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Lineare Operatoren in Hilberträumen

Exercise Sheet 3

(7) Let (X, μ) be a measurable space, $g\colon X \to \mathbb{R}$ a measurable function, and $T := M_g$ be the multiplication operator with the function $g \in L^2(X, \mu)$, where

$$D(T) = \left\{ f \in L^2(X, \mu) \colon gf \in L^2(X, \mu) \right\}, \quad Tf = gf,$$

(i.e., M_g is the *maximal* multiplication operator). Use Stone's formula to show that the spectral family (*Spektralschar*) associated with T is given by $\{E(t)\}$, where $E(t)$ is multiplication with the characteristic function $\chi_{\{x \in X \,:\, g(x) \leqslant t\}}$. **(4 Points)**

(8) Let X be a normed linear space. A sequence $(x_n)_n$ *converges weakly* to $x \in X$ ($x_n \xrightarrow{w} x$) if, for every linear functional $\phi \in X'$, $\phi(x_n) \to \phi(x)$.

(a) Show that every orthonormal sequence (e_n) in a Hilbert space X converges weakly to 0.

(b) Let $(e_n)_n$ be an orthonormal sequence in a Hilbert space X and let $x \in X$, $x \neq 0$. Show that the sequence

$$x_n := \sum_{j=1}^{\infty} \langle e_j, x \rangle \, e_{j+n}$$

converges weakly, but not strongly, to 0. **(2+2 Points)**

(9) Let T be a self-adjoint operator on a Hilbert space X and $D(T)$ its domain. A sequence $(x_n) \subset D(T)$ is called a *singular sequence* (with respect to T and λ) if the following holds.

- $x_n \overset{s}{\not\to} 0$ but $x_n \xrightarrow{w} 0$
- $(T - \lambda)x_n \xrightarrow{s} 0$

Show that if λ is an eigenvalue of T of infinite multiplicity, then there exists a singular sequence with respect to T and λ. **(2 Points)**

(10) Use the spectral theorem to recover the following bound:

$$\|(T - z)^{-1}\| \leqslant \frac{1}{\operatorname{Im} z}.$$

Here, T is self-adjoint and $z \in \mathbb{C}_+$. **(3 Points)**

Abgabe bis 12 Uhr am Mittwoch, 06.05.2026