Dr. Neil Manibo
Fakultät für Mathematik,
Universität Bielefeld

Sommersemester 2026
Bielefeld, 24.06.2026

# Lineare Operatoren in Hilberträumen

## Exercise Sheet 10

**(31)** Let $\tau$ be a regular Sturm–Liouville differential expression in $(a, b)$. Prove that every self-adjoint realisation $A$ of $\tau$ with separable boundary conditions is of the form

$$D(A_{\alpha,\beta}) = \left\{ f \in D(T) \; : \; \begin{matrix} f(a)\cos\alpha - pf'(a)\sin\alpha = 0 \\ f(b)\cos\beta - pf'(b)\sin\beta = 0 \end{matrix} \right\}$$

for some $\alpha, \beta \in [0, \pi)$, with $A_{\alpha,\beta} f = \tau f$ for $f \in D(A_{\alpha,\beta})$.
**Hint:** Separable boundary conditions imply that one has $\mathrm{rank}(B_a) = \mathrm{rank}(B_b) = 1$ in Theorem 4.23 in the lecture.

**(4 Points)**

**(32)** Here, we would like to show part (a) of Theorem 4.27 in the lecture notes:

If $\tau$ is limit point at $x = a$, then $[f, g]_a = 0$, for all $f, g \in D(T)$, where $T$ is the maximal operator corresponding to $\tau$.

(a) Assume that $\tau$ is regular at $x = b$ (Otherwise one can pick a point in $(c, b)$ and consider the restriction of $\tau$ on $(a, c)$). What are the deficiency indices of $T_0$?

(b) Use the answer above to determine $\dim(D(T)/D(T_0))$.
**Hint:** 1st von Neumann Formula and the fact that $(T_0')^* = T$.

(c) Let $u_1, u_2 \in D(T)$, with

| | | |
|---|---|---|
| $u_1(b) = 1$ | $pu_1'(b) = 0$ | $u_1(x) \equiv 0$ around a |
| $u_2(b) = 0$ | $pu_2'(b) = 1$ | $u_2(x) \equiv 0$ around a |

Why are $u_1, u_2$ linearly independent modulo $D(T_0)$?

(d) Use (c) to prove the claim.
**Hint:** Proposition 4.14(b).

**(4 Points)**

**(33)** Consider the radial Sturm–Liouville differential expression

$$\tau_\ell = -\frac{d^2}{dr^2} + \left( \ell(\ell + d - 2) + \frac{1}{4}(d-1)(d-3) \right) \frac{1}{r^2} + V(r)$$

on $C_0^\infty(0, \infty) \subset L^2(0, \infty)$, where $V$ satisfies the following conditions:

- $V \colon (0, \infty) \to \mathbb{R}$ is locally bounded
- $V(r) \geqslant -Cr^2$, for large enough $r$

Show that $\tau_\ell$ is limit point at $x = \infty$, for all $\ell \geqslant 0$ and $d \geqslant 3$.

**(2 Points)**

Abgabe bis 12 Uhr am Mittwoch, 01.07.2026