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Lineare Operatoren in Hilberträumen

Exercise Sheet 4

(11) (a) Let $V \in L^2(\mathbb{R}^n) + L^\infty(\mathbb{R}^n)$ (that is, $V = V_1 + V_2$, with $V_1 \in L^2(\mathbb{R}^n)$ and $V_2 \in L^\infty(\mathbb{R}^n)$) be real-valued. Show that the multiplication operator M_V defines a symmetric operator in $L^2(\mathbb{R}^n)$.

Hint: Here a possible choice for $D(M_V)$ is $C_\mathrm{c}(\mathbb{R}^n)$.

(b) Show that the Laplacian Δ is symmetric in the Sobolev space $D(-\Delta) = H^2(\mathbb{R}^n) \subset L^2(\mathbb{R}^n)$.

(2+2 Points)

(12) (a) Use the result in Problem (7) and the characterisation of elements of the spectrum via the spectral family $\{E(t)\}$ to show that, with $D(-\Delta) = H^2(\mathbb{R}^n) \subset L^2(\mathbb{R}^n)$, one has

$$\sigma(-\Delta) = \sigma_\mathrm{c}(-\Delta) = [0, \infty).$$

Hint: Fourier transformation and Theorem 1.18 in the lecture notes.

(b) Using the characterisation via singular sequences, show that $\sigma(-\Delta) = \sigma_\mathrm{e}(-\Delta)$.
Hint: Consider the sequence $\{u_m\}$ with

$$u_m(x) = \frac{1}{(2\pi)^{n/2}} \int \widehat{u_m}(k) \mathrm{e}^{\mathrm{i}k\cdot x} \, \mathrm{d}x, \text{ with } \widehat{u_m}(k) = (2\pi m)^{n/2} \mathrm{e}^{-m|k-k_0|^2}$$

where $\lambda := |k_0|^2$.

(3+3 Points)

(13) Prove Theorem 1.28 in the lecture notes.

(2 Points)

Abgabe bis 12 Uhr am Mittwoch, 13.05.2026