Aufgabe 1. Beweise folgende Verallgemeinerung des Lemma von Gronwald. Für die stetigen Funktionnen $f, h : [0, a] \to \mathbb{R}$ und die reelle Zahl $a \in \mathbb{R}$ möge gelten:

1. $h(t) \geq 0$ für alle t.
2. $f(t) \leq \alpha + \int_0^t h(s)f(s)\,ds$ für alle t.

Dann ist für alle t

$$f(t) \leq \alpha e^{H(t)}$$

mit $H(t) = \int_0^t h(s)\,ds$.

Lösung: We put

$$F(t) = \alpha + \int_0^t h(s)f(s)\,ds, \quad G(t) = \alpha \exp\left(\int_0^t h(s)\,ds\right),$$

and

$$\Gamma(t) = \frac{F(t)}{G(t)}.$$

Since $G > 0$, the function Γ is well-defined. Note that

$$\Gamma(0) = \frac{\alpha}{\alpha} = 1.$$

We also have

$$\begin{aligned} H'(t) &= \frac{F'(t)G(t) - F(t)G'(t)}{G(t)^2} = \frac{h(t)f(t)G(t) - F(t)h(t)G(t)}{G(t)^2} \\ &= h(t)\frac{f(t) - F(t)}{G(t)} \leq 0, \end{aligned}$$

where the inequality holds due to the assumption $f(t) \leq F(t)$, and hence the function $\Gamma(t)$ is non-increasing, from which we deduce that $F(t) \leq G(t)$, which completes the proof of Grönwall's lemma.
Remark. In the proof, any specific condition of $a \in \mathbb{R}$ is not used, and hence we can put the interval simply $[0, \infty)$. □

Aufgabe 2. Löse die folgenden Anfangswertproblem:

$$\begin{aligned} y'' &= 6y + y', \quad y(0) = 1,\ y'(0) = 0, \\ y'' &= -3y' + 4y, \quad y(0) = 0,\ y'(0) = 1. \end{aligned}$$

Lösung: We first consider the initial value problem:

$$y'' = 6y + y', \quad y(0) = 1,\ y'(0) = 0,$$

which can be rewritten as a first order system:

$$\begin{bmatrix} X' \\ Y' \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ 6 & 1 \end{bmatrix} \begin{bmatrix} X \\ Y \end{bmatrix},$$

where we put $X(t) = y(t)$ and $Y(t) = y'(t)$. We diagonalize the matrix and get

$$\begin{bmatrix} X' \\ Y' \end{bmatrix} = S \begin{bmatrix} -2 & 0 \\ 0 & 3 \end{bmatrix} S^{-1} \begin{bmatrix} X \\ Y \end{bmatrix},$$

where

$$S = \begin{bmatrix} -1 & 1 \\ 2 & 3 \end{bmatrix}, \; S^{-1} = \frac{1}{5} \begin{bmatrix} -3 & 1 \\ 2 & 1 \end{bmatrix}.$$

We put

$$\begin{bmatrix} u \\ v \end{bmatrix} = S^{-1} \begin{bmatrix} X \\ Y \end{bmatrix}$$

and get

$$\begin{bmatrix} u' \\ v' \end{bmatrix} = \begin{bmatrix} -2 & 0 \\ 0 & 3 \end{bmatrix} \begin{bmatrix} u \\ v \end{bmatrix},$$

from which we deduce that $u(t) = C_1 e^{-2t}$ and $v(t) = C_2 e^{3t}$. Then we have

$$y(t) = -C_1 e^{-2t} + C_2 e^{3t}.$$

The initial condition $y(0) = 1,\; y'(0) = 0$ gives

$$\begin{aligned} -C_1 + C_2 &= 1, \\ 2C_1 + 3C_2 &= 0, \end{aligned}$$

which yields $C_1 = -\frac{7}{5},\; C_2 = \frac{2}{5}$.
Now we consider the initial value problem

$$y'' = -3y' + 4y, \quad y(0) = 0,\; y'(0) = 1,$$

or equivalently

$$\begin{bmatrix} X' \\ Y' \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ 4 & -3 \end{bmatrix} \begin{bmatrix} X \\ Y \end{bmatrix},$$

which is via diagonalization rewritten as

$$\begin{bmatrix} X' \\ Y' \end{bmatrix} = S \begin{bmatrix} -4 & 0 \\ 0 & 1 \end{bmatrix} S^{-1} \begin{bmatrix} X \\ Y \end{bmatrix},$$

where $S = \begin{bmatrix} -1 & 1 \\ 4 & 1 \end{bmatrix}$, $S^{-1} = \frac{1}{5}\begin{bmatrix} -1 & 1 \\ 4 & 1 \end{bmatrix}$. We put

$$\begin{bmatrix} u \\ v \end{bmatrix} = S^{-1} \begin{bmatrix} X \\ Y \end{bmatrix}$$

and get

$$\begin{bmatrix} u' \\ v' \end{bmatrix} = \begin{bmatrix} -4 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} u \\ v \end{bmatrix},$$

from which we deduce that $u(t) = C_1 e^{-4t}$ and $v(t) = C_2 e^t$. Then we have

$$y(t) = -C_1 e^t + C_2 e^{3t}.$$

The initial condition $y(0) = 0,\ y'(0) = 1$ gives

$$\begin{aligned} -C_1 + C_2 &= 0, \\ 4C_1 + C_2 &= 1, \end{aligned}$$

which yields $C_1 = \frac{1}{5},\ C_2 = -\frac{1}{5}$. □

Aufgabe 3. Welche Lösungskurve der DGL

$$y'' - 4y' + 4y = 0$$

hat im Punkt $x = 0,\ y = \frac{1}{2}$ einen Extremwert?

Lösung: The solution requires the initial condition $y(0) = \frac{1}{2}$, $y'(0) = 0$, and hence we can rewrite the above initial value problems as follows:

$$\begin{bmatrix} X' \\ Y' \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -4 & 4 \end{bmatrix} \begin{bmatrix} X \\ Y \end{bmatrix},$$

where $X = y$ and $Y = y'$, which can be diagonalized as

$$\begin{bmatrix} X' \\ Y' \end{bmatrix} = S \begin{bmatrix} 2 & 1 \\ 0 & 2 \end{bmatrix} S^{-1} \begin{bmatrix} X \\ Y \end{bmatrix},$$

for some square matrix S. Then we have the solution $y(t) = (at+b)e^{2t}$. The initial conditions give $b = \frac{1}{2}$ and $a = -1$. □

Aufgabe 4. Bestimme die allgemeine Lösung der DGL

$$y''' + 9y'' + 27y' + 27y = 0.$$

Lösung: As the previous problems, we rewrite the above differential equation as the first order system via a process of diagonalization:

$$\begin{bmatrix} X' \\ Y' \\ Z' \end{bmatrix} = S \begin{bmatrix} -3 & 1 & 0 \\ 0 & -3 & 1 \\ 0 & 0 & -3 \end{bmatrix} S^{-1} \begin{bmatrix} X \\ Y \\ Z \end{bmatrix},$$

where we put $X = y,\ Y = y',\ Z = y''$ and

$$S = \begin{bmatrix} 1 & \frac{2}{3} & \frac{1}{3} \\ -3 & -1 & -\frac{1}{3} \\ 1 & 0 & 0 \end{bmatrix},\ S^{-1} = \begin{bmatrix} 0 & 0 & \frac{1}{9} \\ -3 & -3 & -\frac{2}{3} \\ 9 & 6 & 1 \end{bmatrix}.$$

Then we have the solution $X = (at^2 + bt + c)e^{-3t}$. Since the solution y is merely a linear combination of X, Y, Z, the general solutions y is given by $y(t) = (at^2 + bt + c)e^{-3t}$. □