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Lineare Operatoren in Hilberträumen

Exercise Sheet 9

(27) (Sturm's comparison theorem): Let $J = (a, b)$ be an arbitrary interval (finite or infinite). Consider the Sturm–Liouville expression τ with $0 < p(x) \in C^1(J)$, $q(x) \in C(J), r(x) \equiv 1$. Let $u, v \in C^2(J)$ be real-valued. Let x_0, x_1 be consecutive zeros of v (that is, $v(x) \neq 0$ in (x_0, x_1)). Suppose

$$\frac{\tau u}{u} \leqslant \frac{\tau v}{v}$$

for all $x \in (x_0, x_1)$ for which $u(x) \neq 0$. Then exactly one the following statements is true:

(a) $u = cv$, for some constant c

(b) u has a zero in (x_0, x_1).

Hint: Use the properties of the modified Wronskian. **(3 Points)**

(28) Let u, v and τ be as above. We say that the zeros of u and v are *alternating* if, between every two consecutive zeros of u, there is a zero of v, and vice versa. Prove the following.

(a) If u_1, u_2 are linearly independent solutions of $\tau u = 0$, then the zeros of u_1 and u_2 are alternating.

(b) If u is a solution of $\tau u = (pu')' + qu = 0$, v a non-trivial solution of

$$\tau_0 v = (pv')' + q_0 v = 0 \quad \text{with } q_0(x) \leqslant q(x),$$

then between two consecutive zeros of v, there is a zero of u.

(1+1 Points)

(29) An solution to $\tau u = 0$ is called *oscillatory* in J if u has countably infinitely many zeros in J.

(a) Show that if one solution u is oscillatory, then all solutions v must be oscillatory. (Here, we also say that the SL expression is oscillatory in J).

(b) Show that the Bessel differential equation

$$x^2 u'' + xu' + (x^2 - \alpha^2)u = 0$$

is oscillatory in $J = (0, \infty)$, for all $\alpha \in \mathbb{R}$.

Hint: Consider the transformation $s = \log(x)$ and $w(s) = u(e^s)$, and apply 28(b). **(1+3 Points)**

(30) Consider the SL differential equation

$$\tau u = u'' + \frac{\gamma}{x^2} u = 0,$$

where $\gamma \in \mathbb{R}$. For which values of γ is τ oscillatory? Plot examples of $u := u_\gamma$ that show oscillatory behaviour for three values of γ.

(4 Points)

Abgabe bis 12 Uhr am Mittwoch, 24.06.2026