Poisson and Hopf Structures for Finite Groups via Star Product and Convolution

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Abstract

A geometric approach is employed to introduce a nontrivial Poisson bracket for the function space $L^2(G)$ of a finite group G. This allows a tensor formulation of a known family of noncommutative star products $\star_{\lambda_0}$ and the usual convolution product $*_c$. Hopf structures for the same space are derived from matricial elements of irreducible representations.

Keywords:
Geometric mechanics, Deformation Quantization, Fourier analysis

1. Introduction

The mathematical formalism of quantum mechanics usually means introducing a correspondence between the space of functions on a phase space (denoted by $C^\infty(M)$, when the phase space is a manifold M) and the set of Hermitian operators of some Hilbert space $\mathcal{H}$. Quantization of $\mathbb{R}^3$ is a familiar example, where the position and momentum functions $q(x,y,z)$, $p(x,y,z)$ are mapped to $\hat{Q}$ and $\hat{P}$, respectively, where $\hat{Q}(\psi(x,y,z)) = Q.\psi$ and $\hat{P}(\psi(x,y,z)) = \frac{-i}{\hbar}\left(\frac{\partial\psi}{\partial x} + \frac{\partial\psi}{\partial y} + \frac{\partial\psi}{\partial z}\right)$. Bayen, Flato, et. al. introduced in [2] another way to derive a "quantum system" without resorting to mapping to another algebraic structure, by keeping the functions as they are and by introducing a noncommutative product on the function space. The quantization of the classical system is seen to arise from this noncommutativity. An example is the space of formal power series, with functions as the coefficients and the expansion is with respect to a certain formal parameter $\hbar$, together with a new operation called the star product (which is noncommutative and associative), $(C^\infty(M)[[\hbar]], \star)$. This algebra called a deformation quantization of the phase space M [3]. The progress of finding star products for different phase spaces have been successful in the past. De Compte and Wilde, and Fedosov were able to prove that a star product exists for every symplectic manifold (M, ω) [3].

Quantization attempts to look at how classical structures are changed after the phase space undergoes deformation, and how they (Poisson brackets) could be retrieved as a limit of the obtained quantum structures. The construction of quantum moment maps and the deformation of Poisson structures depend heavily on the symplectic and differentiable structures on the manifold. Unfortunately, no such structures are available for finite groups. Explicit formulas for star products on finite and compact groups were presented by Aniello et.al in [5]. In this construction, the family of star products on a compact group G is indexed by the irreducible unitary representations of the group U^λ. The star product for $f, g \in L^2(G)$ is given by

$$f \star_{\lambda_0} g = \operatorname{Tr}\left(\hat{A}\hat{B}U^{\lambda_0}\right) \tag{1}$$

where $\hat{A}, \hat{B}$ are the images of f, g via quantizer maps. For every irreducible unitary representation there corresponds exactly one star product. Endowed with this star product, the group algebra $L^2(G)$ obtains a Lie- Jordan algebra structure, with structure constants given in terms of irreducible characters of the finite group. In papers [1,6] by Marmo, et.al. on the geometrization of quantum mechanics uses dual formulations to construct nontrivial brackets and nonlocal products on Lie algebras. A similar program would be implemented in this paper and the explicit geometric forms of Lie-Jordan and convolution structures would be given. Nontrivial quantum group structures in terms of the matricial elements of irreducible representations would also be derived.

2. Poisson Structure

It is well known that the orthogonalized set of all matricial elements of irreducible unitary representation of a finite group G, denoted by $\left\{d(\lambda)\, a_{ij}^\lambda\right\} = \left\{e_{ij}^\lambda\right\}$ comprise a basis for the function space $L^2(G)$ [7]. Let

$f, g \in L^2(G)$. A nontrivial Lie bracket on $L^2(G)$ under a star product $\star_{\lambda_0}$could then be defined as the commutator of f and g, which reads

$$[f,g]_{\lambda_0} = f \star_{\lambda_0} g - g \star_{\lambda_0} f \tag{2}$$

where $\lambda_0 \in \hat{G}$ corresponds to an irreducible representation of G.

The coordinates for the Hilbert manifold are given by $\{e^{\lambda}_{ij}\}$. Any function $f \in L^2(G)$ could be viewed as a linear funcion on the manifold's coordinates, i.e.

$$f = f\left(e^{\lambda}_{ij}\right) = \sum_{\lambda} \sum_{i,j=1}^{d(\lambda)} \hat{f}(\lambda)_{ij}\, e^{\lambda}_{ji} \tag{3}$$

where f is given in terms of its Fourier expansion.

Since all smooth functions on $L^2(G)$ simplifies to linear functions on $L^2(G)$ via the convolution product of the basis elements, the Lie bracket on $L^2(G)$ could be elevated to a bracket $\{,\}_{\lambda_0}$on $C^{\infty}\left(L^2(G)\right)$ By direct computation, it could easily be shown that the bracket $\{,\}_{\lambda_0}$satisfies the Jacobi and the Leibniz identities, and hence qualifies as a Poisson bracket in $C^{\infty}\left(L^2(G)\right)$ given by

$$\{\alpha, \beta\}_{\lambda_0} = [\alpha, \beta]_{\lambda_0} \tag{4}$$

The star product associated to an irreducible representation is written as

$$(f \star_{\lambda_0} g)(x) = \frac{d(\lambda_0)^{\frac{3}{2}}}{|G|^2} \sum_{i,k=1}^{d(\lambda_0)} \left[\sum_{j=1}^{d(\lambda_0)} \hat{f}(\lambda_0)_{ij}\, \hat{g}(\lambda_0)_{jk}\right] e^{\lambda_0}_{ki}(x) \tag{5}$$

Using (5), the Lie bracket could be expressed as

$$[f,g]_{\lambda_0} = \frac{d(\lambda_0)^{\frac{3}{2}}}{|G|^2} \sum_{i,k=1}^{d(\lambda_0)} \left[\sum_{j=1}^{d(\lambda_0)} \left(\hat{f}(\lambda_0)_{ij}\, \hat{g}(\lambda_0)_{jk} - \hat{g}(\lambda_0)_{ij}\, \hat{f}(\lambda_0)_{jk}\right)\right] e^{\lambda_0}_{ki}(x) \tag{6}$$

where f, g are viewed as elements of $L^2(G)$. It is worth noting that this bracket vanish when $f *_c g = f *_c g$, or when either one of them is a class function. The complex vector space $L^2(G)$ is isomorphic to $L^2(G)_{\mathbb{R}} \times L^2(G)_{\mathbb{R}}$, where $L^2(G)_{\mathbb{R}}$is the vector space of real-valued functions on the group G. When viewed as a product of real manifolds, $L^2(G) \cong L^2(G)_{\mathbb{R}} \times L^2(G)_{\mathbb{R}}$, its coordinates will be given by $\left\{\left(u^{\lambda}_{kl}, v^{\lambda}_{kl}\right)\right\}$. As a real manifold, $L^2(G)_{\mathbb{R}}$ is canonically isomorphic to a tangent space $T_p L^2(G)_{\mathbb{R}}$, where $p \in L^2(G)_{\mathbb{R}}$. This allows the assignment of a contravariant tensor $\frac{\partial}{\partial u^{\lambda}_{ij}}$ to the 1-form du^{λ}_{ij} such that

$$\frac{\partial}{\partial u^{\lambda}_{ij}}\left(du^{\mu}_{kl}\right) = \delta_{ik}\delta_{jl}\delta_{\lambda\mu}. \tag{7}$$

Note that the ij^{th} term of the fourier transform of f with respect to the irreducible representation U^{λ} may be recovered via

$$\hat{f}(\lambda)_{ij} = \frac{\partial}{\partial u^{\lambda}_{ij}}\left(\mathfrak{Re}\,(df)\right) + i\frac{\partial}{\partial u^{\lambda}_{ij}}\left(\mathfrak{Im}\,(df)\right). \tag{8}$$

Using this and (6), a complex 2-tensor Λ could be associated to the bracket $\{,\}_{\lambda_0}$such that,

$$\{f,g\}_{\lambda_0} = \Lambda(df, dg) = \sum_{i,k}^{d(\lambda_0)} \Lambda_{ki}(df, dg) \tag{9}$$

where each Λ_{ki} is explicitly given by

$$
\begin{aligned}
\Lambda_{ki} &= u_{ki}^{\lambda_0}\left[\sum_j\left(\frac{\partial}{\partial u_{ij}^{\lambda_0}}\wedge\frac{\partial}{\partial u_{jk}^{\lambda_0}}-\frac{\partial}{\partial v_{ij}^{\lambda_0}}\wedge\frac{\partial}{\partial v_{jk}^{\lambda_0}}\right)\right]-v_{ki}^{\lambda_0}\left[\sum_j\left(\frac{\partial}{\partial u_{ij}^{\lambda_0}}\wedge\frac{\partial}{\partial v_{jk}^{\lambda_0}}-\frac{\partial}{\partial u_{jk}^{\lambda_0}}\wedge\frac{\partial}{\partial v_{ij}^{\lambda_0}}\right)\right] \\
&\quad +iv_{ki}^{\lambda_0}\left[\sum_j\left(\frac{\partial}{\partial u_{ij}^{\lambda_0}}\wedge\frac{\partial}{\partial u_{jk}^{\lambda_0}}-\frac{\partial}{\partial v_{ij}^{\lambda_0}}\wedge\frac{\partial}{\partial v_{jk}^{\lambda_0}}\right)\right]+iu_{ki}^{\lambda_0}\left[\sum_j\left(\frac{\partial}{\partial u_{ij}^{\lambda_0}}\wedge\frac{\partial}{\partial v_{jk}^{\lambda_0}}-\frac{\partial}{\partial u_{jk}^{\lambda_0}}\wedge\frac{\partial}{\partial v_{ij}^{\lambda_0}}\right)\right].
\end{aligned} \tag{10}
$$

A similar construction could be used to obtain the tensorial form of the Jordan bracket, which arises as the anticommutator

$$
[f,g]_+ = f\star_{\lambda_0} g + g\star_{\lambda_0} f. \tag{11}
$$

This gives the tensor R such that

$$
[f,g]_+ = R\left(df,dg\right)=\sum_{i,k}^{d(\lambda_0)} R_{ki}\left(df,dg\right) \tag{12}
$$

where each R_{ki} could be expressed in the same form as that of Λ_{ki} where the wedge product $\wedge$ would just be replaced by the symmetrized product $\otimes_s$.

Note that the star product $f\star_{\lambda_0} g=\frac{1}{2}\left([f,g]_+ + [f,g]_-\right)$, which allows the expansion of the product in terms of contravariant tensors as

$$
\begin{aligned}
\star_{\lambda_0} &= u_{ki}^{\lambda_0}\left[\sum_j\left(\frac{\partial}{\partial u_{ij}^{\lambda_0}}\otimes\frac{\partial}{\partial u_{jk}^{\lambda_0}}-\frac{\partial}{\partial v_{ij}^{\lambda_0}}\otimes\frac{\partial}{\partial v_{jk}^{\lambda_0}}\right)\right]-v_{ki}^{\lambda_0}\left[\sum_j\left(\frac{\partial}{\partial u_{ij}^{\lambda_0}}\otimes\frac{\partial}{\partial v_{jk}^{\lambda_0}}-\frac{\partial}{\partial u_{jk}^{\lambda_0}}\otimes\frac{\partial}{\partial v_{ij}^{\lambda_0}}\right)\right] \\
&\quad +iv_{ki}^{\lambda_0}\left[\sum_j\left(\frac{\partial}{\partial u_{ij}^{\lambda_0}}\otimes\frac{\partial}{\partial u_{jk}^{\lambda_0}}-\frac{\partial}{\partial v_{ij}^{\lambda_0}}\otimes\frac{\partial}{\partial v_{jk}^{\lambda_0}}\right)\right]+iu_{ki}^{\lambda_0}\left[\sum_j\left(\frac{\partial}{\partial u_{ij}^{\lambda_0}}\otimes\frac{\partial}{\partial v_{jk}^{\lambda_0}}-\frac{\partial}{\partial u_{jk}^{\lambda_0}}\otimes\frac{\partial}{\partial v_{ij}^{\lambda_0}}\right)\right].
\end{aligned} \tag{13}
$$

Note that the convolution product could be written in terms of star products since, $f *_c g$ could be expressed in terms of its Fourier expansion, which yields

$$
f *_c g = \sum_\lambda \frac{|G|^2}{d\left(\lambda\right)^{\frac{3}{2}}} f\star_\lambda g \tag{14}
$$

which is analogous to the decomposition of the left regular representation as a sum of all the irreducible representations of a finite group. This allows the geometric expression of the algebraic operation $*_c$ as a sum of star products, which are all realized in tensor form.

3. Hopf Structures

Classical systems usually spring from a Lie group's action on a symplectic manifold (phase space). A generalization of this framework has been introduced by considering action of quantum groups on noncommutative spaces. Quantum groups are special examples of Hopf algebras, which is a unital algebra with the usual multiplication and with additional structures such as the antipode S, the counit ϵ and the comultiplication Δ [8]. The algebra of complex-valued function on finite groups $\mathcal{F}\left(G\right)=L^2\left(G\right)$ is one of the basic examples of Hopf algebras, where multiplication and comultiplication are defined pointwise. It is worth noting that the expansion of a function $f\in L^2\left(G\right)$ in terms of matricial elements also provides an alternative Hopf structure for $L^2\left(G\right)$ which is similar to the Hopf structure of compact matrix quantum groups (CMQG). The structure for each matrix element could be extended linearly to provide a complete Hopf structure, where $*_c$ is the multiplication, $S(a_{ij}^\lambda)=a_{ij}^{\lambda(-1)}$ is the antipode, $\epsilon\left(a_{ij}^\lambda\right)=\delta_{ij}$ is the counit and the comultiplication is $\Delta\left(a_{ij}^\lambda\right)=\frac{|G|^2}{d\left(\lambda\right)^2}\sum_k a_{ik}^\lambda\otimes a_{kj}^\lambda$. These are indeed Hopf structures as seen in the following commutative diagrams:

$$\begin{tikzcd}
\frac{|G|^2}{d(\lambda)^2}\sum_k a^{\lambda}_{ik}\otimes a^{\lambda}_{kj} & a^{\lambda}_{ij} \arrow[l] \\
L^2(G)\otimes L^2(G) \arrow[d, "\epsilon\otimes id"'] & L^2(G) \arrow[l, "\Delta"'] \arrow[d] \\
\mathbb{C}\otimes L^2(G) \arrow[r, "\cong"'] & L^2(G) \\
\frac{|G|^2}{d(\lambda)^2}\sum_k \delta_{ik}\otimes a^{\lambda}_{kj} \arrow[r] & a^{\lambda}_{ij}
\end{tikzcd}$$

Figure 1: Counit ϵ

$$\begin{tikzcd}
\frac{|G|^2}{d(\lambda)^2}\sum_k a^{\lambda}_{ik}\otimes a^{\lambda}_{kj} & a^{\lambda}_{ij} \arrow[l] \arrow[ddd, bend left, "\eta\circ\epsilon"] \\
L^2(G)\otimes L^2(G) \arrow[d, "S\otimes id"'] & L^2(G) \arrow[l, "\Delta"'] \arrow[d] \\
L^2(G)\otimes L^2(G) \arrow[r] & L^2(G) \\
\frac{|G|^2}{d(\lambda)^2}\sum_k a^{\lambda(-1)}_{ik}\otimes a^{\lambda}_{kj} \arrow[r, "m"] & \delta^{\lambda}_{ij}
\end{tikzcd}$$

Figure 2: Antipode Δ

Future Work

The multiplication $*_c$ in the Hopf algebra $L^2(G)$ could be derived using tensorial language. This follows from the computations presented in this work. This means that actions of quantum groups may also be viewed geometrically and hence introduces the possibility of performing quantum mechanics using a geometric approach. Possible future works would be the construction of moment maps for $L^2(G)$ mapped to its dual $L^2(G)^*$ (since $L^2(G)$ is both a manifold and a Lie algebra), and the derivation of an uncertainty principle for this type of deformation quantization.

Acknowledgments

The authors would like to thank the School of Science an Engineering, Ateneo De Manila University for their support in this work. Mr. Mañibo would like to thank the Department of Science and Technology.

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