Prof. Dr. Barbara Gentz
Christian Wiesel
Faculty of Mathematics
Bielefeld University

SoSe 2017

Probability Theory I - Exercise Sheet 13

Due date: **Friday, July 21, 11:00 h**

Solutions to the assigned problems must be deposited in your tutors mailbox (Katharina von der Lühe: 186, Peter Kuchling: 197, Timo Krause: 59) located in V3-128 no later than 11:00 h on the due date.
Solutions must be completely legible, on A4 paper, in the correct order and stapled, with your name neatly written on the first page.

In the following, let $(X_i)_{i\in\mathbb{N}}$ be a sequence of i.i.d., real valued random variables on a probability space $(\Omega, \mathcal{F}, \mathbb{P})$ and let $S_n = \sum_{i=1}^n X_i$.

Exercise 13.I (8 pts)

Let φ be the characteristic function of X_1. Show that:

a) If $\varphi'(0) \equiv ia$ for some $a \in \mathbb{R}$, then S_n/n converges in probability to a, as $n \to \infty$.

b) If S_n/n converges in probability to some $b \in \mathbb{R}$, as $n \to \infty$, then $\varphi(t/n)^n$ converges to e^{ibt}, as $n \to \infty$.

Exercise 13.II (8 pts)

Making use of the general central limit theorem, show that

$$\lim_{n\to\infty} e^{-n} \sum_{k=1}^{n} \frac{n^k}{k!} = \frac{1}{2}.$$

Hint: Interpret the r.h.s in terms of the distribution of the sum of n independent, Poi(1)-distributed random variables.

Exercise 13.III (8 pts)

i) Let $(Y_n)_n$ and $(Z_n)_n$ be sequences of random variables, s.t. $Y_n \Rightarrow Y$ and $Z_n \Rightarrow c$, where Y is a random variable and c is a constant.

   Show that $Y_n + Z_n \Rightarrow Y + c$, as $n \to \infty$.

ii) Assume that $X_i \geq 0$, $\mathbb{E}[X_i] = 1$ and $\mathrm{var}(X_i) = \sigma^2 \in (0, \infty)$ for all $i \in \mathbb{N}$. Show that

$$\sqrt{S_n} - \sqrt{n} \Rightarrow \frac{\sigma}{2}\mathcal{X}, \qquad \text{as } n \to \infty,$$

   where $\mathcal{X}$ is a $\mathcal{N}(0,1)$-distributed random variable.

In the following, let $(Y_i)_{i\in\mathbb{N}}$ be a sequence of independent, real valued random variables on the probability space $(\Omega, \mathcal{F}, \mathbb{P})$ and let $T_n = \sum_{i=1}^n Y_i$.

Exercise 13.IV (8 pts)

a) Assume that $\mathbb{P}(Y_m = m) = \mathbb{P}(Y_m = -m) = m^{-2}/2$ and for $m \geq 2$

$$\mathbb{P}(Y_m = 1) = \mathbb{P}(Y_m = -1) = (1 - m^{-2})/2, \quad m \in \mathbb{N}.$$

Show that $\mathrm{var}(T_n)/n \to 2$, as $n \to \infty$, but nevertheless $T_n/\sqrt{n} \Rightarrow \mathcal{X}$, as $n \to \infty$, where $\mathcal{X}$ is a $\mathcal{N}(0,1)$-distributed random variable.

b) Assume that $|Y_i| \leq M$ and $\sum_i \mathrm{var}(Y_i) = \infty$.
Show that $(T_n - \mathbb{E}[T_n])/\sqrt{\mathrm{var}(T_n)} \Rightarrow \mathcal{X}$, as $n \to \infty$, where $\mathcal{X}$ is a $\mathcal{N}(0,1)$-distributed random variable.