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Lineare Operatoren in Hilberträumen

Exercise Sheet 1

(1) (a) Let (T_n) be a monotone and bounded sequence of symmetric operators on a Hilbert space X. Then there exists a symmetric operator $T \in B(X)$ such that $T_n \xrightarrow{s} T$.

Hint: Consider the sesquilinear form $s(x, y) := \langle x, (T_n - T_m)y\rangle$ $(m \leqslant n)$ and apply Schwarz' Inequality.

(b) Using part (a), prove Proposition 1.2 in the lecture.

(2+3 Points)

(2) Let X be a Hilbert space and s a bounded sesquilinear form on X (i.e., there exists $C \geqslant 0$ with $|s(x, y)| \leqslant C\|x\|\|y\|$). Then there exists a unique operator $T \in B(X)$ with $\langle y, Tx\rangle = s(y, x)$, for all $x, y \in X$.

Hint: Riesz representation theorem **(3 Points)**

(3) Let X be a (complex) Banach space and T a closed operator on X. Let $\rho(T)$ and $\sigma(T)$ be the resolvent set and spectrum of T, respectively. Prove the following statements.

(a) The set $\rho(T)$ is an open subset of $\mathbb{C}$ and $\sigma(T)$ is a closed set.

(b) The resolvent $R(T, \cdot)$ is analytic in $\rho(T)$, i.e., around every $z_0 \in \rho(T)$, it admits a power series representation

$$R(T, z) = \sum_{n=0}^{\infty} (z - z_0)^n R(T, z_0)^{n+1}.$$

Hint: For (a) and (b), one can invoke the following result on the stability of continuous invertibility:
Let X be a Banach space and $Q, S\colon X \to X$ be linear operators. Suppose Q is bijective and closed. Further, suppose $D(S) \supset D(Q)$ and $\|SQ^{-1}\| < 1$. Then $Q + S$ is also closed and bijective. Moreover

$$(Q + S)^{-1} = \sum_{n=0}^{\infty} (-1)^n Q^{-1}(SQ^{-1})^n.$$

(2+2 Points)

Abgabe bis 12 Uhr am Mittwoch, 22.04.2026