Convergence

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Convergence of sequences

Def: $(z_n)_{n\geq 1} \in \mathbb{C}$ converges to $\ell \in \mathbb{C}$ if $\forall \varepsilon > 0, \exists N \in \mathbb{N}$ st $\forall n \geq N$, $|z_n - \ell| < \varepsilon$

Note: $z_n = x_n + iy_n$ ($x_n, y_n \in \mathbb{R}$), $\ell = p + iq$

Then $z_n \xrightarrow{n\to\infty} \ell$ iff $x_n \to p$ and $y_n \to q$

↳ $|z_n - \ell|^2 = |x_n - p|^2 + |y_n - q|^2$

As in $\mathbb{R}$, $(z_n)_n$ is called a Cauchy sequence if $\forall \varepsilon > 0, \exists N \in \mathbb{N}$ st $\forall n, m \geq N$, $|z_n - z_m| < \varepsilon$

Standard properties/results also carry-over from $\mathbb{R}$, ie sum/difference, multiplication/division, Bolzano-Weierstraß

bounded sequence of complex #'s has convergent subsequence

In particular, Cauchy's Convergence Criterion holds.

↳ $(z_n)_n$ Cauchy iff convergent.

Def: Given a series of complex numbers $\sum_{n=1}^{\infty} z_n$, the series is said to converge if the sequence of partial sums $S_n := \sum_{k=1}^{n} z_k$ converge to some $L \in \mathbb{C}$. Then $L = \sum_{n=1}^{\infty} z_n$.

As in the case of sequences, the series converges iff its real and imaginary parts do, ie w/ $L = P + iQ$, $z_n = x_n + iy_n$

$L = \sum_{n\geq 1} z_n$ iff $\sum_{n\geq 1} x_n = P$ and $\sum_{n\geq 1} y_n = Q$.

Example: For $|z| < 1$, $\sum_{n \geq 0} z^n = \frac{1}{1-z}$

- Sol: First find a formula for partial sums

$$\Rightarrow (z-1)(1 + z + z^2 + \dots + z^n)$$
$$= z + z^2 + \dots + z^{n+1} + -1 + -z + \dots + -z^n$$
$$= z^{n+1} - 1 \quad \Rightarrow \quad \sum_0^n z^k = \frac{z^{n+1}-1}{z-1}$$

For $|z| < 1$ (formula above fails for $z=1$), $\lim_{n\to\infty} z^{n+1} = 0$

& done $\square$

- There are some tests/criterion for convergence.

i) Absolute Convergence

If the series $\sum_{n\geq 1} |z_n|$ converges then so does $\sum_{n \geq 1} z_n$.

(In this case, $\sum_{n\geq 1} z_n$ is called <u>absolutely convergent</u>)

↳ not so much of a test, but rather a stronger convergence

↳ If $\sum_{n\geq 1} z_n$ converges, but $\sum |z_n|$ diverges, we call it <u>conditionally convergent</u>

ii) Ratio Test

Let $L = \lim_{n\to\infty} \left| \frac{z_{n+1}}{z_n} \right|$. If $L < 1$ → series absolutely convergent

$L > 1$ → series <u>diverges</u>

$L = 1$ or d.n.e. → inconclusive

iii) Alternating Series Test (Leibniz Criterion)

Given series of form $\sum_{n \geq 1} (-1)^{n+1} a_n$, and <u>$a_n > 0$</u>.

If a_n is monotonically decreasing and $\lim_{n\to\infty} a_n = 0$,

then the series converges.

Examples

i) $\sum_{n\geq 1} \frac{\sin(n)}{n^3} \Rightarrow \sum_{n\geq 1} \left|\frac{\sin(n)}{n^3}\right| = \sum_{n\geq 1} \frac{|\sin(n)|}{n^3}$

As $|\sin(n)| \leq 1 \Rightarrow \frac{|\sin(n)|}{n^3} \leq \frac{1}{n^3}$

& $\sum_{n\geq 1} \frac{1}{n^3}$ converges, thus so does the original series.

ii) $\sum_{n\geq 0} \frac{(1+i)^n}{2^{n+1}}$

$$L = \lim_{n\to\infty} \left| \frac{(1+i)^{n+1}}{2^{n+2}} \cdot \frac{2^{n+1}}{(1+i)^n} \right| = \lim_{n\to\infty} \left| \frac{1+i}{2} \right| = \frac{\sqrt{2}}{2} = \frac{1}{\sqrt{2}} < 1$$

& $L < 1 \Rightarrow$ convergent (absolutely).

iii) $\sum_{n\geq 1} \frac{i^n}{n} = \frac{i}{1} + \frac{-1}{2} + \frac{-i}{3} + \frac{1}{4} + \dots$

$$= \sum_{n\geq 1} (-1)^n \cdot \frac{1}{2n} + i \sum_{n\geq 0} (-1)^n \frac{1}{2n+1}$$

≥ 0 and monotonically decreasing

& Series converges

$\square$

We are often interested in sequences/series of complex functions.

Def: Given $(f_n)_n$ w/ $f_n : \mathbb{D} \to \mathbb{R}$ or $\mathbb{D} \to \mathbb{C}$, ($\mathbb{D}$ interval in $\mathbb{R}$)

a) $(f_n)_n$ converges pointwise to a function f if

$$\forall x \in \mathbb{D},\ \varepsilon > 0,\ \exists N \in \mathbb{N},\ \forall n \geq N,\ |f_n(x) - f(x)| < \varepsilon$$

$\Longrightarrow$

b) $(f_n)_n$ converges uniformly to a function f if

$\forall \varepsilon > 0, \exists N \in \mathbb{N}, \forall x \in D, n \geq N, |f_n(x) - f(x)| < \varepsilon.$

Def: $\sum_{n \geq 0} f_n(x)$ converges

i) pointwise if the sequence of partial sums $S_n(x) = \sum_{k=0}^{n} f_k(x)$ converges $\forall x \in D$.

ii) uniformly if $S_n(x)$ converges uniformly as $n \to \infty$

iii) absolutely if $\sum_{n \geq 0} |f_n(x)|$ converges $\forall x \in D$

Why bother w/ uniform convergence?

→ It preserves important properties, such as continuity, from the sequence of functions, and passes it to the limit function.

Ex: $f_n(x) = x^n$ on $[0,1]$. Note each f_n is continuous

For $0 \leq x < 1$, $x^{n+1} \leq x^n$ & $\lim_{n \to \infty} x^n = 0$ $x \in [0,1)$

For $x = 1$, $x^n = 1$ $\forall n \in \mathbb{N}$ => $\lim_{n \to \infty} x^n = 1$ $x = 1$

∴ f_n converges (pointwise) to $f(x) = \begin{cases} 0 & 0 \leq x < 1 \\ 1 & x = 1 \end{cases}$

which is not continuous.

If we want our limit function to be continuous, we must ask for uniform convergence =>

Note: We have the following test for convergence for $\sum_{n\geq 1} f_n(x)$

Weierstraß m-test

If $\exists\ (M_n)_n$ w/ $M_n \in \mathbb{R}^+\ \forall n$, such that $|f_n(x)| \leq M_n\ \forall n \geq 1, \forall x \in D,$ and $\sum_{n\geq 1} M_n$ converges then $\sum_{n\geq 1} f_n(x)$ converges absolutely and uniformly.

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