Dr. Neil Manibo

Fakultät für Mathematik,
Universität Bielefeld

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Lineare Operatoren in Hilberträumen

Exercise Sheet 2

(4) Let $H_x = \overline{\operatorname{lin}\{E(t)x \colon x \in \mathbb{R}\}}$, as in Proposition 1.9 in the lecture. Prove that, for every $y \in H_x$, ϱ_y is absolutely continuous with respect to ϱ_x.

(2 Points)

(5) Prove parts (a),(b),(c) and (i) of the spectral calculus in Section 1.2 of the lecture.

Hint: For part (a), construct approximations of u and v using φ_n. For (i), note that $(\gamma_m \circ u)_E x \in D(u_E)$, for all $x \in X$ and $m \in \mathbb{N}$ (why?).

(2+1+2+3 Points)

(6) Prove the *Stieltjes Inversion Formula*:

Let $w \colon \mathbb{R} \to \mathbb{C}$ be a function of bounded variation. Further, suppose that w is upper semicontinuous, with $w(t) \to 0$ as $t \to -\infty$. Consider the function

$$f(z) := \int_{-\infty}^{\infty} \frac{1}{t-z} \, \mathrm{d}w(t) \quad \text{for} \quad z \in \mathbb{C} \setminus \mathbb{R}.$$

(a) For all $t \in \mathbb{R}$, one has

$$w(t) = \lim_{\delta \to 0^+} \lim_{\varepsilon \to 0^+} \frac{1}{2\pi \mathrm{i}} \int_{-\infty}^{t+\delta} \big(f(s + \mathrm{i}\varepsilon) - f(s - \mathrm{i}\varepsilon) \big) \, \mathrm{d}s.$$

In particular, w is uniquely determined by f.

(b) If w is real-valued, one has

$$w(t) = \lim_{\delta \to 0^+} \lim_{\varepsilon \to 0^+} \frac{1}{\pi} \int_{-\infty}^{t+\delta} \operatorname{Im} f(s + \mathrm{i}\varepsilon) \, \mathrm{d}s.$$

Hint: Part (a) follows from part (b) by considering $w = u + \mathrm{i}v$.

(4 Points)

Abgabe bis 12 Uhr am Mittwoch, 29.04.2026