In this seminar, we study the theory of simplicial complexes and some of its applications. The topics include the following: simplicial complexes, simplicial homology, singular homology, simplicial approximation, classification of compact surfaces. The prerequisites are elementary topology and basic group theory. The main references are the lecture notes [1] by Ulrich Brehm and the book [2] by Allen Hatcher. (Many thanks to the authors, both references are available online.) The talks are as follows.

(1) **Simplicial complexes** Define geometric and abstract simplicial complexes and explain the realizations of abstract simplicial complexes. Define morphisms between simplicial complexes and prove basic facts on them. Examples of simplicial complexes. [1] Section 3

(2) **Simplicial homology** Explain the order of the vertices of a simplex. Define the ∆-complex structure on a topological space. Define simplicial homology for a ∆-complex. Explain some explicit examples, e.g., for torus, projective plane. Explain the remarks on [2] Page 107. Define the Barycentric subdivision of a ∆-complex. Remark that the second Barycentric subdivision of a ∆-complex is a simplicial complex (prove this if time permits.) In particular, every ∆-complex is homeomorphic to a simplicial complex. [2] Page 102-107

(3) **Singular homology, part one** Define singular homology $H_*(X, Z)$ for a topological space $X$. Define relative singular homology $H_*(X, Y)$ for a pair of topological spaces $(X, Y)$, where $Y$ is a subspace of $X$. Show that continuous maps of topological spaces induce morphisms of homology groups. Prove some basic properties of homology groups ([2] Propositions 2.6-2.8). Define the reduced homology groups. For a path-connected topological space $X$, construct a morphism $\Pi_1(X) \to H_1(X)$ and explain (without proof) the kernel of this morphism ([2] Theorem 2A.1). [1] Section 6, till Proposition 6.3

(4) **Singular homology, part two** Define homotopy between two chain complexes and prove that homotopic chain complexes have isomorphic homology groups. Recall the notion of homotopy in topology. Prove that homotopic topological spaces have isomorphic homology groups. [1] Section 6, till Korollar 6.7], [2] Page 110-113

(5) **Singular homology, part three** For $Y$ a subspace of $X$, construct the long exact sequence of homology groups

$$\cdots \to H_{n+1}(X,Y) \to H_n(Y) \to H_n(X,Y) \to H_n(X) \to \cdots$$

2.15. Prove the equivalence of simplicial and singular homology.

(6) **Some applications** Let \( f : S^n \to S^n \) be a continuous function, define \( \text{deg}(f) \) and explain the results in [2, Page 134-136]. Find more applications in [2, Page 155] and explain your favorite examples.

(7) **Simplicial approximation, part one** Define simplicial approximation. Show that a continuous map is homotopic to its simplicial approximation. Define simplicial pair and prove a relative version of the last result. [1, Section 4, till Satz 4.9]

(8) **Simplicial approximation, part two** Define the fundamental group of a simplicial complex. Explain the relation between the fundamental group of a simplicial complex and the fundamental group of its underlying topological space. Compute the fundamental group in a special case (more precisely, if \( K \) is a simplicial complex and \( L \) is a subcomplex such that \( K \) is connected and \( L \) is contractible, then \( \tilde{\Pi}_1(K, a_0) \cong F(M)/\langle R \rangle \) for an explicit set \( M \) and a set of explicit relations \( R \)). Explain the example of \( \mathbb{R}P^2 \). [1, Section 4, till Satz 4.14]

(9) **Simplicial approximation, part three** Prove the Seifert-van Kampen Theorem. Show that any group of the form \( F(M)/\langle R \rangle \) is a fundamental group of a path-connected simplicial complex. Examples. Finish [1, Section 4]

(10) **Classification of compact surfaces** Construct the surfaces \( M_g \) (\( g \geq 0 \)) and \( N_h \) (\( h \geq 1 \)) and show that they are pairwise non-homeomorphic. Classify all compact surfaces. [1, Section 5, Satz 4.14]

(11) **Lefschetz fixed point theorem** Prove the Lefschetz Fixed Point Theorem [2, Theorem 2C.3]. Explain some applications of this theorem [2, Page 184].

(12) **Further topics** There are many other topics related to simplicial complexes and homology theory, which we could study in this seminar. For example, CW complexes, homology with coefficients, and the Borsuk-Ulam Theorem... The choices depend on the schedule of the seminar and the interests of the students.

**References**
