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SoSe 2017

Probability Theory I - Exercise Sheet 2

Due date: **Friday, May 5, 11:00 h**

Solutions to the assigned problems must be deposited in your tutors mailbox (Katharina von der Lühe: 186, Julian Femmer: 237, Peter Kuchling: 197, Timo Krause: 59) located in V3-128 no later than 11:00 h on the due date. Solutions must be completely legible, on A4 paper, in the correct order and stapled, with your name neatly written on the first page.

Exercise 2.I (6 pts)

Let $d \in \mathbb{N}$.

(i) Show that a continuous function $f : \mathbb{R}^d \to \mathbb{R}$ is a measurable mapping from $(\mathbb{R}^d, \mathcal{B}(\mathbb{R}^d))$ to $(\mathbb{R}, \mathcal{B}(\mathbb{R}))$.

(ii) Show that $\mathcal{B}(\mathbb{R}^d)$ is the smallest σ-algebra ist, w.r.t which all continuous functions are measurable.

Exercise 2.II (8 pts)

Let $(\Omega, \mathcal{F})$ be a measure space. A map $\varphi : \Omega \to \mathbb{R}$ is called *simple*, if there are an $n \in \mathbb{N}$ and real constants c_m as well as sets $A_m \in \mathcal{F}$, $m \in \{1, \dots, n\}$, s.t.

$$\varphi(\omega) = \sum_{m=1}^{n} c_m \mathbb{1}_{A_m}(\omega).$$

a) Show that the set of $\mathcal{F}$-measurable maps can be identified as the smallest set of maps, which contains all simple functions and is closed under pointwise limits.

b) Let X and Y be real valued random variables on a probability space $(\Omega, \mathcal{F}, \mathbb{P})$. Making use of the result of part a), show that Y is measurable w.r.t. $\sigma(X)$ if and only if there is a measurable map $f : \mathbb{R} \to \mathbb{R}$ s.t. $Y = f(X)$.

Exercise 2.III (10 pts)

(a) Let X be a real random variable with a continuous density f w.r.t. Lebesgue measure. Assume that there is an interval $[\alpha, \beta] \subset \overline{\mathbb{R}} := [-\infty, \infty]$ with $-\infty \leq \alpha < \beta \leq \infty$, s.t. $f = 0$ on $[\alpha, \beta]^c$. Let g be a strictly increasing function, which is differentiable on (α, β) with derivative $g'(t) = \frac{d}{dt} g(t) > 0$ for all $t \in (\alpha, \beta)$.

(i) Show that the density of $g(X)$ is given by $d_g : \mathbb{R} \to \mathbb{R}$, defined as

$$d_g(y) = \begin{cases} \frac{f(g^{-1}(y))}{g'(g^{-1}(y))} & \text{for } y \in (g(\alpha), g(\beta)), \\ 0 & \text{otherwise.} \end{cases}$$

(ii) Let X be a normally distributed random variable with density

$$\varphi(x) = \frac{1}{\sqrt{2\pi\sigma^2}} \exp\left(-\frac{(x-\mu)^2}{2\sigma^2}\right) \qquad \text{for } x \in \mathbb{R},$$

where $\mu \in \mathbb{R}$ and $\sigma > 0$. Employ part (i) in order to calculate the density of the random variable $Y = \exp(X)$.
Remark: The distribution of $\exp(X)$ is called log-*normal distribution.*

(b) Let X be a real random variable with density f w.r.t. the Lebesgue-measure.

(i) Determine the distribution function of the random variable X^2.

(ii) Use differentiation to derive the density of X^2 from its distribution function.

(iii) Determine the density of X^2 for a normally distributed random variable X with density

$$\varphi(x) = \frac{1}{\sqrt{2\pi\sigma^2}} \exp\left(-\frac{(x-\mu)^2}{2\sigma^2}\right) \qquad \text{for } x \in \mathbb{R},$$

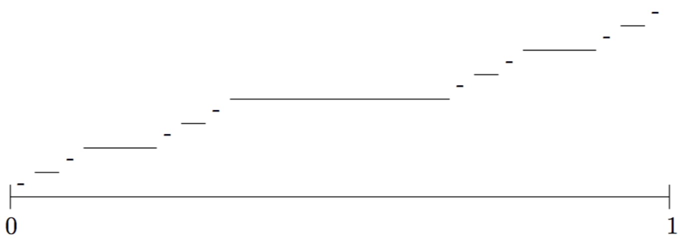
where $\mu \in \mathbb{R}$ and $\sigma > 0$.
Remark: In this case the distribution of X^2 is called χ^2-*distribution.*

Exercise 2.IV (8 pts)

a) Show that a distribution function has at most countably many points where it is discontinuous.

b) The Cantor set C is defined by removing $(1/3, 2/3)$ from $[0, 1]$ and then repeatedly removing the middle third of each interval that remains. We define an associated distribution function by setting[1]

$$F(x) := \begin{cases} 0 & \text{for } x \leq 0 \\ 1 & \text{for } x \geq 1 \\ 1/2 & \text{for } x \in [1/3, 2/3] \\ 1/4 & \text{for } x \in [1/9, 2/9] \\ 3/4 & \text{for } x \in [7/9, 8/9] \\ \vdots & \vdots \end{cases}$$



(i) Show that F is a distribution function.

(ii) Show that F does not have a density w.r.t. Lebesgue-measure.

(iii) Let μ be the measure defined by F. Show that we have $\mu(C^c) = 0$.

[1] cf. Durrett, Probability: Theory and Examples, Example 1.2.4