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# Lineare Operatoren in Hilberträumen

## Exercise Sheet 8

**(24)** Consider the homogeneous differential equation

$$x^2u'' + xu' + (x^2 - \nu^2)u = 0 \quad \text{for } x \in (0, \infty), \text{ where } \nu = \frac{1}{2}.$$

Let $u$ be a solution of the equation above. Show that *any* two consecutive zeros in $(0, \infty)$ of $u(x)$ are exactly at distance $\pi$ apart.

**Hint:** Here one can use the transformation $u = \frac{v}{\sqrt{x}}$. **(3 Points)**

**(25)** Show that the (modified) Wronskian

$$W(u_1, u_2) = \det \begin{pmatrix} u_1(x) & u_2(x) \\ p(x)u_1'(x) & p(x)u_2'(x) \end{pmatrix}$$

does not depend on $x$. Here, $u_1$ and $u_2$ are solutions of the homogeneous SL equation $(\tau - z)u = 0$, for some $z \in \mathbb{C}$.

**Hint:** Consider $u_1(\tau - z)u_2 - u_2(\tau - z)u_1$. **(3 Points)**

**(26)** Pick four statements from the list of True or False questions in the attached list, and explain why these statements are True/False.

**(4 Points)**

Abgabe bis 12 Uhr am Mittwoch, 17.06.2026

**(1) True or false:** Let $S$ be a closed, symmetric operator, $S^*$ its adjoint. If i is an eigenvalue of $S^*$, then every $z \in \mathbb{C}_+$ is also an eigenvalue.

**(2) True or false:** Let $E$ be the spectral projection corresponding to $T$. One then has

$$\lim_{\delta \to 0^+} E(t-\delta) = \lim_{\delta \to 0^+} E(t+\delta)$$

for all $t \in \mathbb{R}$.

**(3) True or false:** Let $T$ be self-adjoint, $W$ be symmetric with $T$-bound $< 1$. Then, $T + W$ is also self-adjoint and has the same spectral family $\{E(t)\}$ as $T$.

**(4) True or false:** A number $\lambda \in \mathbb{R}$ is an element of the spectrum of a self-adjoint operator (with spectral projection $E$) if and only if

$$\dim(E(\lambda+\varepsilon) - E(\lambda-\varepsilon)) = \infty,$$

for every $\varepsilon > 0$.

**(5) True or false:** A symmetric operator $S$ is self-adjoint if and only if its deficiency indices satisfy $\gamma_+ = \gamma_- = 0$.

**(6) True or false**: If $V$ is $T$-compact, then $V$ must also be $T$-spectrally locally compact.

**(7) True or false:**

Let $T$ be a self-adjoint operator with spectral projection $E$. One then has

$$\|E(t+\delta)x - E(t)x\|^2 = \|E(t+\delta)x\|^2 - \|E(t)x\|^2.$$

**(8) True or false:** The following statements are equivalent:

- $V$ has $T$-bound 1
- There exists $a \geqslant 0$ such that

$$\|Vx\| \leqslant a\|x\| + \|Tx\|,$$

for all $x \in D(T)$.

**(9) True or false:** It follows from strong resolvent convergence $T_n \xrightarrow{sr} T$ that

$$\lim_{n \to \infty} \sigma(T_n) = \sigma(T).$$