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Probability Theory I - Exercise Sheet 1

Due date: **Friday, April 28, 11:00 h**

Solutions to the assigned problems must be deposited in your tutors mailbox (Katharina von der Lühe: 186, Julian Femmer: 237, Peter Kuchling: 197, Timo Krause: 59) located in V3-128 no later than 11:00 h on the due date. Solutions must be completely legible, on A4 paper, in the correct order and stapled, with your name neatly written on the first page.

Exercise 1.I (8 pts)

Let $\Omega \neq \emptyset$ be a set.

a) Let I an arbitrary index set. For all $i \in I$ let $\mathcal{A}_i$ be a σ-Algebra on Ω. Show that:

$$\bigcap_{i \in I} \mathcal{A}_i \text{ is a } \sigma\text{-algebra.}$$

b) Let $\mathcal{E} \neq \emptyset$ be a subset of $\mathcal{P}(\Omega)$. Show that there is a smallest σ-algebra $\sigma(\mathcal{E})$ containing $\mathcal{E}$.

c) Let $\Omega = \{1, 2, 3\}$. Give an example of two σ-algebras $\mathcal{F}_1$ and $\mathcal{F}_2$, s.t. $\mathcal{F}_1 \cup \mathcal{F}_2$ is not a σ-algebra.

Exercise 1.II (8 pts)

a) Let $\Omega \neq \emptyset$ be a nonempty set and $\mathcal{A}$ a σ-algebra on Ω, ie. $(\Omega, \mathcal{A})$ is a so-called *measure space*. Let μ be a measure on $(\Omega, \mathcal{A})$ and $\alpha \geq 0$ a constant. Define the map $\alpha\mu : \mathcal{A} \to [0, \infty]$ via $(\alpha\mu)(A) = \alpha\mu(A)$ for all $A \in \mathcal{A}$. Show that $\alpha\mu$ defines a measure.

b) Let $\Omega \neq \emptyset$ be a set and μ_1, μ_2 measures on a measure space $(\Omega, \mathcal{A})$. Show that $\mu_1 + \mu_2$, given by

$$(\mu_1 + \mu_2)(A) := \mu_1(A) + \mu_2(A) \quad \text{für alle } A \in \mathcal{A},$$

defines a measure.

c) Let $\Omega \neq \emptyset$ be a set and $\mathbb{P}_1$ as well as $\mathbb{P}_2$ probability measures on a measure space $(\Omega, \mathcal{A})$. Let furthermore $\lambda_1, \lambda_2 \in [0, 1]$ be given s.t. $\lambda_1 + \lambda_2 = 1$. Show that

$$(\lambda_1\mathbb{P}_1 + \lambda_2\mathbb{P}_2)(A) = \lambda_1\mathbb{P}_1[A] + \lambda_2\mathbb{P}_2[A] \quad \text{für alle } A \in \mathcal{A}$$

defines a probability measure on $(\Omega, \mathcal{A})$.

Exercise 1.III (8 pts)

a) Let the index set I be either given by $I = \{1, 2, \dots, n\}$ for some $n \in \mathbb{N}$ or by $I = \mathbb{N}$. Let $\{A_n\}_{n\in I}$ be a *partition* of Ω, i.e. $A_i \cap A_j = \emptyset$ if $i \neq j$ and $\Omega = \bigcup_{n\in I} A_n$. Show that $\mathcal{A} = \{\bigcup_{n\in J} A_n \big| J \subset I\}$ is a σ-algebra.

b) Let $\Omega = \mathbb{R}$ and let $\mathcal{A}$ be the σ-algebra of all sets $A \subset \Omega$ either being countable or having a countable complement A^c. Let $\mathbb{P} : \mathcal{A} \to [0, 1]$ be defined by

$$\mathbb{P}[A] = \begin{cases} 0 & \text{if } A \text{ countable,} \\ 1 & \text{if } A^c \text{ countable.} \end{cases}$$

In exercise 0.I (to be discussed in the first tutorial session) it will be shown that $\mathbb{P}$ defines a probability measure on $(\Omega, \mathcal{A})$.
Let $\Omega' = \{0, 1\}$, $\mathcal{A}' = \mathcal{P}(\Omega')$ and $T : \Omega \to \Omega'$ be defined as

$$T(\omega) = \begin{cases} 1 & \text{if } \omega \text{ irrational,} \\ 0 & \text{if } \omega \text{ rational.} \end{cases}$$

Show that T is an $\mathcal{A}$-$\mathcal{A}'$-measurable mapping and determine the so-called *image measure* $\mathbb{P}T^{-1}$, defined by $(\mathbb{P}T^{-1})(A') := \mathbb{P}(T^{-1}(A')) = \mathbb{P}(T \in A')$ for all $A' \in \mathcal{A}'$.

Exercise 1.IV (8 pts)

Let $\Omega \neq \emptyset$ be a set. A collection $\mathcal{D}$ of subsets of Ω is called a *Dynkin system*, if:

1) $\Omega \in \mathcal{D}$,

2) $D \in \mathcal{D} \;\Rightarrow\; D^c \in \mathcal{D}$,

3) For every pairwise disjoint family $(D_n)_{n\in\mathbb{N}}$ of elements of $\mathcal{D}$ we have

$$\bigcup_{n=1}^{\infty} D_n \in \mathcal{D}.$$

a) Give an example of a Dynkin system $\mathcal{D}$ for $\Omega = \{1, 2, 3, 4\}$ s.t. $\mathcal{D}$ is <u>not</u> a σ-algebra.

b) Show that a Dynkin system $\mathcal{D}$ on Ω is a σ-algebra if and only if $\mathcal{D}$ is stable w.r.t. intersections, i.e. if and only if for arbitrary $D_1, D_2 \in \mathcal{D}$ we have $D_1 \cap D_2 \in \mathcal{D}$.

c) Let $\mathcal{E}$ be a system of subsets of Ω. Define $\delta(\mathcal{E})$ to be the smallest Dynkin system on Ω, containing $\mathcal{E}$. Prove that the following implication holds:
If $\mathcal{E}$ is stable w.r.t. intersections, then we have: $\delta(\mathcal{E}) = \sigma(\mathcal{E})$.