Aufgabe 1. Betrachte das zeitunabhängige Vektorfeld

$$X : (x, y) \mapsto (-y - x^3, x - y^3).$$

Zeige, dass X linearen Hauptteil hat und berechne dessen Eigenwerte. Untersuche die Stabilitäte der Ruhelage $(0, 0)$. Entscheide, ob das zeitabhängige Vektorfeld

$$X_t : (x, y) \mapsto (-y - tx^3, x - ty^3)$$

ebenfalls einen linearen Hauptteil hat und ob die stationäre Integralkurven $\gamma \equiv 0$ stabil ist.

Lösung: Note that $D_{x,y}X = \begin{bmatrix} \frac{\partial}{\partial x}(-y - x^3) & \frac{\partial}{\partial y}(-y - x^3) \\ \frac{\partial}{\partial x}(x - y^3) & \frac{\partial}{\partial y}(x - y^3) \end{bmatrix} = \begin{bmatrix} -3x^2 & -1 \\ 1 & -3y^2 \end{bmatrix}$. Then

$$X(x, y) = \begin{bmatrix} -y - x^3 \\ x - y^3 \end{bmatrix} = \begin{bmatrix} -3x^2 & -1 \\ 1 & -3y^2 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} + \varphi(x, y),$$

where $\varphi(x, y) = \begin{bmatrix} 2x^3 \\ 2y^3 \end{bmatrix}$. Thus the matrix $A = \begin{bmatrix} -3x^2 & -1 \\ 1 & -3y^2 \end{bmatrix}$ is the linear part of X. The eigenvalues of A at $x = 0, y = 0$ is $\pm i$, purely imaginary. Now we consider the Lyapunov function $f(x, y) = x^2 + y^2$. Since $f(0, 0) = 0$, $f(x, y) > 0$ for $x, y \neq 0$ and $X(f) = -2(x^4 + y^4) \leq 0$, the function f is truly the Lyapunov function for the vector field X. Then the equilibrium point $(0, 0)$ is stable.
For the vector field X_t the linear part is given by the same as the matrix A. Using the Lyapunov function f, the stationary integral curve $\gamma \equiv 0$ is stable for $t > 0$. □

Aufgabe 2. Sei $X : U \to \mathbb{R}^n$ ein lokal Lipschitzstetiges Vektorfeld mit kritischem Punkt $0 \in U$. Sei $f : U \to \mathbb{R}$ differenzierbar. Es sei (u_k) eine Nullfolge in U mit $f(u_k) > 0$. Zeige, dass die Ruhelage 0 instabil ist, sofern mindestens eine der beiden folgenden Bedingungen erfüllt ist:

1. Es gibt ein $\lambda > 0$, so dass $X(f) \geq \lambda f$ auf ganz U gilt.
2. $X(f)(u) > 0$ für alle $u \neq 0$.

Lösung: We let $\epsilon > 0$ be given with $\overline{B_\epsilon(0)} \subset U$. Choose $\delta > 0$ so that $u_k \in B_\delta(0)$ for any $k \geq N$. (Such a N always exist, since the sequence u_k is a zero sequence, i.e., $\lim_{k\to\infty} u_k = 0$.) Then $f(0) \geq 0$. Now we choose $c > 0$ so that

$$\inf_{u \in \partial B_\epsilon(0)} f(u) > c.$$

If we assume the condition 1 or 2 above, we see that $f \circ \gamma$ is strictly increasing, where γ is an integral curve of X. For simplicity, we choose a suitable subsequence of u_k so that the norm $\|u_k\|$ is strictly decreasing to 0 and put this subsequence u_k again. Then for each k we have the integral curve γ_k of X with $\gamma_k(0) = u_k$. Now we assume that the equilibrium point 0 is stable, i.e., $\|\gamma_k(0)\| < \delta$ implies $\|\gamma_k(t)\| < \epsilon$ for all t. Since $f \circ \gamma_k$ is strictly increasing, we have

$$f(\gamma_k(t)) > 100c$$

for sufficiently large t. However, this contradicts to our assumption $\|\gamma_k(t)\| < \epsilon$ and our choice of c. □

Aufgabe 3. Sei $g : \mathbb{R}^2 \to \mathbb{R}$ differentierbar. Betrachte das zeitunabhängige Vektorfeld

$$X : (x, y) \mapsto (y - xg(x, y), -x - yg(x, y)).$$

Zeige, dass die Ruhelage 0 asymptotisch stabil ist, wenn $g > 0$ in einer Umgebung von $(0, 0)$ gilt. Zeige auch, dass die Ruhelage instabil ist, wenn $g < 0$ in einer Umgebung von $(0, 0)$ gilt.

Lösung: Suppose that $g(x, y) > 0$ for all (x, y) in a open neighborhood of the origin $(0, 0)$. Then the function $f(x, y) = x^2 + y^2$ is the Lyapunov function for the vector field X and hence the equilibrium point $(0, 0)$ is stable. Note that $X(f) = -2(x^2 + y^2)g(x, y)$.
Now we assume that $g(x, y) < 0$ for (x, y) in a open neighborhood of the origin $(0, 0)$. Since $X(f) > 0$, by the previous problem the equilibrium point $(0, 0)$ is instable. □

Aufgabe 4. Bestimme die kritischen Punkte (Ruhelagen / stationäre Integralkurven) des zeitunabhängigen Veltorfelds

$$X : (x, y) \mapsto (y, \alpha x + \beta \sin y)$$

für alle Parameterwerte $\alpha, \beta \neq 0$. Diskutiere die Stabilität, asymptotische Stabilität und Instabilität der stationäre Integralkurven.

Lösung: The linear part of the vector field X is given by

$$A = \begin{bmatrix} 0 & 1 \\ \alpha & \beta \cos y \end{bmatrix},$$

whose eigenvalues at $x = y = 0$ are given by

$$\lambda = \frac{\beta \pm \sqrt{\beta^2 + 4\alpha}}{2}.$$

1. $\beta^2 + 4\alpha < 0$: If $\beta < 0$ then the stationary integral curve $\gamma \equiv 0$ is asymptotically stable. If $\beta > 0$, then $\gamma \equiv 0$ is instable.

2. $\beta^2 + 4\alpha \geq 0$: If $\beta > 0$, then $\gamma \equiv 0$ is instable, since at least one eigenvalue is negative. Now we consider the case $\beta < 0$. If $\beta + \sqrt{\beta^2 + 4\alpha} > 0$ then $\gamma \equiv 0$ is again instable. On the other hand, if $\beta + \sqrt{\beta^2 + 4\alpha} < 0$, then $\gamma \equiv 0$ is asymptotically stable, since all the eigenvalues are positive.

□