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Lineare Operatoren in Hilberträumen

Exercise Sheet 5

(14) Show that the following are equivalent:

(i) There exist constants $a, b \geqslant 0$ such that

$$\|Vx\| \leqslant a\|x\| + b\|Tx\| \quad \text{for all } x \in D(T) \subset D(V).$$

(ii) For every $\varepsilon > 0$, there exists $a, b \geqslant 0$ such that

$$\|Vx\|^2 \leqslant a(1+\varepsilon^{-2})\|x\|^2 + b(1+\varepsilon^2)\|Tx\|^2 \quad \text{for all } x \in D(T) \subset D(V).$$

(2 Points)

(15) Show that the condition $E(t) = E(t-)$ is crucial in Theorem 2.27(i) by considering the Hilbert space $X = \mathbb{C}^2$ and the sequence $\{A_n\}$ of (symmetric) matrices, with

$$A_n = \frac{1}{n}\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.$$

(3 Points)

(16) Let $X = \ell_2$ be the space of square-summable sequences in $\mathbb{R}$, i.e.,

$$\ell_2 = \Big\{ (x_n)_{n\in\mathbb{N}} : \; x_n \in \mathbb{R} \;\text{ and }\; \sum_{n=1}^{\infty} x_n^2 < \infty \Big\}.$$

Let T_n be the orthogonal projection onto the subspace spanned by the first n components, and let E_n and E be the spectral projection corresponding to T_n and I, respectively. Show that the following holds.

(a) $T_n \overset{s}{\to} I$ and $E_n(t) \overset{s}{\to} E(t)$ for all $t \in \mathbb{R}$.

(b) $\sigma(T_n) \not\to \sigma(I)$ and $\sigma_{\mathrm{e}}(T_n) \not\to \sigma_{\mathrm{e}}(I)$.

(4 Points)

(17) Let T be a positive, self-adjoint operator. Suppose S is a symmetric operator with $D(T) \subset D(S)$ and $\|Sx\| \leqslant \|Tx\|$, for all $x \in D(T)$. Show that

$$|\langle x, Sx\rangle| \leqslant \langle x, Tx\rangle, \quad \text{for all } x \in D(T).$$

(Here, T: positive means $\langle x, Tx\rangle \geqslant 0$, for all $x \in D(T)$.)

Hint: Consider $V := \alpha S$ with $\alpha \in (-1, 1)$ and apply Theorem 2.10. **(2 Points)**

Abgabe bis 12 Uhr am Mittwoch, 20.05.2026