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SoSe 2017

Probability Theory I - Exercise Sheet 10

Due date: **Friday, June 30, 11:00 h**

Solutions to the assigned problems must be deposited in your tutors mailbox (Katharina von der Lühe: 186, Peter Kuchling: 197, Timo Krause: 59) located in V3-128 no later than 11:00 h on the due date.
Solutions must be completely legible, on A4 paper, in the correct order and stapled, with your name neatly written on the first page.

In the following exercises, let $(X_i)_{i\in\mathbb{N}}$ be i.i.d. random variables on a probability space $(\Omega, \mathcal{F}, \mathbb{P})$ and set $S_n := \sum_{i=1}^n X_i$.

Exercise 10.I (8 pts)
Assume
$$\frac{S_n}{n^{1/p}} \to 0 \text{ a.s. for some } p > 0.$$

Show that $\mathbb{E}(|X_1|^p) < \infty$.

Remark: This statement corresponds to the converse of Theorem 16.3 from the lecture.

Exercise 10.II (8 pts)
Use Exercise 9.I to show that
$$\frac{S_n}{n} \to 0 \quad \text{in probability,}$$
implies
$$\frac{\max_{1\le m\le n} S_m}{n} \to 0 \quad \text{in probability.}$$

Exercise 10.III (8 pts)

Let $(a_n)_n$ be a sequence in $\mathbb{R}$ s.t. $a_n \nearrow \infty$ and $\frac{a_{2^n}}{a_{2^{n-1}}}$ is bounded.

i) Use Exercise 9.I to show that

$$\frac{S_n}{a_n} \to 0 \text{ in probability} \quad \text{and} \quad \frac{S_{2^n}}{a_{2^n}} \to 0 \quad \text{a.s.}$$

together imply

$$\frac{S_n}{a_n} \to 0 \quad \text{a.s.}$$

ii) Suppose in addition that $\mathbb{E}(X_1) = 0$ and $\mathbb{E}(X_1^2) < \infty$. Show that

$$\frac{S_n}{n^{1/2}(\log_{\mathbf{2}} n)^{1/2+\epsilon}} \to 0 \quad \text{a.s.}$$

Exercise 10.IV (8 pts)

a) Let X_1 be a Poisson distributed random variable with mean 1.

  i) Show that S_n has a Poisson distribution with mean n.

  ii) Use Stirling's formula to show that $\frac{k_n - n}{\sqrt{n}} \to x$ implies

$$\sqrt{2\pi n}\,\mathbb{P}(S_n = k_n) \to \exp\left(-\frac{x^2}{2}\right).$$

b) Suppose $\mathbb{P}(X_1 = 1) = \mathbb{P}(X_1 = -1) = \frac{1}{2}$. Show that

$$\frac{1}{2n} \log \mathbb{P}(S_{2n} \geq 2na) \to -\gamma(a) \qquad \forall a \in (0,1),$$

where $\gamma(a) = \frac{1}{2}\left\{(1+a)\log(1+a) + (1-a)\log(1-a)\right\}.$