IHARA’S LEMMA FOR SHIMURA CURVES OVER TOTALLY REAL FIELDS AND MULTICIPACITY TWO

CHUANGXUN CHENG

Abstract. In this paper, under some technical conditions on the coefficients, we prove Ihara’s lemma for Shimura curves over totally real fields. We then construct examples where the multiplicities of Galois representations in the cohomology groups of Shimura curves are two. These examples provide some evidence for the BDJ conjecture.

1. Introduction

Ihara’s lemma is a result about the kernel of a certain map between cohomology groups (or Jacobians) of modular curves with different level structures. In the Taylor-Wiles system method, Ihara’s lemma is a key ingredient to extend an $R = T$ theorem in the minimal case to the nonminimal case.

In [9], the authors proved an analogue of Ihara’s lemma for Shimura curves over $\mathbb{Q}$. The argument cannot apply to general totally real fields case directly because in the later case the Shimura curves are in general not moduli spaces of abelian varieties and we do not have a universal object to construct the crystals as in [9]. Nevertheless, in [23], the author showed how we could construct crystals over Shimura curves from the crystals over unitary Shimura curves. In this paper, we formulate an analogue of Ihara’s lemma for Shimura curves over totally real fields and prove it in certain cases by combining the methods in [9] and [23]. The result is Theorem 1.2 below.

In [5], by the Diamond-Taylor-Wiles method, the author proved an $R = T$ theorem and a multiplicity one result for Shimura curves over totally real fields (see Theorem 1.3 below.) The results in [5] deal primary with the minimal case. For those coefficients to which we may apply Theorem 1.2, we can extend the results in [5] from minimal case to nonminimal case. In particular, we have Theorem 1.5 below. Starting with this result, we then construct examples where the multiplicity one fails. More precisely, by adapting the method in [22], we construct examples where the multiplicities of Galois representations in the cohomology groups of Shimura curves are two (see Theorem 1.6 below.)

To state our main results, we first fix some notation. Let $F$ be a totally real field of degree $n = [F : \mathbb{Q}] > 1$, $\text{Hom}_F(F, \mathbb{R}) = \{\tau_1, \ldots, \tau_n\}$. Let $S_D$ be a finite set of finite places of $F$ such that $|S_D| \equiv n - 1 \pmod{2}$. Let $D$ be a quaternion algebra over $F$ which is ramified at $\{\tau_2, \ldots, \tau_n\} \cup S_D$. (There is a unique one up to isomorphism.) Let $G = \text{Res}_{F/\mathbb{Q}} D^\times$ be the algebraic group over $\mathbb{Q}$ attached to $D^\times$. Let $L$ be a finite extension of $F$ which splits $D$. We also assume that $L$ is sufficiently large. In particular, we assume that $L$ contains the image of each embedding $F \to \bar{\mathbb{Q}}$.

Let $p > 3$ be a rational prime number unramified in $F$. Assume that $p$ factors as $p = p_1 p_2 \cdots p_r$. We also fix a bijection between $\text{Hom}(F, \mathbb{R})$ and $\coprod_{p | p} \text{Hom}(F_p, \bar{\mathbb{Q}}_p)$, denote
this set by $S$. We fix a prime $\lambda \mid p$ of $L$. Let $O$ be the ring of integers of $L_\lambda$ and $\kappa$ the residue field.

Fix a maximal order $O_D$ of $D$ and isomorphisms $O_D \otimes_{O_F} O_F = M_2(O_F)$ for finite primes $v \nmid S_D$. Let $K_0 \subset (D \otimes_F \mathbb{A}_F)^	imes$ be the open compact subset such that the $v$-component $(K_0)_v = (O_D \otimes_{O_F} O_F)_v$. In particular, $(K_0)_v = GL_2(O_F)$ for finite primes $v \nmid S_D$. Define $K_0(N)$, for an ideal $N$ prime to $S_D$, to be the subgroup of $K_0$ consisting of those $u$ for which $u_v$ congruent to $\begin{pmatrix} * & * \\ 0 & * \end{pmatrix}$ mod($\text{ord}_v(N)$) for every $v \mid N$.

1.1. The curves. Let $X$ be the $G(\mathbb{R})$-conjugacy class of the map

$$h : \mathbb{C}^\times \to G(\mathbb{R}) \cong \text{GL}_2(\mathbb{R}) \times \mathbb{H}^\times \times \cdots \times \mathbb{H}^\times,$$

which maps $a + ib$ to $\left(\begin{array}{cc} a & -b \\ b & a \end{array} \right), 1, \ldots, 1$. The conjugacy class $X$ is naturally identified with the union of the upper and lower half plane by the map $ghg^{-1} \mapsto g(i)$, where $g(i) = x + iy$

if $g = \left(\begin{array}{cc} a & b \\ c & d \end{array} \right), \cdots$.

Let $M = M(G, X) = (M_H)_H$ be the canonical model defined over $F$ of the Shimura variety defined by $G$ and $X$. (Here $H$ runs through the open compact subgroups of $G(\mathbb{A}_F)$.) Each $M_H$ is proper and smooth but not necessarily geometrically connected over $F$, and $M_H(\mathbb{C}) \cong G(\mathbb{Q}) \backslash X \times G(\mathbb{A}_F) / H$.

For each $H$ and $H'$ and $g \in G(\mathbb{A}_F)$ with $g^{-1}H'g \subset H$ and $H$ sufficiently small (see [3, Lemma 1.4.1.1]), there is an etale map $g_0 : M_H \to M_H$ which on complex points coincides with the one induced by right multiplication by $g$ in $G(\mathbb{A}_F)$. For a normal subgroup $H'$ of $H$, the etale cover $g_1 : M_H \to M_H$ is Galois, and mapping $g^{-1} \mapsto g_0$ defines an isomorphism of $H/H'$ with a group of covering maps.

The weight homomorphism $w : \mathbb{G}_{m,R} \to G_R$ attached to $h$ is given by

$$w(r) = (r^{-1}, 1, \ldots, 1) \quad r \in \mathbb{R}.$$  

This is not defined over $\mathbb{Q}$ if $n > 1$. Then we cannot expect a description of $M_H(\mathbb{C})$ as a moduli space of abelian varieties. Indeed, if $M_H(\mathbb{C})$ is a moduli space, we could describe $h$ as given by the Hodge structure of an abelian variety, which is rational.

Suppose that $H$ and $H'$ are sufficiently small open compact subgroups of $G(\mathbb{A}_F)$, and $g \in G(\mathbb{A}_F)$. Let $\mathcal{F}$ be a sheaf over the Shimura curves attached to a certain module (see [7, Section 6.1]). There is a natural identification of sheaves on $M_H \cap gH'g^{-1} : \mathcal{F}_g^{-1} \mid_{M_H \cap gH'g^{-1}} = \mathcal{F}_g^{-1} \mid_{M_H \cap gH'g^{-1}}$. Here $\mathcal{F}_R$ means that we consider $\mathcal{F}$ as a sheaf over the curve $M_R$. Then define

$$[HgH'] : H^j(M_H' \otimes \tilde{F}, \mathcal{F}_{H'}) \to H^j(M_H' \cap g^{-1}Hg \otimes \tilde{F}, \mathcal{F}_{H'} \mid_{M_H' \cap g^{-1}Hg})$$

$$[HgH'] : H^j(M_H' \cap g^{-1}Hg \otimes \tilde{F}, \mathcal{F}_{H'} \mid_{M_H' \cap g^{-1}Hg})$$

$$= H^j(M_H' \cap g^{-1}Hg \otimes \tilde{F}, \mathcal{F}_{H'} \mid_{M_H' \cap g^{-1}Hg})$$

$$= H^j(M_H \otimes \tilde{F}, \mathcal{F}_H),$$

(1.1)
where the first arrow is the restriction map, the second arrow is induced from $\varrho : M_{gH'g^{-1}H} \to M_g$ and the last arrow is the trace map. See [15, Section 15] for more details. In this paper, we are interested in the case where $H_v = H'_v = GL_2(\mathcal{O}_{F_v})$ for $v \mid p$ and $\mathcal{F}$ is associated with a $\prod_{v \mid p} GL_2(\mathcal{O}_{F_v})$-module.

Let $H = H'$. If $q$ is a prime of $\mathcal{O}_F$ which is unramified in $D$ and does not divide $p$, let $\omega_q \in \mathbb{A}_F^\infty$ be such that $\omega_q$ is a uniformizer at $q$ and is 1 at every other place. Then write

$$T_q = [H \left( \begin{smallmatrix} \omega_q & 0 \\ 0 & 1 \end{smallmatrix} \right) H].$$

If also $H_q = GL_2(\mathcal{O}_q)$, define

$$S_q = [H \left( \begin{smallmatrix} \omega_q & 0 \\ 0 & \omega_q \end{smallmatrix} \right) H].$$

If $H = K_0(N)$, denote by $T_\mathcal{O}(H, \mathcal{F})$ the $\mathcal{O}$-algebra generated by $T_q$ for $q \nmid NS_D$ and $S_q$ for $q$ with $H_q = GL_2(\mathcal{O}_q)$. Write $T_\mathcal{A}(H, \mathcal{F}) = T_\mathcal{O}(H, \mathcal{F}) \otimes A$ for any $\mathcal{O}$-algebra $A$. Write $U_{\omega_q}$ if $q \mid N$.

A maximal ideal of $T_\mathcal{A}(H, \mathcal{F})$ is Eisenstein if it contains $T_v - 2$ and $S_v - 1$ for all but finitely many primes $v$ of $F$ which split completely in some finite abelian extension of $F$.

1.2. **Ihara’s Lemma.** Let $U \subset K_0$ be a sufficiently small open compact subgroup such that $U_v = (\mathcal{O}_D \otimes \mathcal{O}_{F_v})^\times$ for almost all $v$ and for all $v \mid pS_D$. Let $q$ be a finite prime of $F$ such that $q \nmid pS_D$ and $U_q = GL_2(\mathcal{O}_q)$. Let $U_0(q)$ be the subgroup of $U$ defined by $U_0(q)_v = U_v$ if $v \neq q$ and

$$U_0(q)_q = \{ \left( \begin{array}{cc} a & b \\ c & d \end{array} \right) \in U_q | \left( \begin{array}{cc} a & b \\ c & d \end{array} \right) \equiv \left( \begin{array}{cc} * & * \\ 0 & * \end{array} \right) (\text{mod } q) \}.$$ 

Let $V$ be an irreducible mod $p$ representation of $GL_2(\mathcal{O}_F/p) = \prod_{v \mid p} GL_2(k_v)$ given by $V = \otimes_{v \mid p} \otimes_{D \sim k_v} \text{Symm}^{b_r-2}$ with $2 \leq b_r \leq p + 1$. Let $H = U$ or $U_0(q)$. We attach to $V$ a locally constant sheaf

$$\mathcal{F}_V = G(\mathbb{Q})/(X \times G(\mathbb{A}^\infty)) \times V/H$$

over $M_H$. Let $\eta_q \in (D \otimes \mathbb{A}^\infty)^\times$ be the element $\left( \begin{array}{cc} 1 & 0 \\ 0 & \omega_q \end{array} \right)$. Then we have $\eta_q^{-1}U_0(q)\eta_q \subset U$ and two maps $\varrho_1, \varrho_\eta : M_{U_0(q)} \to M_U$.

**Conjecture 1.1.** Assume that the genus of $M_U$ is greater than 1, then the following map is injective:

$$H^1_{et}(M_U \otimes \bar{F}, \mathcal{F}_V)_m \oplus H^1_{et}(M_U \otimes \bar{F}, \mathcal{F}_V)_m \to H^1_{et}(M_{U_0(q)} \otimes \bar{F}, \mathcal{F}_V)_m$$

$$(f_1, f_2) \mapsto 1_* f_1 + \left( \begin{array}{cc} 1 & 0 \\ 0 & \omega_q \end{array} \right) \ast f_2.$$ 

Here $m$ is a non-Eisenstein maximal ideal of the Hecke algebra.

One of the main results we prove in this paper is the following theorem.
Theorem 1.2. If $\sum_{\tau \in \mathcal{S}} b_{\tau} - 1 \leq p - 2$, then Conjecture 1.1 holds.

1.3. Multiplicity two. We first recall the statement of the multiplicity one result in [5]. Let $N$ be a squarefree ideal of $F$ such that $(N, pS_D) = 1$.

Let $M_{K_0(N)}$ be the Shimura curve over $F$ attached to $(G, X)$ with level $K_0(N)$. Let $\sigma : \text{GL}_2(\mathcal{O}_F/p) \to \text{Aut}(W_{f, p})$ be a regular Serre weight, $\mathcal{F}_V$ be the sheaf attached to $V = \sigma^{op} = \otimes_{\nu} \otimes_{\nu,k} F_{\nu} \text{Symm}^{b_{\nu} - 2}$ (regularity means that $2 \leq b_{\nu} - 1 \leq p - 2$), $m$ be a maximal non-Eisenstein ideal of the corresponding Hecke algebra. The following result is [5] Theorem 5.3.

Theorem 1.3. Let $\tilde{\rho} : G_F \to \text{GL}_2(\tilde{F}_p)$ be a modular Galois representation of weight $\sigma$ with conductor dividing $pNS_D$. Assume that $\tilde{\rho}$ satisfies the following conditions

1. the restriction $\tilde{\rho}|_{G_{F'}}$ is irreducible, where $F' = F(\sqrt{-1})^{(p-1)/2}$;
2. if $v | p$, then $\text{End}_{\tilde{\mathcal{F}}_p}(\tilde{\rho}|_{G_{F_v}}) = \tilde{F}_p$;
3. if $v | N$, then $\tilde{\rho}$ is ramified at $v$;
4. if $v | S_D$, and $\text{Norm}(v)^2 \equiv 1 \pmod{p}$, then $\tilde{\rho}$ is ramified at $v$.

Then $\dim_{\tilde{F}_p} \text{Hom}_{G_{F'}}(\tilde{\rho}, H^1_{et}(M_{K_0(N)} \otimes \tilde{F}, \mathcal{F}_V)_m) \leq 1$.

Remark 1.4. (a) We have the obvious embedding $H^1_{et}(M_{K_0(N)} \otimes \tilde{F}, \mathcal{F}_V)[m] \to H^1_{et}(M_{K_0(N)} \otimes \tilde{F}, \mathcal{F}_V)_m$. It induces an isomorphism (see [2] Lemma 4.10)

$$\text{Hom}_{G_{F'}}(\tilde{\rho}, H^1_{et}(M_{K_0(N)} \otimes \tilde{F}, \mathcal{F}_V)[m]) \cong \text{Hom}_{G_{F'}}(\tilde{\rho}, H^1_{et}(M_{K_0(N)} \otimes \tilde{F}, \mathcal{F}_V)_m).$$

On the other hand, by Eichler-Shimura relation, we know that $H^1_{et}(M_{K_0(N)} \otimes \tilde{F}, \mathcal{F}_V)[m] \cong \tilde{\mathcal{F}}^a$ for some Galois representation $\tilde{\tau} : G_F \to \text{GL}_2(\tilde{F}_p)$ and some integer $a$ (see for example [1]). Then the above theorem says that if $a > 0$ and $\tilde{\rho} \cong \tilde{\tau}$, then $a = 1$.

(b) The theorem is proved by using Diamond’s refined Taylor-Wiles system [3]. The condition (1) is necessary for this argument and the corresponding Galois deformation problem. The condition (3) is here because we do not have a general Ihara’s lemma for Shimura curves over totally real fields. The condition (4) is essential for having a multiplicity one result, as we shall see in Theorem 1.6.

Theorem 1.5. Let $\tilde{\rho} : G_F \to \text{GL}_2(\tilde{F}_p)$ be a modular Galois representation of weight $\sigma$ with conductor dividing $pNS_D$ such that $\sum_{\tau \in \mathcal{S}} b_{\tau} - 1 \leq p - 2$. Assume that $\tilde{\rho}$ satisfies the following conditions

1. the restriction $\tilde{\rho}|_{G_{F'}}$ is irreducible, where $F' = F(\sqrt{-1})^{(p-1)/2}$;
2. if $v | p$, then $\text{End}_{\tilde{\mathcal{F}}_p}(\tilde{\rho}|_{G_{F_v}}) = \tilde{F}_p$;
3. if $v | N$, then $\tilde{\rho}$ is ramified at $v$;
4. if $v | S_D$, and $\text{Norm}(v)^2 \equiv 1 \pmod{p}$, then $\tilde{\rho}$ is ramified at $v$.

Then $\dim_{\tilde{F}_p} \text{Hom}_{G_{F'}}(\tilde{\rho}, H^1_{et}(M_{K_0(N)} \otimes \tilde{F}, \mathcal{F}_V)_m) \leq 1$.

Proof. By [2] Lemma 2.3, we may assume that all the $a_{\tau}$ are 0. Let $\mathbb{T}(K_0(N), \mathcal{F})$ be the Hecke algebra attached to $H^1_{et}(M_{K_0(N)} \otimes \tilde{F}, \mathcal{F}_V)$. Then

$$\dim_{\tilde{F}_p} \text{Hom}_{G_{F'}}(\tilde{\rho}, H^1_{et}(M_{K_0(N)} \otimes \tilde{F}, \mathcal{F}_V)_m) = 1$$

if and only if $H^1_{et}(M_{K_0(N)} \otimes \tilde{F}, \mathcal{F}_V)_m$ is free of rank two over $\mathbb{T}(K_0(N), \mathcal{F})_m$. Combining Theorem 1.2 and Theorem 1.3 by a standard argument in Taylor-Wiles method (see for example [24] Section 3) and [14] Section 5.3, we can remove condition (3) in Theorem 1.3. 

Let $p$ and $q$ be two primes of $F$ such that $(pq, S_D) = 1$ and $pq | N$. Let $M_{K_0(N)}$ be the Shimura curve attached to $D$ with level $K_0(N)$ as in Theorem 1.5. Let $D'$ be another quaternion algebra over $F$ such that $D'$ is ramified at $p$, $q$ and at the primes where $D$ is ramified. Fix a maximal order $O_{D'}$ of $D'$ and isomorphisms $O_{D'} \otimes_F O_{F_v} = M_2(O_{F_v})$ for finite primes $v | pq$. Let $M'_O$ be the Shimura curve attached to $D'$ with level $O$, where $O_v = K_0(N)_v$ if $v | p,q$. Then in section 4, we study the Galois structure on cohomology $\text{Sh}_n$. In section 3, we prove Theorem 1.2 following the strategy of [9]. Then in section 4, we prove Theorem 1.2 following the strategy of [9].

**Remark 1.7.** We prove this theorem by adapting the idea of Ribet [22], where the author proved a multiplicity two result for the Jacobians of Shimura curves over $\mathbb{Q}$. The idea is used in [20] to prove a level-lowering result for Galois representations attached to Hilbert modular forms. In fact, we use some arguments in [20]. These ideas also appeared in for example [9] and [15].

**1.4. Notation.** If $L$ is a perfect field we will let $\bar{L}$ denote the algebraic closure of $L$ and $G_L$ its absolute Galois group $Gal(\bar{L}/L)$. If $L$ is a number field, we let $\mathbb{A}_L$ denote the ring of adeles over $L$, and $\mathbb{A}^\infty_L$ denote the ring of finite adeles over $L$. If $L = \mathbb{Q}$, we write $\mathbb{A}$ and $\mathbb{A}^\infty$ for $\mathbb{A}_\mathbb{Q}$ and $\mathbb{A}^\infty_\mathbb{Q}$, respectively.

Let $F$, $p$ be as above. For any prime $v$ of $F$, let $F_v$ be the completion of $F$ at $v$, $O_{F_v}$ the ring of integers of $F_v$, $k_v$ the residue field of $O_{F_v}$, $\varpi_v$ a uniformizer of $O_{F_v}$, and $\text{Frob}_v \in \text{Gal}(F_v/F_v)$ an arithmetic Frobenius element. Write $I_{F_v} \subset G_{F_v}$ for the inertia group at prime $v$.

Let $\Sigma$ be a set of primes of $F$. If a group $U$ has the form $U = \prod_{v \in \Sigma} U_v$, and $J$ is an ideal which is a product of some elements in $\Sigma$, we will write $U^J$ for the subgroup of $U$ given by $U^J = \prod_{v \in \Sigma, v | J} U_v$ and $U_J$ for the subgroup of $U$ given by $U_J = \prod_{v \in \Sigma, v \not| J} U_v$.

This paper is organized as follows. In section 2 we recall some basic properties of Shimura curves and prove some lemmas we need. In section 3 we prove Theorem 1.2 following the strategy of [9]. Then in section 4 we study the Galois structure on cohomology groups of Shimura curves and prove Theorem 1.6.

## 2. Construction of certain sheaves on Shimura curves

In this section, we recall some facts about Shimura curves over totally real fields, the construction of unitary Shimura curves, and the construction of certain sheaves on them. The details can be found in [4] and [23]. We also review facts about reductions of Shimura curves, which are proved in [4, 15], and [26].

### 2.1. Unitary Shimura curves.

Let $E_0 = \mathbb{Q}(\sqrt{-a})$ be an imaginary quadratic extension of $\mathbb{Q}$ splits $p$. Fix embedding $E_0 \rightarrow \mathbb{C}$ and $E_0 \rightarrow \mathbb{Q}_p$. Let $E = E_0 F = F \otimes_{\mathbb{Q}} E_0$, $B = D \otimes_F E = D \otimes_{\mathbb{Q}} E_0$. Let $G''$ be the algebraic group over $\mathbb{Q}$ of $D^\times \times F^\times \cong D^\times . E^\times \subset B^\times$. Let $\nu := \text{Nrd}_{D/F} \times \text{Norm}_{E/F} : G'' \rightarrow F^\times$ be the product of the reduced norm and norm. Let $G'$ be the algebraic group over $\mathbb{Q}$ of the inverse image of $Q^\times \subset F^\times$ of the surjective
map $\nu$. Fix a prime $\mathfrak{p} | p$ of $F$. Then $\mathfrak{p}$ splits in $E$ and we fix a prime $\mathfrak{P}'$ of $E$ which divides $\mathfrak{p}$. Therefore $F_{\mathfrak{P}} = E_{\mathfrak{P}}$, $\mathcal{O}_{F_{\mathfrak{P}}} = \mathcal{O}_{E_{\mathfrak{P}'}^{\prime}}$.

We consider the $G'(\mathbb{R})$-conjugacy class $X'$ ($G''(\mathbb{R})$-conjugacy class $X''$) of

\[h' : \mathbb{C}^x \to G'(\mathbb{R}) \subseteq G''(\mathbb{R}) = GL_2(\mathbb{R}) \cdot \mathbb{C}^x \times \mathbb{H}^x \cdot \mathbb{C}^x \times \cdots \times \mathbb{H}^x \cdot \mathbb{C}^x\]

\[z = x + iy \mapsto \left(\begin{array}{cc} x & y \\ -y & x \end{array} \right)^{-1}, 1 \otimes z^{-1}, \cdots, 1 \otimes z^{-1}\]

The conjugacy classes $X'$, $X''$ have natural structures of complex manifolds and are isomorphic to the upper half plane $\mathcal{H}^+$ and to the union of upper and lower half planes $\mathcal{H}$, respectively. Let $M' = M(G', X')$, $M'' = M(G'', X'')$ be the canonical models of the Shimura varieties defined over the reflex field $E$. The inclusions $G \to G''$ and $G' \to G''$ induce morphisms of Shimura varieties

\[M \to M'' \leftarrow M'.\]

### 2.2. Moduli interpretation of $M'$

We recall the moduli interpretation of $M'$ following [4] and [23]. Let the involution $*$ on $B = D \otimes_F E$ be the tensor product of the main involution of $D$ and the conjugation of $E$ and let $\psi$ be the non-degenerate alternating form on $B$ defined by

\[\psi(x, y) = \text{Tr}_{E/F}(\sqrt{-a} \text{Tr}_{B/E} x y^*).\]

Let $\mathcal{O}_B$ be a maximal order in $B$ stable under the involution $*$. An abelian scheme $A$ over a scheme $S$ is called an $\mathcal{O}_B$-abelian scheme over $S$ when a ring homomorphism $m : \mathcal{O}_B \to \text{End}_S(A)$ is given. Let $\text{Lie}(A)$ denote the locally free $\mathcal{O}_S$-module $\text{Lie}(A/S) = \text{Hom}_{\mathcal{O}_S}(\mathcal{O}_S^1 \otimes_{\mathcal{O}_S} A, \mathcal{O}_S)$, where $0 : S \to A$ denotes the 0-section. When $S$ is a scheme over $\text{Spec} E$, for an $\mathcal{O}_B$-abelian scheme $A$ over $S$, we define direct summands $\text{Lie}^2 A \supset \text{Lie}^{1,2} A$ of the $\mathcal{O}_B \otimes_\mathbb{Z} \mathcal{O}_S = B \otimes_\mathbb{Q} \mathcal{O}_S$-module $\text{Lie}(A)$ as follows. The submodule $\text{Lie}^2 A$ is defined to be the submodule on which the action of $E_0 \subset B$ and that of $E_0 \subset \mathcal{O}_S$ are conjugate to each other over $\mathbb{Q}$. Similarly, $\text{Lie}^{1,2} A$ is the submodule where the action of $E \subset B$ and that of $E \subset \mathcal{O}_S$ are conjugate to each other over $F$. They are the same as the tensor products $\text{Lie}^2 A = \text{Lie}(A) \otimes_{E_0 \otimes_{E_0} E_0} E_0$, $\text{Lie}^{1,2} A = \text{Lie}(A) \otimes_{E \otimes E} E$ and hence are direct summands. If $A$ is an $\mathcal{O}_B$-abelian scheme, the dual $A^*$ is considered as an $\mathcal{O}_B$-abelian scheme by the composite map

\[m^* : \mathcal{O}_B \xrightarrow{\text{opp}} \mathcal{O}_B^{\text{opp}} \xrightarrow{m} \text{End}(A)^{\text{opp}} \xrightarrow{\text{opp}} \text{End}(A^*)\]

where opp denotes the opposite ring. A polarization $\theta \in \text{Hom}(A, A^*)^{\text{sym}}$ of an $\mathcal{O}_B$-abelian scheme $A$ is called an $\mathcal{O}_B$-polarization if it is $\mathcal{O}_B$-linear.

Let $K' \subset G'(\mathbb{A}^\infty)$ be a sufficiently small open compact subgroup. Assume that $K' \subset (\mathcal{O}_B \otimes \mathbb{Z})^\times$. Let $\hat{T} \subset B \otimes \mathbb{A}^\infty$ be an $\mathcal{O}_B \otimes \mathbb{Z}$-lattice satisfying $\psi(\hat{T}, \hat{T}) \subset \mathbb{Z}$. We define a functor $M'_{K', \hat{T}}$ on the category of schemes over $E$ as follows. For a scheme $S$ over $E$, let $M'_{K', \hat{T}}(S)$ be the set of isomorphism classes of the triples $(A, \theta, \zeta)$ consisting of the following data

- An $\mathcal{O}_B$-abelian scheme $A$ over $S$ of dimension $4n$ such that $\text{Lie}^2(A) = \text{Lie}^{1,2} A$ and that it is a locally free $\mathcal{O}_S$-module of rank two.
- An $\mathcal{O}_B$-polarization $\theta \in \text{Hom}(A, A^*)^{\text{sym}}$ of $A$.  


A $K'$-equivalent class $\zeta$ of an $O_B \otimes \hat{\mathcal{Z}}$-linear isomorphism $\iota : \hat{T}(A) \to \hat{T}$ such that there exists a $\hat{\mathcal{Z}}$-linear isomorphism $\iota'$ making the following commutative diagram

$$
\begin{array}{ccc}
\hat{T}(A) \times \hat{T}(A) & \xrightarrow{(1, \theta_\ast)} & \hat{T}(A) \times \hat{T}(A^\ast) \\
\downarrow \iota \times \iota & & \downarrow \iota' \\
\hat{T} \times \hat{T} & \xrightarrow{\text{Tr} \psi} & \hat{\mathcal{Z}}
\end{array}
$$

By [4], the functor $M'_{K', \hat{T}}$ is represented by the scheme $M'_{K'}$. This functor is independent of the choice of $\hat{T}$ up to unique canonical isomorphism. Moreover, there is an integral version of this moduli description (see [4, Section 2.6].) We still write $M'_{K'}$ for the moduli space over $O_{E_\ast}$. Let $A_{K'}$ over $O_{E_\ast}$ be the universal abelian scheme over $M'_{K'}$. Let $\pi_{K'} : A_{K'} \to M'_{K'}$ be the structure map. Then $(\pi_{K'})_\ast \Omega^1_{A_{K'}/M'_{K'}}$ and $R^1(\pi_{K'})_\ast O_{A_{K'}}$ are $O_B \otimes O_{M'_{K'}}$-modules locally free of rank one. Note that we have a decomposition

$$O_B \otimes O_L \cong O_{B_1} \oplus O_{B_2} \oplus \cdots \oplus O_{B_n} \oplus O_{B_n^2} \oplus \cdots \oplus O_{B_n^2}$$

where $B_i^2$ is isomorphic to $M_2(L)$. Let $e_i$ denote the idempotent whose $B_i^2$ component is

$$\begin{pmatrix}
1 & 0 \\
0 & 0
\end{pmatrix}
$$

and the other components are zero. Then $e_i(\pi_{K'})_\ast \Omega^1_{A_{K'}/M'_{K'}}$ and $e_iR^1(\pi_{K'})_\ast O_{A_{K'}}$ are locally free $O_L$-modules of rank one.

Let $\omega'_i = e_i(\pi_{K'})_\ast \Omega^1_{A_{K'}/M'_{K'}} \otimes O$ and $H'_i = e_iR^1(\pi_{K'})_\ast O_{A_{K'}} \otimes O$.

**Lemma 2.1.** Assume that $K'$ has no level structure at the prime $\Psi'$. Then

1. $H'_i \cong (\omega'_i)^{-1}$.
2. The exact sequence

$$0 \to (\pi_{K'})_\ast \Omega^1_{M'_{K'}/O_{E_{\Psi'}}} \to \Omega^1_{A_{K'}/O_{E_{\Psi'}}} \to \Omega^1_{A_{K'}/M'_{K'}} \to 0$$

induces an isomorphism

$$\omega'_i : \Omega^1_{M'_{K'}/O_{E_{\Psi'}}} \cong H'_i.$$

**Proof.** (1) It suffices to prove that $(\pi_{K'})_\ast \Omega^1_{A_{K'}/M'_{K'}}$ and $R^1(\pi_{K'})_\ast O_{A_{K'}}$ are dual to each other. Let $A_{K'}^\dagger$ be the dual abelian scheme of $A_{K'}$, the principal polarization gives an isomorphism $\text{Lie}(A_{K'}) = \text{Lie}(A_{K'}^\dagger)$. Since we have an isomorphism of $O_B$-sheaves $R^1(\pi_{K'})_\ast O_{A_{K'}} \cong \text{Lie}(A_{K'}^\dagger)$, the lemma follows.

(2) Since $K'$ has no level at $\Psi'$, both $M'_{K'}$ and $A_{K'}$ are smooth over $O_{E_{\Psi'}}$. Therefore the sequence is exact. Taking the long exact sequence of this short exact sequence, we obtain the following map

$$(\pi_{K'})_\ast \Omega^1_{A_{K'}/M'_{K'}} \to R^1(\pi_{K'})_\ast (\pi_{K'})_\ast \Omega^1_{M'_{K'}/O_{E_{\Psi'}}} \cong \Omega^1_{M'_{K'}/O_{E_{\Psi'}}} \otimes R^1(\pi_{K'})_\ast O_{A_{K'}}.$$

By the same argument as in [9] Lemma 7, this map is an isomorphism. Thus the result follows.

**Remark 2.2.** This is also proved in [18] Proposition 3.1.
2.3. Sheaves on $M_K$. By the construction in [23 Section 6.2], the universal $\mathcal{O}_B$-abelian scheme $A'$ over $M'$ extends to an $\mathcal{O}_B$-abelian scheme $A''$ over $M''$. Let $m = \sum_{i \in S} b_i - 1$. Let $A'''$ be the $m$-fold self fibre product of $A''$ over $M''$. Now we construct our sheaves as at the beginning of [23 Section 6.3]. Let $A$ (resp. $Y$) be the base change of $A''$ (resp. $A'''$) by $M \to M''$. Let $\pi_K$ be the structure map $\pi_K : A_K \to M_K$. Then we may define an algebraic correspondence $e$ on $Y$ such that $eR^q(\pi_K)_* \kappa = F_V$ if $q = m$. In fact, $e$ is defined by $e_i$ where we have $F_V \cong \bigotimes_{i \in S} \text{Symm}^{b_i - 2} \otimes \text{det}^a(e_i R^1(\pi_K)_* \kappa)$.

Similarly, $e_i(\pi_K)_* \Omega^1_{A_K/M_K}$ and $e_i R^1(\pi_K)_* \mathcal{O}_{A_K}$ are locally free $\mathcal{O}_S$-modules of rank one. Let $\omega_i = e_i(\pi_K)_* \Omega^1_{A_K/M_K} \otimes \mathcal{O}$ and $\mathcal{H}_i = e_i R^1(\pi_K)_* \mathcal{O}_{A_K} \otimes \mathcal{O}$.

Lemma 2.3. Assume that $K$ has no $\Psi$-level structure, then

(1) $\mathcal{H}_i \cong \omega_i^{-1}$.
(2) The exact sequence

$$0 \to (\pi_K)^* \Omega^1_{M_K/\mathcal{O}_F} \to \Omega^1_{A_K/\mathcal{O}_F} \to \Omega^1_{A_K/M_K} \to 0$$

induces an isomorphism

$$\omega_i \sim \Omega^1_{M_K/\mathcal{O}_F} \otimes \mathcal{H}_i.$$

Proof. Those properties in Lemma 2.1 extend to the objects over $M''$. The lemma then follows by pulling back.

By Riemann-Roch, we have the following corollary.

Corollary 2.4. Assume that $K$ has no $\Psi$-level structure, then the degree of $\omega_i$ is $(g_K - 1)$. Here $g_K$ is the genus of $M_K$.

2.4. Supersingular and ordinary points on the special fibre of $M_K$. Before we prove the main theorem, we review some properties of reductions of Shimura curves.

Let $\wp$ be a finite prime of $F$, $K$ be an open compact subgroup of $G(\mathbb{A}^\infty)$. We suppose that $K$ factors as $K_{\wp} H$. Then we have the following results. The first one is proved in [4]. The second one is indicated in [3] and the detail proof is given in [15].

Theorem 2.5. (1) Suppose $K_{\wp}$ is the subgroup $K_{\wp}^0 = GL_2(\mathcal{O}_{\wp})$. Then, if $H$ is sufficiently small, there exists a model $\mathbf{M}_{0,H}$ of $M_K$ defined over $\mathcal{O}_{F,\wp}$. This model is proper and smooth.

(2) Suppose $K_{\wp}$ is the subgroup $K_{\wp}^n$ of matrices congruent to $I$ modulo $\wp^n$. Then, if $H$ is sufficiently small, there exists a regular model $\mathbf{M}_{n,H}$ of $M_K$ with a map to $\mathbf{M}_{0,H}$. The morphism $\mathbf{M}_{n,H} \to \mathbf{M}_{0,H}$ is finite and flat.

Theorem 2.6. (1) Suppose $K_{\wp}$ is the group

$$\Gamma_0(\wp) = \{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in GL_2(\mathcal{O}_{F,\wp}) \mid c \in \wp \}.$$ 

Then, if $H$ is sufficiently small, there exists a regular model $\mathbf{M}_{\Gamma_0(\wp),H}$ of $M_K$ defined over $\mathbf{M}_{0,H}$. The morphism $\mathbf{M}_{\Gamma_0(\wp),H} \to \mathbf{M}_{0,H}$ is finite and flat.

(2) The special fibre $\mathbf{M}_{\Gamma_0(\wp),\bar{H}} \times \bar{k}_{\wp}$ is isomorphic to a union of two copies of $\mathbf{M}_{0,H} \times \bar{k}_{\wp}$ intersecting transversely above a finite set of points $\Sigma_H$. 

The points in $\Sigma_H$ are the supersingular points of $M_{0,H} \times \bar{k}_v$. We can describe this set in another way. Let $D$ be another quaternion algebra over $F$ ramified at primes in $S_D \cup \{\wp\} \cup \{\tau | \tau | \infty\}$. So it is totally definite. Let $G = \text{Res}_{F/Q} \bar{D}^\times$ be the corresponding algebraic group over $\mathbb{Q}$, then there is a bijection

$$\Sigma_H \cong \bar{G}(\mathbb{Q}) \backslash \bar{G}(\mathbb{A}^\times) / H \times \mathcal{O}_D^\times$$

where the second isomorphism is induced by the reduced norm $\bar{D}_D \rightarrow F^\times_D$. Since $\bar{D}$ is totally definite, $\Sigma_H$ is a finite set.

According to \cite{4} and \cite{15}, the action of $GL_2(F_\wp)$ on the inverse system of Shimura curves descends to an action on the set $\Sigma_H$ of supersingular points, and the action factors through $\det : GL_2(F_\wp) \rightarrow F^\times_\wp$. Further, if we normalize the reciprocity map of class field theory so that the arithmetic Frobenius elements correspond to uniformizers, then an element $\sigma \in W(F_\wp/F_\wp)$ acts on the set $\Sigma_H$ in exactly the same way as the element $[\sigma] \in F^\times_\wp$ corresponding to $\sigma$ by class field theory.

Let $K = GL_2(O_v)H$ and $\sharp(\Sigma_H)$ be the number of supersingular points on $M_{0,H} \times \bar{k}_v$. Let $g_R$ be the genus of any geometric fibre of $M_R$. Then we have the following lemma.

**Lemma 2.7.** $\sharp(\Sigma_H) = (\text{Norm}(\wp) - 1)(g_{GL_2(O_v)H} - 1)$.

**Proof.** The argument is similar to the proof of \cite[Lemma 6]{9}. We compute $g_{\Gamma_0(\wp)H}$ in two ways. On one hand, in characteristic 0, we have a map $M_{\Gamma_0(\wp)H} \rightarrow M_{GL_2(O_v)H}$ which is of degree $\text{Norm}(\wp) + 1$. Therefore, $g_{\Gamma_0(\wp)H} - 1 = (\text{Norm}(\wp) + 1)(g_{GL_2(O_v)H} - 1)$. On the other hand, modulo $\wp$, $M_{\Gamma_0(\wp)H}$ is singular. One can still define its arithmetic genus and it is equal to $g_{\Gamma_0(\wp)H}$ since $M_{\Gamma_0(\wp)H}$ is flat. Applying Riemann-Hurwitz formula to the map $(M_{\Gamma_0(\wp)H})_\wp \rightarrow (M_{GL_2(O_v)H})_\wp$, we obtain the following equation

$$2g_{\Gamma_0(\wp)H} - 2 = 2(2g_{GL_2(O_v)H} - 2) + 2\sharp(\Sigma_H).$$

Comparing the two equations, the lemma follows. \hfill \square

2.5. $p$-adic uniformization of Shimura curves. The discussion in this subsection is needed in section \cite{3}. Let $v$ be a finite place of $F$ at which $D'$ is ramified. Here $D'$ is the quaternion algebra defined in section \cite{13}. Let $P$ be an open compact subgroup of $(D' \otimes_F H_F)^\times$ such that $P = (\mathcal{O}_{D'} \otimes \mathcal{O}_F \mathcal{O}_F)^\times$. Then we have a Shimura curve $M'_P$ which is defined over $\mathbb{F}_v$.

Let $F_v^{nr}$ be the maximal unramified extension of $F_v$, let $\Omega_{F_v}$ be Drinfeld’s upper half plane over $F_v$, and $\Omega = \Omega_{F_v} \otimes F_v^{nr}$. We let $g \in GL_2(F_v)$ act on $\Omega$ via the natural (left) action on $\Omega_{F_v}$ and the action of $\text{Frob}_v$ on $F_v^{nr}$. We also let $n \in \mathbb{Z}$ act on $\Omega$ through the action of $\text{Frob}_v^n$ on $F_v^{nr}$. This gives an $F_v$-rational action of $GL_2(F_v) \times \mathbb{Z}$ on $\Omega$. Moreover, the $F_v$-analytic space $GL_2(F_v) \backslash (\Omega \times (P^{nr}(D' \otimes_F H_F)^\times / G'(\mathbb{Q})))$ algebraizes canonically to a scheme $\mathfrak{X}_S$ over $F_v$. Let $M'$ and $\mathfrak{X}$ be the inverse limits of $M'_P$ and $\mathfrak{X}_P$ over all $P$. Then a special case of \cite[Theorem 5.3]{26} gives the following theorem. (See also \cite[Section 1]{17}.)

**Theorem 2.8.** There exists a $(D' \otimes_F H_F^{nr})^\times \times \mathbb{Z}$-equivariant, $F_v$-rational isomorphism

$$M' \otimes F_v \cong \mathfrak{X}.$$
In particular, we have an $F_v$-rational isomorphism
\[ M'_p \otimes_F F_v \cong X_P. \]
Furthermore, there exists an integral model $M'_p$ for $M'_p$ over $\mathcal{O}_{F_v}$, and the above isomorphism can be extended to schemes over $\mathcal{O}_{F_v}$.

We apply the above theorem to the case $P = O$ and $v = q$ to study the singular points of the mod $q$ reduction of $M'_O$.

The special fibre of $M'_O$ has non-degenerate quadratic singular points. The dual graph $\mathfrak{G}$ attached to the special fibre of $M'_O \otimes k_q$ is the quotient
\[ GL_2(F_q)^+ \setminus (\Delta \times (O^q \backslash G'(\mathbb{A}^\infty)^q / G'(\mathbb{Q}))), \]
where $\Delta$ is the well-known tree attached to $SL_2(F_q)$, $GL_2(F_q)^+$ is the kernel of the map
\[ \mu : GL_2(F_q) \to \mathbb{Z}/2\mathbb{Z} \]
defined by the formula
\[ \mu(\gamma) = \text{ord}_q(\det \gamma) \pmod{2}. \]
The singular points of $M'_O \otimes k_q$ correspond to the edges of $\mathfrak{G}$. Let $D'$ be a definite quaternion algebra over $F$ ramified at places in $S_D \cup \{p\} \cup \{v \mid v|\infty\}$. (Note that $D'$ is ramified at $q$ and unramified at $\tau_1$, and $D'$ is unramified at $q$ and ramified at $\tau_1$.) Let $G' = \text{Res}_{\mathbb{Q}}^{F'}(D')^\times$ be the associated algebraic group. Let $S$ be an open compact subgroup of $G'((\mathbb{A}^\infty))$ such that $S_v = O_v$ if $v \neq q$, and $S_q = \Gamma_0(q)$ where
\[ \Gamma_0(q) = \{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in GL_2(O_{F_q}) \mid c \in q \}. \]
Then the edges of $\Delta$ are in one-to-one correspondence with $GL_2(F_q)^+/(S_q F^\times)$. Therefore, the edges of $\mathfrak{G}$ are bijective to $G'(\mathbb{Q}) \backslash G'(\mathbb{A}) / S$.

Remark 2.9. Let us consider the curves $M_{K_0(N)}$ and $M'_O$ and their reductions $M_{K_0(N)}$ (mod $p$) and $M'_O$ (mod $q$). By the definition of the quaternion algebras $D$ and $D'$, we see that there is a bijection between the set of singular points of $M_{K_0(N)}$ (mod $p$) and the set of singular points of $M'_O$ (mod $q$).

3. Ihara’s lemma

In this section, we prove Theorem 1.2 following the strategy of [9]. Some of the arguments are exactly the same as in [9]. Recall that $U$ has no level structure at $pq$. Fix $\mathfrak{P} | p$. From the two natural maps $\alpha_1, \alpha_2 : M_{U_0(q)} \rightarrow M_U$ we get a map
\[ \alpha := \alpha_1^* + \alpha_2^* : H^0(M_U \times k_{\mathfrak{P}}, \omega_i^{\otimes b_i})^2 \rightarrow H^0(M_{U_0(q)} \times k_{\mathfrak{P}}, \omega_i^{\otimes b_i}). \]

Lemma 3.1. If $1 \leq b_i \leq p - 2$, then $\alpha$ is injective.

Proof. Assume that $(f_1, f_2)$ is in the kernel. The map is $U_{\omega_q}$-equivariant if we let $U_{\omega_q}$ act on the left hand side by the matrix \( \begin{pmatrix} T_3 & -1 \\ S_q \text{Norm}(q) & 0 \end{pmatrix} \). Here, $T_3$ and $S_q$ are $q$-th Hecke operators on $H^0(M_U \times k_{\mathfrak{P}}, \omega_i^{\otimes b_i})$, $U_{\omega_q}$ is the $q$-th Hecke operator on $H^0(M_{U_0(q)} \times k_{\mathfrak{P}}, \omega_i^{\otimes b_i})$. We may assume that $(f_1, f_2)$ is an eigenvector of $U_{\omega_q}$. Similarly, we may assume that $f_1$ is an eigenform of $S_q$. This implies that $f_1$ is a multiple of $f_2$. Therefore $\alpha_2^*(f_2) = -\alpha_1^*(f_1)$ is
a multiple of $\alpha_1^*(f_2)$. Now if $x$ is a point such that $f_2(x) = 0$, then by the above analysis, 
\[ f_2(\eta_q(x)) = 0 \]
where $\eta_q = \begin{pmatrix} 1 & 0 \\ 0 & \omega_q \end{pmatrix}$. By the following lemma and Lemma 2.7, the degree of 
\[ \omega_q^{\otimes b} \]
is either 0 or $\geq (\mathrm{Norm}(p) - 1)(g_U - 1)$. This is impossible because of Corollary 2.4. Therefore $f_2 = 0$ and $f_1 = f_2 = 0$.

\[ \Box \]

**Definition 3.2.** Let $x$ be a point of $M_U \times k_p$. The $q$-orbit of $x$ is defined to be $H_q(x) = \{ \eta_q^n(x) | n \in \mathbb{Z} \}$.

**Lemma 3.3.** (1) If $x$ is ordinary, then $H_q(x)$ has infinitely many elements.

(2) If $x$ is supersingular, then $H_q(x)$ is the set of all supersingular points.

**Proof.** (1) By [4] Section 1.3], the set of connected components of $M_U$ is bijective to the set 
\[ T(\mathbb{Q}) \backslash T(\mathbb{A}^\infty) / \mathrm{Nrd}(U) \]
where $T = \mathrm{Res}_{F'/F} \mathbb{G}_{m,F}$ and $\eta_q$ acts on this set as the multiplication by $\det \eta_q$. Assume that $\eta_q^m$ (for some integer $m$) acts trivially on this set, it suffices to show that the set $\{ \eta_q^m(x) | n \in \mathbb{Z} \}$, which is a subset of the intersection of $H_q(x)$ and the connected component containing $x$, is infinite. By [4] Proposition 4.5.4, we know that every irreducible component of $M_U \times F'$ is isomorphic to an irreducible component of $M'_U \times F'$ for some open compact subgroup $U'$ and some unramified extension $F'/F_p$. Let $z \in M'_U \times k_p$ corresponds to $x$ under this identification. It is ordinary, which means that the corresponding abelian variety $A_z$ is ordinary. The set $\{ \eta_q^m(x) | n \in \mathbb{Z} \}$ is bijective to the set of abelian varieties which are $q^m$-isogenous to $A_z$. From [4] Section 11.4, we know that $A$ is ordinary if and only if $\mathrm{End}_D A = E'$ where $E'$ is a quadratic extension of $E$. (Also, $A$ is supersingular if and only if $\mathrm{End}_D A \cong \bar{D}$ for some definite quaternion algebra over $F$.) Therefore, we may apply the argument in [4] Lemma 8 to prove the statement.

(2) This part is now clear since we know the set of supersingular points and the action of $\eta_q$ from subsection 2.4.

As in [9] Section 4], we interpret the above results in the content of crystalline cohomology. For a smooth proper scheme $M$ over $\mathcal{O}_{F_p}$ and a non-negative integer $a$, let $\mathcal{MF}^\nabla_{[a]}(M)$ be the category of torsion $F'$-crystals as defined in [12]. By [12] Theorem 6.2, we have an object in $\mathcal{MF}^\nabla_{[1]}(M_U \times \mathcal{O}_{F_p})$ defined by $R^1 \pi_*(\Omega^\bullet_{A_U/M_U} \otimes \mathcal{O}_{F_p}) \otimes \kappa$. This has a natural action of $\mathcal{O}_B$, so we can define $e_i R^1 \pi_*(\Omega^\bullet_{A_U/M_U} \otimes \mathcal{O}_{F_p}) \otimes \kappa$. Let $E_i$ denote the object in $\mathcal{MF}^\nabla_{[1]}(M_U \times \mathcal{O}_{F_p})$ corresponding to $e_i R^1 \pi_*(\Omega^\bullet_{A_U/M_U} \otimes \mathcal{O}_{F_p}) \otimes \kappa$ and $E_{b_i}$ denote the object in $\mathcal{MF}^\nabla_{[b_i-1]}(M_U \times \mathcal{O}_{F_p})$ defined by $\mathrm{Sym}^{b_i-2} E_i$. Finally, let $E = \oplus_{r \in S} E_r$. By [12] Theorem 2.1, it is an object in $\mathcal{MF}^\nabla_{[m]}(M_U \times \mathcal{O}_{F_p})$. Note that we assumed that $m = \sum (b_i - 1) \leq p - 2$. Write $E(U) = H^1(M_U \times \mathcal{O}_{F_p}, \mathcal{E} \otimes \Omega^\bullet_{M_U/M_U} \otimes \mathcal{O}_{F_p}) = H^1(M_U \times k_p, \mathcal{E} \otimes \Omega^\bullet_{M_U/k_p})$. By [12] Theorem 4.1, it is an object in $\mathcal{MF}^\nabla_{[m]}(\mathcal{O}_E)$ with coefficient $\kappa \otimes_{\mathcal{O}_p} \mathcal{O}_F/p \cong \oplus_{r \in S} \kappa$. We may decompose $E(U)$ as

\[ E(U) = \oplus_{r \in S} E(U)_r \]

where each $E(U)_r$ is a free $\kappa$-module.
Lemma 3.4.

\[
gr^j E(U)_r \cong \begin{cases} 
H^1(M_U \times k_\bar{p}, \omega^{2-b_r}_r) & j = 0 \\
H^0(M_U \times k_\bar{p}, \omega^{b_r}_r) & j = b_r - 1 \\
0 & \text{otherwise}
\end{cases}
\]

Proof. By Faltings’ comparison theorem in [12], we have

\[
gr^j E(U)_r \cong H^1(M_U \times k_\bar{p}, \omega^{2-b_r}_r) \quad \text{and} \quad H^0(M_U \times k_\bar{p}, \omega^{b_r}_r)
\]

By the same argument as in [9, Lemma 10], we have that

\[
\text{gr}^j (\mathcal{E}_{b_r} \otimes \Omega^\bullet_{M_U/k_\bar{p}}) \cong \begin{cases} 
\omega^{2-b_r}_r & j = 0 \\
\omega^{b_r-2}_r \otimes \Omega^1_{M_U/k_\bar{p}} & j = b_r - 1 \\
0 & \text{otherwise}
\end{cases}
\]

The lemma follows. \qed

The two natural maps \(\alpha_1\) and \(\alpha_2\) induce morphisms \(\alpha^*_i,DR : E(U) \to E(U_0(q))\) in \(\mathcal{M}F_{[0,p-2]}(O_E)\). We let \(\alpha_{DR}\) denote the map

\[
\alpha^*_1 + \alpha^*_2,DR : E(U)^2 \to E(U_0(q)).
\]

Lemma 3.5. If \(2 \leq b_r \leq p - 2\), then \(\text{gr}^j(\text{Ker}(\alpha_{DR})) = 0\) for \(j \neq 0\).

Proof. It suffices to prove that \(\text{gr}^j(\text{Ker}(\alpha_{DR})) = 0\) for \(j \neq 0\). This follows from Lemma 3.1 and Lemma 3.4. \qed

We can now prove Theorem 1.2.

Proof of Theorem 1.2. The proof is the same as that of [9, Theorem 4]. We sketch it here. Let \(\Xi\) be the following map

\[
\Xi : H^1_{\text{et}}(M_U \otimes \bar{F}, \mathcal{F}_V) \oplus H^1_{\text{et}}(M_U \otimes \bar{F}, \mathcal{F}_V) \to H^1_{\text{et}}(M_{U_0(q)} \otimes \bar{F}, \mathcal{F}_V).
\]

Let \(\mathcal{V}\) be the functor defined below [9, Corollary 6]. Then we have \(\mathcal{V}(\alpha_{DR}) = \Xi\), and therefore \(\mathcal{V}(\text{Ker} \alpha_{DR}) = \text{Ker} \Xi\). Any Jordan-Holder constituent of \(\text{Ker} \Xi\) induces a Jordan-Holder constituent of \(H^1_{\text{et}}(M_U \otimes \bar{F}, \mathcal{F}_V)\). Suppose we have a Jordan-Holder constituent \(W\) of \(H^1_{\text{et}}(M_U \otimes \bar{F}, \mathcal{F}_V)\), then it must be a Jordan-Holder constituent of some \(T(U,V)\)-subquotient of \(H^1_{\text{et}}(M_U \otimes \bar{F}, \mathcal{F}_V)\) annihilated by a maximal ideal \(m\). If \(m\) is non-Eisenstein, then \(W\) is contained in some irreducible rank two Galois representation \(r_m : G_F \to GL_2(T(U,V)/m)\). Therefore \(W\) must be the whole \(r_m\). Since \(r_m|_{G_{F_p}}\) is of the form \(\mathcal{V}(W)\) where \(W\) has a two step filtration, the theorem follows. \qed

One immediate corollary is Ihara’s lemma for some \(p\)-adic sheaves. If \(W\) is a free \(O\)-module, we know that the natural map

\[
H^1_{\text{et}}(M_H \otimes \bar{F}, \mathcal{F}_W)_m \otimes \kappa \to H^1_{\text{et}}(M_H \otimes \bar{F}, \mathcal{F}_W \otimes \Omega^\kappa)_m
\]

is an isomorphism. (See for example [5, Lemma 1.1].) Assume that \((k_r)_{r \in S} \in \mathbb{Z}^n\) with \(2 \leq k_r \leq p + 1\) and all \(k_r\) are of the same parity. Let \(W\) be the \(GL_2(O_p)\) representation defined by

\[
W = \otimes_{v \mid p} \otimes_{r:F_v \to \mathbb{Q}_p} (\text{Symm}^{k_r-1} \mathcal{O}_{F_v}^2) \otimes \mathcal{O}.
\]
Then we have the following result.

**Theorem 3.6.** If the genus of $M_U$ is greater than 1, and $\sum_{\tau \in S}(k_{\tau} - 1) \leq p - 2$, then the following map is injective:

$$H^1_{et}(M_U, \mathcal{F}_W)_m \oplus H^1_{et}(M_U, \mathcal{F}_W)_m \rightarrow H^1_{et}(M_{U_0(q)}, \mathcal{F}_W)_m$$

$$(f_1, f_2) \mapsto 1_s f_1 + \begin{pmatrix} 1 & 0 \\ 0 & \omega_q \end{pmatrix} \cdot f_2.$$  

Here $m$ is a non-Eisenstein maximal ideal of the Hecke algebra.

4. Examples with multiplicity two

In this section, we study certain exact sequences following Carayol [3] and Jarvis [15]. For simplicity, we will use $M_K$ to denote the integral model $\mathcal{M}_K$ (as stated in subsections 2.4 and 2.5) if $M_K$ exists.

4.1. Exact sequences of $M_{K_0(N)}$. First, we consider the Shimura curve $M_{K_0(N)}$ and the sheaf $\mathcal{F} = \mathcal{F}_V$. Let $H = K_0(N)^p$, then $K_0(N) = \Gamma_0(p)H$. The exact sequence of vanishing cycles for the proper morphism $M_{\Gamma_0(p), H} \otimes \mathcal{O}_{F_p} \rightarrow \text{Spec} \mathcal{O}_{F_p}$ and the sheaf $\mathcal{F}$ is

$$0 \rightarrow H^1_{et}(M_{\Gamma_0(p), H} \otimes \overline{k}_p, \mathcal{F}) \rightarrow H^1_{et}(M_{\Gamma_0(p), H} \otimes \overline{F}_p, \mathcal{F})$$

$$\rightarrow \bigoplus_{x \in \Sigma_H} (R^1\Phi_{\eta}, \mathcal{F})_x \rightarrow H^2_{et}(M_{\Gamma_0(p), H} \otimes \overline{k}_p, \mathcal{F})$$

$$\rightarrow H^2_{et}(M_{\Gamma_0(p), H} \otimes \overline{F}_p, \mathcal{F}) \rightarrow \cdots,$$

(4.1)

where $\Sigma_H$ consists of a finite number of non-degenerate quadratic points.

Write

$$L(H) = \text{Ker}(H^1_{et}(M_{\Gamma_0(p), H} \otimes \overline{k}_p, \mathcal{F}) \rightarrow H^2_{et}(M_{\Gamma_0(p), H} \otimes \overline{F}_p, \mathcal{F})),$$

$$Z(H) = H^1_{et}(M_{\Gamma_0(p), H} \otimes \overline{k}_p, \mathcal{F}),$$

$$M(H) = H^1_{et}(M_{\Gamma_0(p), H} \otimes \overline{F}_p, \mathcal{F}),$$

$$X(H) = \bigoplus_{x \in \Sigma_H} (R^1\Phi_{\eta}, \mathcal{F})_x,$$

$$\tilde{X}(H) = \text{Ker}(X(H) \rightarrow L(H)).$$

Then we have a short exact sequence

$$0 \rightarrow Z(H) \rightarrow M(H) \rightarrow \tilde{X}(H) \rightarrow 0.$$  

(4.2)

We construct a second exact sequence, based on the comparison between the cohomology of the special fibre and the cohomology of the normalization of the special fibre. Recall we have a map $r : M_{0, H} \otimes \overline{k}_p \sqcup M_{0, H} \otimes \overline{k}_p \rightarrow M_{\Gamma_0(p), H} \otimes \overline{k}_p$, and $M_{\Gamma_0(p), H} \otimes \overline{k}_p$ can be regarded as two copies of $M_{0, H} \otimes \overline{k}_p$ glued together transversally above each supersingular point of $M_{0, H} \otimes \overline{k}_p$. As $r$ is an isomorphism away from supersingular points, there is an exact sequence of sheaves

$$0 \rightarrow \mathcal{F} \rightarrow r_* r^* \mathcal{F} \rightarrow \mathcal{G} \rightarrow 0,$$
where $G$ is a skyscraper sheaf supported on $\Sigma_H$. Taking the long exact sequence, we have

$$0 \to H^0_{\text{et}}(M_{\Gamma_0(p), H} \otimes \tilde{k}_p, \mathcal{F}) \to H^0_{\text{et}}(M_{\Gamma_0(p), H} \otimes \tilde{k}_p, r_* r^* \mathcal{F})$$

$$\to H^1_{\text{et}}(M_{\Gamma_0(p), H} \otimes \tilde{k}_p, \mathcal{G}) \to H^1_{\text{et}}(M_{\Gamma_0(p), H} \otimes \tilde{k}_p, \mathcal{F})$$

$$\to H^1_{\text{et}}(M_{\Gamma_0(p), H} \otimes \tilde{k}_p, r_* r^* \mathcal{F}) \to 0.$$  \hfill (4.3)

Write

$$\tilde{L}(H) = \text{Im}(H^0_{\text{et}}(M_{\Gamma_0(p), H} \otimes \tilde{k}_p, r_* r^* \mathcal{F}) \to H^0_{\text{et}}(M_{\Gamma_0(p), H} \otimes \tilde{k}_p, \mathcal{G})),$$

$$Y(H) = H^0_{\text{et}}(M_{\Gamma_0(p), H} \otimes \tilde{k}_p, \mathcal{G}) = \bigoplus_{x \in \Sigma_H} \mathcal{G}_x,$$

$$R(H) = H^1_{\text{et}}(M_{\Gamma_0(p), H} \otimes \tilde{k}_p, r_* r^* \mathcal{F}),$$

$$\tilde{Y}(H) = Y(H)/\tilde{L}(H).$$

Then we have another short exact sequence

$$0 \to \tilde{Y}(H) \to Z(H) \to R(H) \to 0.$$  \hfill (4.4)

The following lemma collects several properties of the exact sequences (4.2) and (4.4).

**Lemma 4.1.** (1) The sequences (4.2) and (4.3) are equivariant for the Hecke action and the Galois action.

(2) $H^1_{\text{et}}(M_{\Gamma_0(p), H} \otimes \tilde{k}_p, r_* r^* \mathcal{F}) \cong H^1_{\text{et}}(M_{0, H} \otimes \tilde{k}_p, \mathcal{F})^2$.

(3) There is an isomorphism

$$H^1_{\Sigma_{\mathbb{H}}}(M_{\Gamma_0(p), H} \otimes \tilde{k}_p, \mathcal{F}) \cong \bigoplus_{x \in \Sigma_{\mathbb{H}}} H^1_{x}(M_{\Gamma_0(p), H} \otimes \tilde{k}_p, R\Psi_{\eta} \mathcal{F}).$$

(4) There is an isomorphism

$$H^1_{\text{et}}(M_{\Gamma_0(p), H} \otimes \tilde{k}_p, \mathcal{G}) \cong H^1_{\Sigma_{\mathbb{H}}}(M_{\Gamma_0(p), H} \otimes \tilde{k}_p, \mathcal{F}).$$

(5) There is an isomorphism

$$N : X(H)(1) \to Y(H).$$

(6) $\tilde{L}(H)_m = 0$, $L(H)_m = 0$.

**Proof.** These results are proved in [13 Sections 16-18]. \hfill \Box

**Remark 4.2.** (1) We have an isomorphism

$$H^1_{\text{et}}(M_{0, H} \otimes \tilde{k}_p, \mathcal{F}) \cong H^1_{\text{et}}(M_{0, H} \otimes \tilde{F}_p, \mathcal{F}),$$

because $M_{0, H}$ has good reduction at $p$. Therefore, we also have

$$R(H) \cong H^1_{\text{et}}(M_{0, H} \otimes \tilde{F}_p, \mathcal{F})^2.$$

(2) Jarvis [15] proved more than those listed in the lemma. In particular, [15] Proposition 17.4] gives the following diagram which describes the action of the inertia group and is very important in our application:

$$M(H) \xrightarrow{\tau^{-1}} M(H)$$

$$\downarrow \beta \quad \quad \quad \uparrow \alpha$$

$$X(H) \xrightarrow{\text{Var}(\tau)} Y(H)$$

Here $\tau$ is an element in the inertia group, and $\text{Var}(\tau)$ is the variation map.
By the lemma, \( \tilde{X}(H)_m = X(H)_m \) and \( \tilde{Y}(H)_m = Y(H)_m \). Then combining exact sequences (4.2) and (4.4), we have the following diagram

\begin{align*}
0 & \quad \downarrow \\
Y(H)_m & \\
0 & \quad \downarrow \\
Z(H)_m & \longrightarrow M(H)_m \longrightarrow X(H)_m \longrightarrow 0 \\
R(H)_m & \quad \downarrow \\
0 & 
\end{align*}

(4.6)

Together with an isomorphism \( N : (X(H)_m)(1) \rightarrow Y(H)_m \).

Fix an embedding \( \mathbb{F} = \mathbb{T}(K,F)_m/m \rightarrow \overline{\mathbb{F}}_p \). Tensoring the above diagram with \( \overline{\mathbb{F}}_p \), we obtain

\begin{align*}
Y(H) & \otimes \overline{\mathbb{F}}_p \\
Z(H) & \otimes \overline{\mathbb{F}}_p \longrightarrow M(H) \otimes \overline{\mathbb{F}}_p \longrightarrow X(H) \otimes \overline{\mathbb{F}}_p \longrightarrow 0 \\
R(H) & \otimes \overline{\mathbb{F}}_p \\
0 & 
\end{align*}

(4.7)

Together with an isomorphism \( N : X(H)(1) \otimes \overline{\mathbb{F}}_p \rightarrow Y(H) \otimes \overline{\mathbb{F}}_p \).

**Lemma 4.3.** If \( M(H) \otimes \overline{\mathbb{F}}_p \rightarrow X(H) \otimes \overline{\mathbb{F}}_p \) is an isomorphism, then \( \bar{\rho} \) is unramified at \( p \).

**Proof.** We can compute the action of the inertia group \( I_{F_p} \). An element \( \tau \in I_{F_p} \) acts by

\begin{align*}
M(H) \otimes \overline{\mathbb{F}}_p & \quad \xrightarrow{\tau^{-1}} \\
X(H) \otimes \overline{\mathbb{F}}_p & \quad \xrightarrow{\text{Var}(\tau)} \\
Y(H) \otimes \overline{\mathbb{F}}_p & 
\end{align*}

\begin{align*}
\beta & \downarrow \\
\alpha & 
\end{align*}

(4.8)

If \( \beta \) is an isomorphism, then by the above diagram we see that \( \alpha \) is the zero map. Therefore, \( \tau - 1 = 0 \), i.e., the action is unramified. \( \Box \)

We have the following lemma, which corresponds to [22] Proposition 1.

**Lemma 4.4.** \( \dim_{\overline{\mathbb{F}}_p} X(H) \otimes \overline{\mathbb{F}}_p \leq 2 \). The dimension is 2 if and only if \( \bar{\rho} \) is unramified at \( p \) and \( \text{Frob}_p \) acts as \( \pm 1 \).
**Proof.** Let $W_p$ be the Atkin-Lehner operator on $M_{K_0(N)}$ defined by the matrix \(egin{pmatrix} 0 & 1 \\ -\varpi_p & 0 \end{pmatrix}\) (see [16] Section 3), $U_{\omega_p}$ the $p$-th Hecke operator on $M_{K_0(N)}$. By [15] Theorem 10.2, the involution $W_p$ permutes the two components of $M_{K_0(N)} \otimes k_p$. Note that $X(H) \otimes \bar{F}_p$ is an unramified $G_{\bar{F}_p}$-module and $\text{Frob}_p$ acts on $X(H) \otimes \bar{F}_p$ as $\pm \text{Norm}(p)$ by the discussion in subsection 2.4.

We have a surjection $M(H) \otimes \bar{F}_p \to X(H) \otimes \bar{F}_p$ with $\dim M(H) \otimes \bar{F}_p = 2$ and $\dim X(H) \otimes \bar{F}_p \geq 1$. If $\dim X(H) \otimes \bar{F}_p = 2$, then $M(H) \otimes \bar{F}_p \cong X(H) \otimes \bar{F}_p$. The $G_{\bar{F}_p}$-action on $M(H) \otimes \bar{F}_p$ is unramified. Taking the determinant of $\text{Frob}_p$, we see $\text{Norm}(p)^2 \equiv \text{Norm}(p)$ (mod $\mathfrak{m}$) and therefore $\text{Frob}_p$ acts as $\pm 1$.

Assume now that $\bar{\rho}$ is unramified at $p$ and $\text{Frob}_p$ acts as $\pm 1$. Taking determinant, we see that $\text{Norm}(p) \equiv 1$ (mod $\mathfrak{m}$). Define map $A$ by

$$A : H^1_{\text{et}}(M_{K_0(N)/p} \otimes \bar{k}_p, \mathcal{F})_\mathfrak{m} \to H^1_{\text{et}}(M_{K_0(N)} \otimes \bar{k}_p, \mathcal{F})_\mathfrak{m}$$

$$(f, g) \mapsto 1* f + (\eta_p)* g$$

and map $B$ induced by the normalization map

$$B : H^1_{\text{et}}(M_{K_0(N)} \otimes \bar{k}_p, \mathcal{F})_\mathfrak{m} \to H^1_{\text{et}}(M_{K_0(N)/p} \otimes \bar{k}_p, \mathcal{F})^2_\mathfrak{m}.$$ 

An easy computation shows that $B \circ A = \left( \begin{array}{cc} \text{Id} & \text{Ver} \\ \text{Ver} & \text{Id} \end{array} \right)$ where $\text{Ver}$ is the transpose of the Frobenius. Since each component of $M_{K_0(N)} \otimes k_p$ is rational over $k_p$, $U_{\omega_p}$ acts on $\Sigma_H$ as the same way as $\text{Frob}_p$. By [3] Theorem 0.7A or [20] Proposition 7], the action of $\text{Frob}_p$ on $Z(H)_\mathfrak{m}$ is the same as that of $\text{Norm}(p)U_{\omega_p}$. Thus $U_{\omega_p} \equiv \pm 1$ (mod $\mathfrak{m}$) on $H^1_{\text{et}}(M_{K_0(N)} \otimes \bar{k}_p, \mathcal{F})_\mathfrak{m} \otimes \bar{F}_p$. Moreover, $U_{\omega_p} + W_p$ factors through $H^1_{\text{et}}(M_{K_0(N)} \otimes \bar{k}_p, \mathcal{F})_\mathfrak{m} \otimes \bar{F}_p \to H^1_{\text{et}}(M_{K_0(N)/p} \otimes \bar{k}_p, \mathcal{F})_\mathfrak{m} \otimes \bar{F}_p$ (see [16] Proposition 3.1) hence $U_{\omega_p} + W_p$ acts as 0 on $Y(H) \otimes \bar{F}_p$. Thus $W_p \equiv \mp 1$ (mod $\mathfrak{m}$).

Let $C$ be the diagonal map (resp. anti-diagonal map)

$$C : H^1_{\text{et}}(M_{K_0(N)/p} \otimes \bar{k}_p, \mathcal{F})_\mathfrak{m} \to H^1_{\text{et}}(M_{K_0(N)/p} \otimes \bar{k}_p, \mathcal{F})^2_\mathfrak{m}$$

if $\text{Frob}_p$ acts as $-1$ (resp. as $+1$). Note that $H^1_{\text{et}}(M_{K_0(N)/p} \otimes \bar{F}, \mathcal{F})_\mathfrak{m} \otimes \bar{F}_p \to H^1_{\text{et}}(M_{K_0(N)} \otimes \bar{F}, \mathcal{F})_\mathfrak{m} \otimes \bar{F}_p$ is surjective because $\bar{\rho}$ is irreducible. In particular, $A \otimes \bar{F}_p$ is surjective. Since $W_p(A(f, g)) = A(g, f)$, we see that the following map is a surjection on to $\text{Im}(B \otimes \bar{F}_p) \circ (A \otimes \bar{F}_p)) = \text{Im}(B \otimes \bar{F}_p) = R(H) \otimes \bar{F}_p$.

$$C' : H^1_{\text{et}}(M_{K_0(N)/p} \otimes \bar{k}_p, \mathcal{F})_\mathfrak{m} \to H^1_{\text{et}}(M_{K_0(N)/p} \otimes \bar{k}_p, \mathcal{F})^2_\mathfrak{m} \otimes \bar{F}_p.$$ 

The composition $(A \otimes \bar{F}_p) \circ C'$ is surjective. Also $\text{Ver} \equiv \pm 1$ (mod $\mathfrak{m}$) since $\text{Norm}(p) \equiv 1$ (mod $\mathfrak{m}$), it is easy to check that $(B \otimes \bar{F}_p) \circ (A \otimes \bar{F}_p) \circ C'(H^1_{\text{et}}(M_{K_0(N)/p} \otimes \bar{k}_p, \mathcal{F})_\mathfrak{m}) = 0$. Therefore $(A \otimes \bar{F}_p)(R(H) \otimes \bar{F}_p) \subset \text{Ker}(B \otimes \bar{F}_p) = Y(H) \otimes \bar{F}_p$. Hence $Z(H) \otimes \bar{F}_p \subset Y(H) \otimes \bar{F}_p$ and maps to 0 in $M(H) \otimes \bar{F}_p$. The lemma follows. □

4.2. **Exact sequences of $M'_D$.** Next, we describe the corresponding picture for the curve $M'_D$. Note that now we consider the reduction mod $q$. The exact sequence of vanishing
cycles for the proper morphism \( M'_O \otimes \mathcal{O}_{F_q} \to \text{Spec} \mathcal{O}_{F_q} \) and the sheaf \( \mathcal{F} \) is
\[
0 \to H^1_{\text{et}}(M'_O \otimes \overline{k}_q, \mathcal{F}) \to H^1_{\text{et}}(M'_O \otimes \overline{F}_q, \mathcal{F}) \\
(4.9) \quad \xrightarrow{\oplus} (R^1\Phi_q\mathcal{F})_x \to H^2_{\text{et}}(M'_O \otimes \overline{k}_q, \mathcal{F}) \to H^2_{\text{et}}(M'_O \otimes \overline{F}_q, \mathcal{F}) \to \cdots,
\]
where \( \Sigma_O \) denotes the set of singular points for the \( \overline{k}_q \)-scheme \( M'_O \otimes \overline{k}_q \). Note that \( \Sigma_O \) consists of a finite number of non-degenerate quadratic points.

Write
\[
\begin{align*}
L(O) &= \text{Ker}(H^2_{\text{et}}(M'_O \otimes \overline{k}_q, \mathcal{F}) \to H^2_{\text{et}}(M'_O \otimes \overline{F}_q, \mathcal{F})), \\
Z(O) &= H^1_{\text{et}}(M'_O \otimes \overline{k}_q, \mathcal{F}), \\
M(O) &= H^1_{\text{et}}(M'_O \otimes \overline{F}_q, \mathcal{F}), \\
X(O) &= \bigoplus_{x \in \Sigma_O} (R^1\Phi_q\mathcal{F})_x, \\
\bar{X}(O) &= \text{Ker}(X(O) \to L(O)).
\end{align*}
\]
Then we have a short exact sequence
\[
(4.10) \quad 0 \to Z(O) \to M(O) \to \bar{X}(O) \to 0.
\]
We construct a second exact sequence. Let \( r \) be the normalization map of the special fibre of \( M'_O \). We define \( \mathcal{G}' \) via the following exact sequence of sheaves
\[
0 \to \mathcal{F} \to r_* r^* \mathcal{F} \to \mathcal{G}' \to 0.
\]
Then \( \mathcal{G}' \) is a skyscraper sheaf supported on \( \Sigma_O \). Taking the long exact sequence, we obtain
\[
(4.11) \quad 0 \to H^0_{\text{et}}(M'_O \otimes \overline{k}_q, \mathcal{F}) \to H^0_{\text{et}}(M'_O \otimes \overline{k}_q, r_* r^* \mathcal{F}) \to H^0_{\text{et}}(M'_O \otimes \overline{k}_q, \mathcal{G}') \\
\quad \to H^1_{\text{et}}(M'_O \otimes \overline{k}_q, \mathcal{F}) \to H^1_{\text{et}}(M'_O \otimes \overline{k}_q, r_* r^* \mathcal{F}) \to 0.
\]
Write
\[
\begin{align*}
\bar{L}(O) &= \text{Im}(\alpha : H^0_{\text{et}}(M'_O \otimes \overline{k}_q, r_* r^* \mathcal{F}) \to H^0_{\text{et}}(M'_O \otimes \overline{k}_q, \mathcal{G}')), \\
\bar{Y}(O) &= H^1_{\text{et}}(M'_O \otimes \overline{k}_q, \mathcal{F}) = \bigoplus_{x \in \Sigma_O} \mathcal{G}'_x, \\
R(O) &= H^1_{\text{et}}(M'_O \otimes \overline{k}_q, r_* r^* \mathcal{F}), \\
\bar{Y}(O) &= Y(O)/\bar{L}(O).
\end{align*}
\]
Then we have another short exact sequence
\[
(4.12) \quad 0 \to \bar{Y}(O) \to Z(O) \to R(O) \to 0.
\]

**Lemma 4.5.** (1) \( H^i_{\text{et}}(M'_O \otimes \overline{k}_q, r_* r^* \mathcal{F}) \cong H^i_{\text{et}}(\mathbb{P}^1 \otimes \overline{k}_q, \mathcal{F})^b \), for some positive integer \( b \).

(2) There is an isomorphism
\[
H^1_{\Sigma_O}(M'_O \otimes \overline{k}_q, \mathcal{F}) \cong \bigoplus_{x \in \Sigma_O} H^1_x(M'_O \otimes \overline{k}_q, \mathcal{R}\Psi_q \mathcal{F}).
\]

(3) There is an isomorphism
\[
H^0_{\text{et}}(M'_O \otimes \overline{k}_q, \mathcal{G}') \cong H^1_{\Sigma_O}(M'_O \otimes \overline{k}_q, \mathcal{F}).
\]

(4) There is an isomorphism
\[
N : X(O)(1) \to Y(O).
\]

(5) \( \bar{L}(O)_m = 0, L(O)_m = 0, R(O)_m = 0 \).
Proof. (1) We know that reduction $\Omega \mod p$ is a union of $\mathbb{P}^1$. By Theorem 2.8, the special fibre $M_0 \otimes \kappa_q$ is several copies of $\mathbb{P}^1$ glued together transversally above the singular points. The normalization of the special fibre is a disjoint union of finitely many $\mathbb{P}^1$. The statement follows.

(2)(3)(4) The arguments are the same as the arguments in Lemma 4.1.

(5) The first two terms vanish because they come from the 0th and 2nd cohomology groups and $m$ is a non-Eisenstein ideal. The third term vanishes because the 1st cohomology group of $\mathbb{P}^1$ vanishes. □

Combining exact sequences (4.10) and (4.12), we obtain the following short exact sequence

$$0 \to Y(O)_m \to M(O)_m \to X(O)_m \to 0,$$

together with an isomorphism $N: (X(O)_m(1) \otimes \overline{F}_p) \sim \to Y(O)_m \otimes \overline{F}_p$.

**Lemma 4.6.** There is an isomorphism of Hecke modules

$$Y(H)_m \cong Y(O)_m.$$

**Proof.** We deduce this from the exact sequence (3.14) of [20]. Rewriting that sequence in our setting, we have

$$0 \to Y(H')_m^2 \to Y(H)_m \to Y(O)_m \to 0.$$

Here $Y(H')$ corresponds to the group $Y(H)$ for the curve $M_{K'}$ where $K' = K_0(N/q)$. Note that $\rho$ is ramified at $q$, hence $m$ is not $q$-old. On the other hand, $M_{K'}$ has no level structure at $q$ and $Y(H')$ is $q$-old. Therefore, $Y(H')_m = 0$ and the claim follows. □

4.3. **Proof of Theorem 1.6.** We combine the results in the above sections to prove a general multiplicity two result.

**Lemma 4.7.** $\rho$ is unramified at $q$ if and only if there is an isomorphism

$$M(O) \otimes \overline{F}_p \sim X(O) \otimes \overline{F}_p.$$

**Proof.** By the diagram in Remark 4.2, we can compute the action of the inertia group $I_{F_p}$. An element $\tau \in I_{F_p}$ acts by

$$M(O) \otimes \overline{F}_p \xrightarrow{\tau} M(O) \otimes \overline{F}_p \xrightarrow{\alpha \cdot N} Y(O) \otimes \overline{F}_p$$

If $\rho$ is unramified at $q$, then $M(O) \otimes \overline{F}_p$ is unramified at $q$. Thus $\alpha N \beta = \tau - 1 = 0$ for all $\tau \in I_{F_p}$. This is the same as $\alpha N \beta = 0$ since $Var(\tau) = -\epsilon(\tau)N$. Now $\beta$ is a surjection and $N$ is an isomorphism, so $\alpha = 0$. Thus by the exact sequence, $\beta$ is an isomorphism.

On the other hand, if $\beta$ is an isomorphism, then by the exact sequence we see that $\alpha$ is the zero map. Therefore, $\tau - 1 = 0$, i.e., the action is unramified. □

We have assumed that $\rho$ is ramified at $q$, then we have the following lemma.
Lemma 4.8. \( \dim_{\hat{\mathbb{F}}_p} M(O) \otimes \bar{\mathbb{F}}_p = 2 \dim_{\hat{\mathbb{F}}_p} X(O) \otimes \bar{\mathbb{F}}_p \).

Proof. By the exact sequences constructed above, we have

\[
\dim_{\hat{\mathbb{F}}_p} M(O) \otimes \bar{\mathbb{F}}_p \leq \dim_{\hat{\mathbb{F}}_p} X(O) \otimes \bar{\mathbb{F}}_p + \dim_{\hat{\mathbb{F}}_p} Y(O) \otimes \bar{\mathbb{F}}_p
\]
\[
= 2 \dim_{\hat{\mathbb{F}}_p} Y(O) \otimes \bar{\mathbb{F}}_p = 2 \dim_{\hat{\mathbb{F}}_p} Y(H) \otimes \bar{\mathbb{F}}_p
\]
\[
= 2 \dim_{\hat{\mathbb{F}}_p} X(H) \otimes \bar{\mathbb{F}}_p \leq 2 \dim_{\hat{\mathbb{F}}_p} M(H) \otimes \bar{\mathbb{F}}_p \leq 4
\]

By the above lemma, we see that \( \dim_{\hat{\mathbb{F}}_p} M(O) \otimes \bar{\mathbb{F}}_p = \dim_{\hat{\mathbb{F}}_p} X(O) \otimes \bar{\mathbb{F}}_p = 2 \) cannot happen since \( \bar{\rho} \) is ramified at \( q \). The lemma follows. \( \square \)

Proof of Theorem 1.6. \( \dim_{\hat{\mathbb{F}}_p} \text{Hom}(\bar{\rho}, H^1_c(M_\rho \otimes \hat{F}, \mathcal{F})) \leq 2 \) follows from equation (4.15). Since \( \dim_{\hat{\mathbb{F}}_p} Y(H) \otimes \bar{\mathbb{F}}_p = \dim_{\hat{\mathbb{F}}_p} Y(O) \otimes \bar{\mathbb{F}}_p \), we have \( \dim_{\hat{\mathbb{F}}_p} X(H) \otimes \bar{\mathbb{F}}_p = \dim_{\hat{\mathbb{F}}_p} X(O) \otimes \bar{\mathbb{F}}_p \). In particular, \( \dim_{\hat{\mathbb{F}}_p} X(O) \otimes \bar{\mathbb{F}}_p \leq 2 \). The dimension is 2 if and only if \( \text{Gal}(\hat{F}_p/F_p) \)-action is unramified and \( \text{Frob}_p \) acts as \( \pm 1 \). Theorem 1.6 is now clear from Lemmas 4.8 and 4.9. \( \square \)

Remark 4.9. Now we have a multiplicity two result. If we start with a Shimura curve such that the cohomology group has multiplicity two, we can do the same thing as above (as long as we have enough primes in the level): construct a new Shimura curve by changing local invariants, compare the corresponding cohomology groups, etc.. Then we may get a multiplicity 4 result similar to Theorem 1.6. In this way we may construct Shimura curves with multiplicity \( 2^n \). This is closely related to the conjectures in [2 Section 4].

Acknowledgements The author would like to thank Matthew Emerton, for his support, encouragement, and many enlightening discussions. He would like to thank Brian Conrad, Fred Diamond, and Mark Kisin for helpful conversations and correspondences, Panagiotis Tsaknias for pointing out a mistake in an early version of this paper.

References

[6] Cherednik, I.V. Uniformization Of Algebraic Curves by Discrete Arithmetic Subgroups of \( \text{PGL}_2(k_w) \) With Compact Quotient.


