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SoSe 2017

Probability Theory I - Exercise Sheet 3

Due date: **Friday, May 12, 11:00 h**

Solutions to the assigned problems must be deposited in your tutors mailbox (Katharina von der Lühe: 186, Julian Femmer: 237, Peter Kuchling: 197, Timo Krause: 59) located in V3-128 no later than 11:00 h on the due date. Solutions must be completely legible, on A4 paper, in the correct order and stapled, with your name neatly written on the first page.

Exercise 3.I (8 pts)

Let $(\Omega, \mathcal{F}, \mu)$ be a measure space with σ-finite μ, i.e. there exist $(E_n)_n \subset \mathcal{F}$ with $E_n \uparrow \Omega$ and $\mu(E_n) < \infty$, $\forall\, n \geq 1$.

a) Let $f \geq 0$ and $a \wedge b := \min(a, b)$; show that:

$$\int_{E_n} f \wedge n \, d\mu \uparrow \int f \, d\mu.$$

b) Let $f_n : \Omega \to \mathbb{R}$ and $f : \Omega \to \mathbb{R}$ be integrable, i.e. $\int |f_n| d\mu, \int |f| d\mu < \infty$. Prove the following statement, without making use of the theorem of dominated convergence:

Bounded convergence theorem
Assume that:

- For all $\varepsilon > 0, A \in \mathcal{F}$ with $\mu(A) < \infty$: $\mu\big(\{|f - f_n| > \varepsilon\} \cap A\big) \longrightarrow 0$ as $n \to \infty$, which defines the so-called convergence of $f_n \xrightarrow{n\to\infty} f$ in measure μ.
- Furthermore assume that there exist a set $E \in \mathcal{F}$ with $\mu(E) < \infty$ and an $M \in \mathbb{N}$ with $|f_n| \leq M$, $|f| \leq M$ and $f_n = 0$ on E^c for all $n \geq 1$.

Then $\lim_{n\to\infty} \int f_n \, d\mu$ exists and

$$\lim_{n\to\infty} \int f_n \, d\mu = \int f \, d\mu.$$

Exercise 3.II (8 pts)

Prove the following theorem:

Fatou's Lemma

For a (σ-finite) measure space $(\Omega, \mathcal{F}, \mu)$ and measurable function $f_n \geq 0$, $n \geq 1$ on $(\Omega, \mathcal{F})$, we have:

$$\liminf_{n\to\infty} \int f_n \, d\mu \geq \int \liminf_{n\to\infty} f_n \, d\mu.$$

Exercise 3.III (8 pts)

Let X be a real, integrable random variable on a probability space $(\Omega, \mathcal{A}, \mathbb{P})$ and let $\varphi : \mathbb{R} \to \mathbb{R}$ be an $\mathcal{A}$-measurable, strictly convex mapping, i.e.

$$\lambda\varphi(x) + (1-\lambda)\varphi(y) > \varphi(\lambda x + (1-\lambda)y) \quad \text{for all } \lambda \in (0,1).$$

Moreover let $\varphi(X)$ be integrable w.r.t. $\mathbb{P}$ and assume that in the Jensen inequality we have equality, i.e. $\varphi(\mathbb{E}[X]) = \mathbb{E}[\varphi(X)]$.

Show that $X = \mathbb{E}[X]$ $\mathbb{P}$-a.s..

Exercise 3.IV (8 pts)

Let X standard normally distributed and Y be Poisson-distributed with parameter λ (i.e. $\mathbb{P}(Y = n) = e^{-\lambda}\frac{\lambda^n}{n!}$ for $n = 0, 1, 2, \ldots$) on a probability space $(\Omega, \mathcal{A}, \mathbb{P})$.

Show that for all $k \in \mathbb{N}$, X^k and Y^k are integrable w.r.t $\mathbb{P}$, and calculate the k-th moments $\mathbb{E}[X^k]$ and $\mathbb{E}[Y^k]$.