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SoSe 2017

Probability Theory I - Exercise Sheet 4

Due date: **Friday, May 19, 11:00 h**

Solutions to the assigned problems must be deposited in your tutors mailbox (Katharina von der Lühe: 186, Julian Femmer: 237, Peter Kuchling: 197, Timo Krause: 59) located in V3-128 no later than 11:00 h on the due date. Solutions must be completely legible, on A4 paper, in the correct order and stapled, with your name neatly written on the first page.

Exercise 4.I (8 pts)

Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a probability space and $A, B \in \mathcal{F}$. Moreover let $(S, \mathcal{S})$ and $(R, \mathcal{R})$ be measurable spaces.

a) Let $A, B \in \mathcal{F}$ be independent. Show that this implies the independence of the following pairs of sets:

   (i) A^c and B,

   (ii) A and B^c,

   (iii) A^c and B^c.

b) Show that the independence of A and B is equivalent to the independence of the corresponding indicator functions $\mathbb{1}_A(\cdot)$ and $\mathbb{1}_B(\cdot)$.

c) Let $X : \Omega \to S$ and $Y : \Omega \to R$ be independent random variables. Conclude that $\sigma(X)$ and $\sigma(Y)$ are independent sigma-algebras.

d) Let $\mathcal{F}_X, \mathcal{F}_Y \subset \mathcal{F}$ be sub-σ-algebras of $\mathcal{F}$ such that $X : \Omega \to S$ is $\mathcal{F}_X - \mathcal{S}$-measurable and $Y : \Omega \to R$ is $\mathcal{F}_Y - \mathcal{R}$-measurable (e.g. $\mathcal{F}_X = \sigma(X), \mathcal{F}_Y = \sigma(Y)$). Assume that $\mathcal{F}_X$ and $\mathcal{F}_Y$ are independent σ-algebras, and show that X and Y are independent random variables.

Exercise 4.II (8 pts)
Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a probability space.

a) Let E_i be a countable set for all $i \in \{1, ..., n\}$, equipped with the σ-algebra $\mathcal{E}_i := \mathcal{P}(E_i)$. Furthermore, let $X_i : \Omega \to E_i$ be a random variable for all $i \in \{1, ..., n\}$. Show that the random variables $X_1, \dots, X_n$ are independent if and only if

$$\mathbb{P}\big[X_1 = x_1, \dots, X_n = x_n\big] = \prod_{i=1}^{n} \mathbb{P}[X_i = x_i], \qquad \text{for all } x_i \in E_i, i \in \{1, \dots, n\}.$$

b) Let $X_1, \dots, X_n$ be real valued random variables. Show the equivalence of the following two statements:

(i) $X_1, \dots, X_n$ are independent and $(X_1, \dots, X_n)$ has a probability density $f_{(X_1,\dots,X_n)} : \mathbb{R}^n \to [0, \infty)$.

(ii) X_i has a probability density $f_{X_i} : \mathbb{R} \to [0, \infty)$ for all $i \in \{1, ..., n\}$ and

$$f_{(X_1,\dots,X_n)}(x_1, \dots, x_n) = \prod_{i=1}^{n} f_{X_i}(x_i) \quad \text{ for all } x_1, \dots, x_n \in \mathbb{R}$$

defines a probability density for $(X_1, \dots, X_n)$.

Exercise 4.III (8 pts)

Let $X, X_1, \dots, X_n$ be independent, real valued random variables. Show that:

a) The pair X, X is independent if and only if X is constant $\mathbb{P}$-a.s.

b) The random variables $X_1, \dots, X_n$ are constant $\mathbb{P}$-a.s. if and only if $\sum_{i=1}^{n} X_i$ is constant $\mathbb{P}$-a.s.

Exercise 4.IV (8 pts)
Let $X_1, \dots, X_n$ be independent and identically distributed random variables s.t. X_i is uniformly distributed on (a, b) for all $i = 1, \dots, n$. Show that the random variables $\min_{i\in 1,\dots,n} X_i$ and $\max_{i\in 1,\dots,n} X_i$ possess densities and give an explicit expression.

Remark regarding densities (Problems II b) and IV):
A nonnegative, measurable function $f : \mathbb{R}^n \to \mathbb{R}$ is a density for the $\mathbb{R}^n$-valued random variable $(X_1, \dots, X_n)$ if $\mathbb{P}((X_1, \dots, X_n) \in A) = \int_A f(x)\, dx$ for all $A \in \mathcal{B}(\mathbb{R}^n)$.