!},\ j \geq 0.$$

Hence we see that

$$[\exp(tJ_m(\lambda))]_{1(j+3)} = \sum_{k=j+1}^{\infty} \frac{t^k(k+1)\cdots(k+j)}{(k-2)!(j+2)!}\lambda^{k-2}$$

For $i > 2$, we obtain the explicit components via an obvious symmetry of the matrix. Then we use the power series formula: $e^x = \sum_{k\geq 0} \frac{x^k}{k!}$, for all $x \in \mathbb{R}$ to obtain the explicit formula of the matrix $\exp(tJ_m(\lambda))$. For example, if $i = j = 1, \cdots, m$, then

$$[\exp(tJ_m(\lambda))]_{ij} = \sum_{k=0}^{\infty} \frac{1}{k!}(t\lambda)^k = e^{t\lambda}.$$

If $i > j$ then we have $[\exp(tJ_m(\lambda))]_{ij} = 0$. If $j - i = 1$, then

$$\begin{aligned}
[\exp(tJ_m(\lambda))]_{ij} &= \sum_{k=0}^{\infty} \frac{1}{k!}t^k k\lambda^{k-1} = \sum_{k=1}^{\infty} \frac{1}{k!}kt^k\lambda^{k-1} = \sum_{k=1}^{\infty} \frac{t^k\lambda^{k-1}}{(k-1)!} \\
&= t\sum_{k=1}^{\infty} \frac{(t\lambda)^{k-1}}{(k-1)!} = te^{t\lambda}.
\end{aligned}$$

If $j - i = 2$, then

$$\begin{aligned}
[\exp(tJ_m(\lambda))]_{ij} &= \sum_{k=0}^{\infty} \frac{1}{k!}t^k T_{k-1} \\
&= \sum_{k=0}^{\infty} \frac{1}{k!}t^k \frac{k(k-1)}{2}\lambda^{k-2} \\
&= \frac{1}{2}\sum_{k=2}^{\infty} \frac{t^k\lambda^{k-2}}{(k-2)!} = \frac{t^2}{2}e^{t\lambda}.
\end{aligned}$$

If $j - i = 3$, then

$$\begin{aligned}
[\exp(tJ_m(\lambda))]_{ij} &= \sum_{k=2}^{\infty} \frac{t^k(k+1)}{(k-2)!3!}\lambda^{k-2} \\
&= \sum_{k=0}^{\infty} \frac{t^{k+2}(k+3)}{3!k!}\lambda^k \\
&= \frac{t^2}{6}\sum_{k=0}^{\infty} \frac{t^k(k+3)}{k!}\lambda^k = \frac{t^2}{6}\left(\sum_{k=0}^{\infty} \frac{k}{k!}(t\lambda)^k + \sum_{k=0}^{\infty} \frac{3}{k!}(t\lambda)^k\right) \\
&= \frac{t^2}{6}(t\lambda + 3)e^{t\lambda}.
\end{aligned}$$

In this way, one can obtain the explicit formula of the matrix. □

Aufgabe 2. Zeige, dass für Blockdiagonalmatrizen gilt

$$\exp\left(\begin{bmatrix} A_1 & & & \\ & A_2 & & \\ & & \ddots & \\ & & & A_\ell \end{bmatrix}\right) = \begin{bmatrix} \exp(A_1) & & & \\ & \exp(A_2) & & \\ & & \ddots & \\ & & & \exp(A_\ell) \end{bmatrix}.$$

Lösung: It suffices to show that

$$\begin{bmatrix} A & 0 \\ 0 & B \end{bmatrix}^k = \begin{bmatrix} A^k & 0 \\ 0 & B^k \end{bmatrix},$$

where A and B are block matrices. By inductive argument, it is enough to show the above identity for $k = 2$. We first consider $A = a \in \mathbb{R}$ and $B \in M_{n\times n}(\mathbb{R})$, and put

$$M = \begin{bmatrix} a & 0 \\ 0 & B \end{bmatrix}.$$

Them $[M]_{ij} = m_{ij} = a$ for $i = j = 1$, $m_{1j} = m_{i1} = 0$ for $i, j \geq 2$, and $m_{ij} = [B]_{(i-1)(j-1)} = b_{(i-1)(j-1)}$, for $i, j = 2, \ldots, n+1$. Now we compute M^2. We see that

$$[M^2]_{11} = \sum_{i=1}^{n+1} m_{1i} m_{i1} = m_{11} m_{11} = a^2,$$

$$[M^2]_{1j} = \sum_{k=1}^{n+1} m_{1k} m_{kj} = m_{11} m_{1j} = 0, \ j \geq 2,$$

$$[M^2]_{i1} = \sum_{k=1}^{n+1} m_{ik} m_{k1} = m_{i1} m_{11} = 0, \ i \geq 2,$$

$$[M^2]_{ij} = \sum_{k=1}^{n+1} m_{ik} m_{kj} = \sum_{k=2}^{n+1} m_{ik} m_{kj} = \sum_{k=1}^{n} b_{(i-1)k} b_{k(j-1)} = [B^2]_{(i-1)(j-1)},$$

and hence

$$M^2 = \begin{bmatrix} a^2 & 0 \\ 0 & B^2 \end{bmatrix}.$$

Now we consider the general case $A \in M_{n\times n}(\mathbb{R})$ and $B \in M_{m\times m}(\mathbb{R})$. We put

$$M = \begin{bmatrix} A & 0 \\ 0 & B \end{bmatrix},$$

where $[M]_{ij} = m_{ij} = a_{ij}$ for $1 \leq i, j \leq n$, and $m_{ij} = b_{ij}$ for $i, j = n+1, \cdots, n+m$. We compute M^2. We see that for $1 \leq i, j \leq n$

$$[M^2]_{ij} = \sum_{k=1}^{n+m} m_{ik} m_{kj} = \sum_{k=1}^{n} m_{ik} m_{kj} = \sum_{k=1}^{n} a_{ik} a_{kj} = [A^2]_{ij},$$

and for $n+1 \leq i, j \leq n+m$,

$$[M^2]_{ij} = \sum_{k=1}^{n+m} m_{ik}m_{kj} = \sum_{k=n+1}^{n+m} m_{ik}m_{kj} = \sum_{k=n+1}^{n+m} b_{ik}b_{kj} = [B^2]_{ij},$$

which shows that

$$M^2 = \begin{bmatrix} A^2 & 0 \\ 0 & B^2 \end{bmatrix}.$$

□

Aufgabe 3. Trage die nötigen Rechtfertigungen für die Rechnung

$$\exp(PAP^{-1}) = P\exp(A)P^{-1}$$

nach.

Lösung: At least $\|A\|$ should be bounded. □

Aufgabe 4. Berechne

$$\exp\left(t\begin{bmatrix} -1 & 3 & -3 \\ 3 & 0 & 2 \\ 3 & -2 & -4 \end{bmatrix}\right).$$

Gib auch die allgemeine Lösung der homogenen linearen DGL

$$\begin{bmatrix} x'(t) \\ y'(t) \\ z'(t) \end{bmatrix} = \begin{bmatrix} -1 & 3 & -3 \\ 3 & 0 & 2 \\ 3 & -2 & -4 \end{bmatrix} \begin{bmatrix} x(t) \\ y(t) \\ z(t) \end{bmatrix}$$

an.

Lösung: We first note the diagonalization of the given matrix:

$$\begin{bmatrix} -1 & 3 & -3 \\ 3 & 0 & 2 \\ 3 & -2 & -4 \end{bmatrix} = \begin{bmatrix} -1 & 0 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 0 \end{bmatrix} \begin{bmatrix} -1 & 0 & 0 \\ 0 & 2 & 1 \\ 0 & 0 & 2 \end{bmatrix} \begin{bmatrix} -1 & 0 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 0 \end{bmatrix}^{-1}.$$

Then we see that

$$\exp\left(t\begin{bmatrix} -1 & 3 & -3 \\ 3 & 0 & 2 \\ 3 & -2 & -4 \end{bmatrix}\right) = \begin{bmatrix} -1 & 0 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 0 \end{bmatrix} \exp\left(t\begin{bmatrix} -1 & 0 & 0 \\ 0 & 2 & 1 \\ 0 & 0 & 2 \end{bmatrix}\right) \begin{bmatrix} -1 & 0 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 0 \end{bmatrix}^{-1}.$$

By the previous problems, we see that

$$\begin{bmatrix} -1 & 0 & 0 \\ 0 & 2 & 1 \\ 0 & 0 & 2 \end{bmatrix}^k = \begin{bmatrix} (-1)^k & 0 & 0 \\ 0 & 2^k & k2^{k-1} \\ 0 & 0 & 2^k \end{bmatrix},$$

and hence

$$\exp\left(t\begin{bmatrix}-1 & 3 & -3\\ 3 & 0 & 2\\ 3 & -2 & -4\end{bmatrix}\right) = \begin{bmatrix}-1 & 0 & 1\\ 1 & 1 & 1\\ 1 & 1 & 0\end{bmatrix}\begin{bmatrix}e^{-t} & 0 & 0\\ 0 & e^{2t} & te^{2t}\\ 0 & 0 & e^{2t}\end{bmatrix}\begin{bmatrix}-1 & 0 & 1\\ 1 & 1 & 1\\ 1 & 1 & 0\end{bmatrix}^{-1}$$
$$= \begin{bmatrix}-1 & 0 & 1\\ 1 & 1 & 1\\ 1 & 1 & 0\end{bmatrix}\begin{bmatrix}e^{-t} & 0 & 0\\ 0 & e^{2t} & te^{2t}\\ 0 & 0 & e^{2t}\end{bmatrix}\begin{bmatrix}-1 & 1 & -1\\ 1 & -1 & 2\\ 0 & 1 & -1\end{bmatrix}$$
$$= \begin{bmatrix}-1 & 0 & 1\\ 1 & 1 & 1\\ 1 & 1 & 0\end{bmatrix}\begin{bmatrix}-e^{-t} & e^{-t} & -e^{-t}\\ e^{2t} & (t-1)e^{2t} & (2-t)e^{2t}\\ 0 & e^{2t} & -e^{2t}\end{bmatrix}$$
$$= \begin{bmatrix}e^{-t} & e^{2t}-e^{-t} & e^{-t}-e^{2t}\\ e^{2t}-e^{-t} & te^{2t}+e^{-t} & (1-t)e^{2t}-e^{-t}\\ e^{2t}-e^{-t} & (t-1)e^{2t}-e^{-t} & (2-t)e^{2t}-e^{-t}\end{bmatrix}.$$

Now we solve the DGL. We put

$$\begin{bmatrix}u(t)\\ v(t)\\ w(t)\end{bmatrix} = \begin{bmatrix}-1 & 0 & 1\\ 1 & 1 & 1\\ 1 & 1 & 0\end{bmatrix}^{-1}\begin{bmatrix}x(t)\\ y(t)\\ z(t)\end{bmatrix}.$$

Then we have

$$\begin{bmatrix}u'(t)\\ v'(t)\\ w'(t)\end{bmatrix} = \begin{bmatrix}-1 & 0 & 0\\ 0 & 2 & 1\\ 0 & 0 & 2\end{bmatrix}\begin{bmatrix}u(t)\\ v(t)\\ w(t)\end{bmatrix}.$$

Then the solutions are easily given by

$$\begin{bmatrix}u(t)\\ v(t)\\ w(t)\end{bmatrix} = \begin{bmatrix}e^{-t}\\ (at+b)e^{2t}\\ (ct+d)e^{2t}\end{bmatrix}.$$

Finally we obtain

$$\begin{bmatrix}x(t)\\ y(t)\\ z(t)\end{bmatrix} = \begin{bmatrix}-1 & 0 & 1\\ 1 & 1 & 1\\ 1 & 1 & 0\end{bmatrix}\begin{bmatrix}u(t)\\ v(t)\\ w(t)\end{bmatrix} = \begin{bmatrix}-1 & 0 & 1\\ 1 & 1 & 1\\ 1 & 1 & 0\end{bmatrix}\begin{bmatrix}e^{-t}\\ (at+b)e^{2t}\\ (ct+d)e^{2t}\end{bmatrix}.$$

$\square$