Prof. Dr. Barbara Gentz
Christian Wiesel
Faculty of Mathematics
Bielefeld University

SoSe 2017

Probability Theory I - Exercise Sheet 12

Due date: **Friday, July 14, 11:00 h**

Solutions to the assigned problems must be deposited in your tutors mailbox (Katharina von der Lühe: 186, Peter Kuchling: 197, Timo Krause: 59) located in V3-128 no later than 11:00 h on the due date.
Solutions must be completely legible, on A4 paper, in the correct order and stapled, with your name neatly written on the first page.

In the following, let $(\Omega, \mathcal{F}, \mathbb{P})$ be a probability space.

Exercise 12.I (8 pts)

a) Make use of characteristic functions to determine the distribution of $X_1 + X_2$, where X_1 and X_2 are independent random variables, satisfying
   i) $X_1 \sim \mathcal{N}(\mu_1, \sigma_1^2)$, $X_2 \sim \mathcal{N}(\mu_2, \sigma_2^2)$;
   ii) $X_1 \sim \mathrm{Poi}(\lambda_1)$, $X_2 \sim \mathrm{Poi}(\lambda_2)$.

b) Let $X_1, X_2, \dots$ be i.i.d. real valued random variables with distribution μ and let N be a random variable with $N \sim \mathrm{Poi}(\lambda)$, independent of $X_1, X_2, \dots$.
   Determine the characteristic function of the random sum $S = \sum_{i=1}^{N} X_i$.

Exercise 12.II (8 pts)

Let μ be a probability measure on $\mathbb{R}$ and $\varphi(t) = \int e^{itx}\mu(\mathrm{d}x)$ its characteristic function. Prove the following statements:

a) Discrete FOURIER-inversion: $\mu(\{x\}) = \lim_{T\to\infty} \frac{1}{2T} \int_{-T}^{T} e^{-itx}\varphi(t)\,\mathrm{d}t \quad \text{for } x \in \mathbb{R}.$

b) PLANCHEREL equality: $\int_{\mathbb{R}} \mu(\{x\})\mu(\,\mathrm{d}x) = \lim_{T\to\infty} \frac{1}{2T} \int_{-T}^{T} |\varphi(t)|^2\,\mathrm{d}t.$

c) If $\mathbb{P}(X \in h\mathbb{Z}) = 1$ for $h > 0$, then it follows that $\varphi(2\pi/h + t) = \varphi(t)$, and therefore

$$\mathbb{P}(X = x) = \frac{h}{2\pi} \int_{-\pi/h}^{\pi/h} e^{-itx}\varphi(t)\,\mathrm{d}t \quad \text{for } x \in h\mathbb{Z}.$$

Exercise 12.III (8 pts)

a) Let X be a real valued random variable with characteristic function $\varphi_X(t)$.

   Show that $\varphi_X(t)$ is real valued, if and only if X and $-X$ have the same distribution.

b) Let X_n be normally distributed with expectation value γ_n and variance $\sigma_n^2 > 0$ for all $n \in \mathbb{N}$. Assume that $X_n \Rightarrow X$ to some random variable X, as $n \to \infty$.

   Show that there are $\gamma \in \mathbb{R}$ and $\sigma^2 \in [0, \infty)$ s.t. $\gamma_n \to \gamma$ and $\sigma_n^2 \to \sigma^2$, as $n \to \infty$.

   Assuming now that $\sigma^2 > 0$, show that X is normally distributed with expectation value γ and variance σ^2.

c) Assume that X_n, Y_n are independent for $1 \leq n \leq \infty$ and satisfy $X_n \Rightarrow X_\infty$ and $Y_n \Rightarrow Y_\infty$, as $n \to \infty$. Show that $X_n + Y_n \Rightarrow X_\infty + Y_\infty$, as $n \to \infty$.

Exercise 12.IV (8 pts)

a) Let $(\mu_i)_{i \in I}$ be a tight family of measures, i.e. $\sup_i \mu_i([-M, M]^{\mathrm{c}}) \to 0$, as $M \to \infty$.

   Show that the corresponding characteristic functions $(\varphi_i)_{i \in I}$ are uniformly equicontinuous, i.e. for every $\varepsilon > 0$ there is a $\delta > 0$, s.t. for all h with $|h| < \delta$ we have: $|\varphi_i(t + h) - \varphi_i(t)| < \varepsilon$.

b) Show that: If $\mu_n \Rightarrow \mu_\infty$ for $n \to \infty$, then the characteristic functions $(\varphi_n)_n$ converge uniformly on every compact set to some function φ_∞, i.e. for every compact set $K \subset \mathbb{R}$ we find: $\sup_{s \in K} |\varphi_n(s) - \varphi_\infty(s)| \to 0$, as $n \to \infty$.