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SoSe 2017

Probability Theory I - Exercise Sheet 11

Due date: **Friday, July 7, 11:00 h**

Solutions to the assigned problems must be deposited in your tutors mailbox (Katharina von der Lühe: 186, Peter Kuchling: 197, Timo Krause: 59) located in V3-128 no later than 11:00 h on the due date.
Solutions must be completely legible, on A4 paper, in the correct order and stapled, with your name neatly written on the first page.

In the following, let $(X_i)_{i\in\mathbb{N}}$ be i.i.d. random variables on a probability space $(\Omega, \mathcal{F}, \mathbb{P})$.

Exercise 11.I (*Convergence of maxima*) (8 pts)
Let $F(\cdot)$ denote the distribution function of X_1. Define $M_n := \max\{X_1, \ldots, X_n\}$, $n \in \mathbb{N}$.

a) Show that
$$\mathbb{P}(M_n \le x) = [F(x)]^n, \quad \forall x \in \mathbb{R}.$$

b) Let $\alpha > 0$ and assume that X_1 has distribution function
$$F(x) = \begin{cases} 1 - x^{-\alpha} & \text{if } x \ge 1, \\ 0 & \text{otherwise.} \end{cases}$$
Show that for $y > 0$
$$\mathbb{P}\left[\frac{M_n}{n^{\frac{1}{\alpha}}} \le y\right] \xrightarrow[n\to\infty]{} \exp\left(-y^{-\alpha}\right).$$

c) Let $\beta > 0$ and assume that X_1 has distribution function
$$F(x) = \begin{cases} 1 - |x|^{\beta} & \text{if } x \in [-1, 0], \\ 1 & \text{if } x > 0, \\ 0 & \text{otherwise.} \end{cases}$$
Show that for $y < 0$:
$$\mathbb{P}\left[n^{\frac{1}{\beta}} M_n \le y\right] \xrightarrow[n\to\infty]{} \exp\left(-|y|^{\beta}\right).$$

d) Assume that X_1 has the distribution function

$$F(x) = \begin{cases} 1 - \exp(-x) & \text{if } x \geq 0, \\ 0 & \text{otherwise.} \end{cases}$$

Show that for all $y \in \mathbb{R}$

$$\mathbb{P}\left[M_n - \log(n) \leq y\right] \xrightarrow[n\to\infty]{} \exp\left[-\exp\left(-y\right)\right].$$

Remark: *The limits that appear above are called the* ***extreme value distributions****. The last one is called the* ***double exponential*** *or* ***Gumbel distribution****.*

Exercise 11.II (8 pts)

a) Show that the discrete uniform distribution on $\{\frac{1}{n}, \frac{2}{n}, \ldots, \frac{n-1}{n}, 1\}$ converges weakly to the uniform distribution on $[0, 1]$, as $n \to \infty$.

b) Let $\alpha > 0$ and X_1 be uniformly distributed on $[0, \alpha]$. Let $Y_n := \max\{X_1, \ldots, X_n\}$ and $Z_n := n(\alpha - Y_n)$ for $n \in \mathbb{N}$.

   Show that the distribution of Z_n converges weakly to the exponential distribution with parameter $1/\alpha$, as $n \to \infty$.

Exercise 11.III (8 pts)

i) **Lévy Metric** on the space of distribution functions:

   Let F and G be distribution functions. Define

$$\rho(F, G) := \inf\Big\{\varepsilon \geq 0 : F(x - \varepsilon) - \varepsilon \leq G(x) \leq F(x + \varepsilon) + \varepsilon \text{ for all } x \in \mathbb{R}\ \Big\}.$$

   a) Show that $\rho(\cdot, \cdot)$ defines a metric on the space of distribution functions.

   b) Let $(F_n)_{n\in\mathbb{N}}$ be a family of distribution functions. Show that

$$\rho(F_n, F) \xrightarrow[n\to\infty]{} 0 \quad \text{if and only if} \quad F_n \overset{}{\underset{n\to\infty}{\Longrightarrow}} F.$$

ii) **Ky Fan metric** on the space of random variables:

   Let X and Y be random variables on $(\Omega, \mathcal{F}, \mathbb{P})$. Define

$$\alpha(X, Y) = \inf\left\{\varepsilon \geq 0 \mid \mathbb{P}\left(|X - Y| > \varepsilon\right) \leq \varepsilon\right\}.$$

   Show that if $\alpha(X, Y) = c$ for some $c \geq 0$, then the corresponding distributions have Lévy distance $\rho(F, G) \leq c$.

Exercise 11.IV (*Preparation for a mini-presentation*) (8 pts)

Prepare a short talk on the topic given below. You should present the material in your own words without relying on your notes. The presentation should be made using the blackboard, writing down keywords and the essential formulae.

Durrett: Example 3.2.5. Birthday problem.